I. SYLLABUS

This course will provide an introduction to classical mechanics and electricity and magnetism with an emphasis on the use of calculus. The book that will be used is *Physics for Scientists and Engineers* (3rd Edition, Prentice Hall) by Douglas Giancoli. The course is taught on the basis of lecture notes which can be obtained from my web site (http://www.physics.niu.edu/∼veenendaal/252.htm). The course will cover the following subjects (the G... refer to Chapters in Giancoli). The final grade will be based on 50% of the biweekly homeworks (to be posted on the web site) and 50% for the final exam.

- Length scales
- The concepts of differentiating and integrating
  - Differentiation
  - Integration
  - A little history of calculus
  - Formal approach to differentiating
  - Rules for differentiating
  - Derivatives of some elementary functions
  - Formal approach to integrating
- Mechanics: What came before
- Motion in one dimension (G.2)
  - Constant acceleration
  - Gravitation
  - Braking distance
  - Time-dependent acceleration
- Kinematics in two dimensions (G.3)
  - Vectors
- Newton's laws (G.4,5)
- Circular motion
- Gravitation (G.6)
  - Density of the Earth
- Work and Energy (G.7,8)
  - One dimension
  - Three dimensions
- Conservative forces
  - Gravity
- Friction
- The fundamental forces
  - Gravitational and electric forces
  - Strong force
  - Weak force
  - What do we learn from all this?

- Conservation of momentum (G.9)
- Electric charge and electric field (G.21).
- Gauss's law (Chapter G.22).
- Electric potential (Chapter G.23).
- Capacitance and dielectrics (Chapter G.24).
- Magnetism and applications (G.27, 28).
- Induction (G.29).
- $*$ Inductance and AC Circuits (G.30,31).
- Maxwell's equations (G.32).

The course will have biweekly graded homeworks and a final exam.

II. LENGTH SCALES

As humans we are generally concerned with length scales of the order of feets or meters. Units were originally defined to be relevant to humans. The origin of foot in not hard to guess. The English word for inch comes from the Latin uncia meaning "one twelth part." Therefore, twelve inches in a foot. Note that the word ounce has the same origin (one twelfth of a pound). In many languages, the word for inch is related to another human body part: the

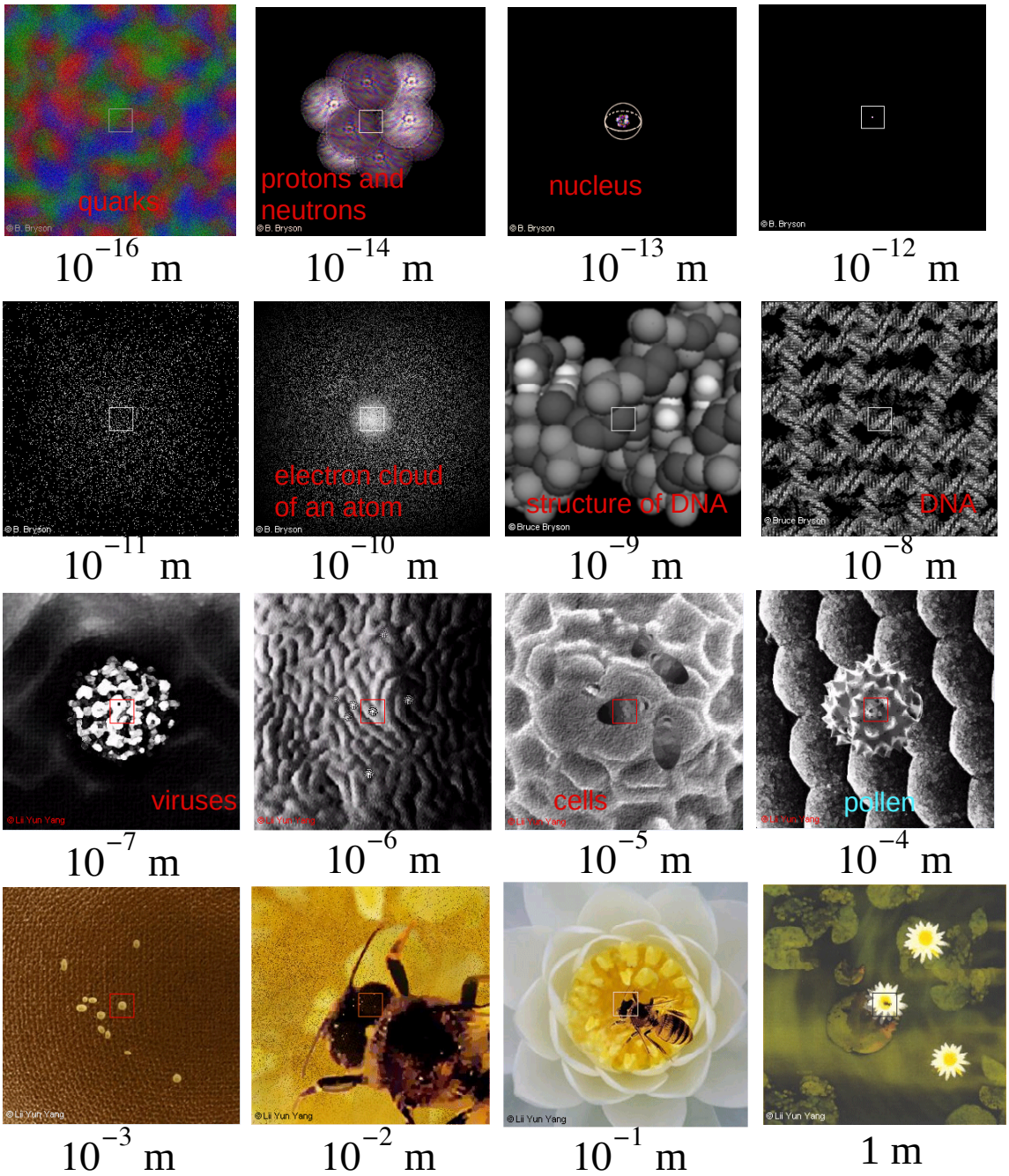


FIG. 1: Typical length scales less than a meter. Copyright Bruce Bryson (www.wordwizz.com/pwrsof10.htm).

thumb (Dutch: *duim* is both thumb and inch, French: *pouce*, Sanskrit: *Angulam* is inch and *Anguli* is finger, similar examples are available in Italian, Spanish, Swedish, etc.). You might also wonder, why 12 inch to a foot and not say 10 index fingers to a foot? We are nowadays so used to the decimal system (although some countries are not that fond of the metric system), that we prefer to think in tens, and we have tended to forget about the great advantages of non-metric systems. Consider a dozen of eggs. Twelve can be divided by 2, 3, 4, and 6. Try to do that with ten eggs. The same convenience existed with measures of fluids: 2 cups=1 pint, 2 pints= 1 quart, 4 quarts =1 gallon, 8 gallons =1 bushel. Even computers agree that the binary system is preferable (101101101).

The metric system was introduced by the French and spread around Europe through the French revolution and Napoleon. However, Napoleon never invaded England and the United States, and the metric system never really took hold there. Napoleon himself was not a big fan ("Nothing is more contrary to the organization of the mind, memory, and imagination. The new system will be a stumbling block and a source of difficulties for generations to come. It is just tormenting the people with trivia"). The French Academy of Sciences, which included distinguished scientist such as Lagrange and Laplace considered a suitable unit for length. They took the earth's circumference as a measure defining the "metre" as one ten millionth of a quarter of the earth's meridian passing through (obviously) Paris. Again, they took note of the size of humans. They could have defined the earth's meridian as 1 meter and we would all be dealing with hundreds of nanometers. The determination of the meter was an arduous task performed by astronomers Jean Baptiste Delambre and P. F. A. Mecahin between 1792 and 1798 amidst the turmoil of the French revolution. They slightly misjudged the flattening of the earth. The quadrant is now known to be 10,001,957 meters.

It is interesting to note that the metric system was not rejected outright by the United States. A thorough investigation was prepared for Secretary of State John Quincy Adams in 1821. The reasons for not adapting the metric system was that the metric system was only used sporadically in France at the time and eventually forced on them by legislation in the 1840s. In addition, the United States with its much shorter history had much less disparate measures than France.

The long history of measures even allows one to relate the size of the space shuttle rocket boosters to a horse's ass: The Solid-Rocket Boosters (SRB) are the two rockets attached to the side of the space shuttle. They were contructed in Utah and had to be transported by train to the launch site. The engineers would have liked to make them a bit bigger, but the train had to go through a tunnel which was only slightly wider than the railroad track and the railroad gauge is 4 feet and $8\frac{1}{2}$ inches.

Hmmm, odd number why?

Well, the English built them like that.

Sure, but why did the English do that

The first trains were built by the same people who built pre-railroad tramways, and that was the gauge they used.

Why did they use that gauge?

Well, before tramways, they built wagons and they used that wheel spacing.

Why did they have to use that odd wheel spacing?

Well, the road in England are very old and if you use a different wheel spacing than the old wheel spacing, wheels would often break off.

Amazing, who built these old roads then?

The first roads in Europe were built for war chariots for the Roman Legions and all the chariots in Imperial chariot had all the same wheel spacing.

And why did they choose that wheel spacing

Well, the Roman chariots were just wide enough to accomodate the back-ends of two war horses.

The establishment of standards is almost a science in itself. At first the standard was a platinum bar, which was 0.2 mm short, because of the misjudgement in the Earth's flattening. Later new bars were made of platinum mixed with 10% iridium. Of course, it still required that people had to go to somewhere to compare their bars with the standard bar. To avoid all uncertainty the meter has now been defined as the distance the light travels in vacuum in

1/299,792,458'th of a second. Other units are less easily standardized. The kilogram is still defined as "the mass of the international prototype of the kilogram". Again made of the platinum-iridium alloy.

Most people are use to dealing with length scales from millimeters to kilometers. However, smaller and larger than that our sense of scales gets a bit lost. Figure 1 shows the length scales less than a meter. Everybody will recognize the bee or at least some insect up to 1 cm (10^{-2} m), but most people will get lost recognizing the pollen on the bee's eye at the millimeter scale ($10^{-}3$) meter or identify the prickly looking sphere as a pollen at (10^{-4}) meter. When we

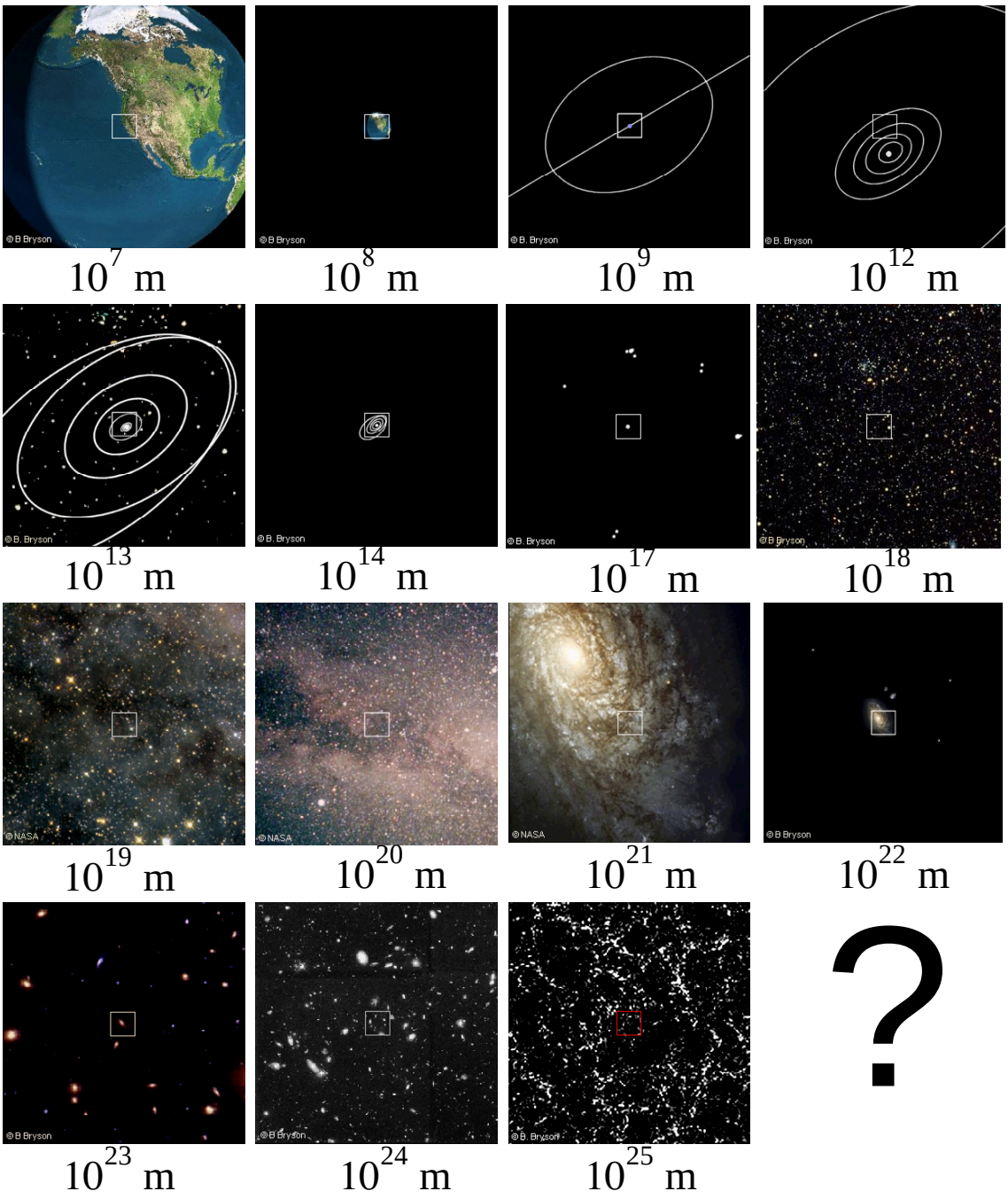


FIG. 2: Typical length scales greater than a meter. Copyright Bruce Bryson (www.wordwizz.com/pwrsof10.htm).

get down to the micrometer scale (10^{-6}), we are at the scale of the cells that make up animals and plants.

At the 10^{-7} scale, we are dealing with viruses. It is also the length scale of the smallest feature sizes on your the chips that you find in you computer, cell phone, etc. It is also the the area of the visible light. Human can see wavelengths between 3.8 and 7.8 10^{-7} m (or 380 to 780 nm). That this corresponds to the minimum feature sizes on your computer chips is no coincidence. Visible light is used in lithography to print the strucures on the silicon. The technology using visible light, which employs lenses is very advanced. The number of transistors of an integrated circuit has roughly doubled every 24 months since 1971. This is commonly known has Moore's law. This is mainly due to a decrease in feature sizes as a result of the progress in lithography. However, feature size are now close and sometimes even smaller than the wavelength of the light and it becomes very complex to create smaller feature size with visible light. We can try to make something with radiation with a smaller wavelength (for example, ultraviolet or x-rays). However, for wavelengths in this regions it is very difficult to make lenses. This is strongly related to the the fact that optical microscopes also fail around 0.2 μm. This has nothing to do with how well we can make microscopes, but with the properties of the light. You might compare this with waves in the water. For small ripples, putting your hand in the water can disturb the wave. However, large waves are barely affected by your presence. Therefore, below 10^{-7} m, we are basically blind. However, scientists can still produces images of things up to 0.1-1 nanometer (10^{-10}-10^{-9} m) using electron microscopes. Below that, we have to use schematic pictures to imagine things.

At 10^{-8} m, we are on the length scales of molecules, such as DNA, which is a very large molecule consisting of two very long polymers in which our genetic information is stored. The DNA can be broken down in smaller parts (10^{-9} m). The molecules themselves are made up out of atoms. At 10^{-10} m (0.1 nm or 1 Ångstrom), we are entering inside the atom. What is depicted is a representation of the electron cloud. Often electrons are depicted as little balls. This idea is appealing, but becomes very confusing when you start describing them with quantum mechanics, the theory for small particles. Quantum mechanics tells you that the electron are spread all over the place. We can basically only give a probability for finding an electron at a certain position. This is kind of indicated by the dots in the figure. The more dots, the higher the probability of finding an electron. The nucleus, we only find at a length scale of 10^{-14} m. The size of a proton is 1.5 10^{-15} m. Again, in the figure, protons and neutrons are indicated as little balls. This is the way we like to think about it, because this the way we experience objects on our length scales and how we visualize things. Note that there is a lot of nothing inside the atoms. The atom is basically a very dense nucleus with a very diffuse cloud of electrons around it. When we go to even smaller length scales (10^{-16}), we find that the protons and neutrons themselves are made up out of other particles known as quarks. The figure gives a blurry picture of red, green, and blue, which is an artist impression of quarks which has physically very little meaning. We will very briefly discuss quarks later on, and see that quarks are given a color (red, green, blue), but that has no relation to the real colors (note that the length scale is nine orders of magnitude smaller), we could have given it names such as apple, orange, and banana. It is almost impossible for humans to visualize a quark and probably the blurry aspect of the picture is the best part of it.

What does physics describe here? Physics comes from the Greek *phusikos*, meaning natural and is concerned with the fundamental laws in nature. Obviously, physics deals with the most elementary parts here: the quarks, the nucleus, and atoms. Here we are dealing with interactions which a the territory of physicists: the strong and weak interactions, in combination with electromagnetic interactions. However, it also encompasses things that are best described with these laws. Now here things become blurry. This is a result of the fact that our division into different disciplines is our own invention and not something that nature intended. The division into different disciplines is a nineteenth century creation. Before that, all natural sciences fell under the term natural philosopy. Isaac Newton's famous 1687 book is known as Mathematic Principles of Natural Philosophy. The journal *Philosophical Magazine*, founed in 1798, is a physics journal and not a philosophical journal. Therefore, when we get to molecules and solids, the division between physics and chemistry becomes less clear. When we talk about chemical reactions, we think of chemistry. However, chemists also like to describe molecules and solids from the basic quantum-mechanical equations. Is that physics or chemistry. There are journals called the *Journal of Chemical Physics* and the *Journal of Physical Chemistry*. When it comes to solids, the situation is entirely unclear. Synthesis of new materials seems chemical, but physicists also do it. Magnetism is done by both physicists and chemists, although superconductivity seems more a physicists' thing. What about DNA and other proteins. The chemists might say "Chemistry," because they are just big molecules (biochemistry). The biologists say "Biology," because it deals with life. However, look at the discovery of DNA. This was a combined effort of the experimentalists who measured the X-ray diffraction and the theorists who determined the structure from their data. The experimentalist were Maurice Wilkins and Rosalind Franklin. Maurice Wilkins was a physicist who was hired John Randall a physicist in charge of the biophysics laboratory at King's College London. Wilkins arranged for a three year fellowship for Rosalind Franklin, a physical chemist, to work on the structure of DNA. Francis Crick was a physicist turned biologist who worked at the Cavendish laboratory (Cambridge's Department of Physics). The Cavendish was led by Sir Lawrence Bragg, who had one the Nobel prize in physics in 1915 (at the age of 25) for his analysis of X-ray diffraction patterns from crystals. He was determined that the Cavendish should determine the structure of DNA before the American chemist Linus Pauling. Crick was

joined by the American biologist James Watson, who had a bachelor in zoology before switching to genetics. As is clear the key investigators had a wide variety of backgrounds. This interdisciplinary team effort (as we would call it nowadays) was crucial in the determination of the double helix structure of DNA. A distinguishing feature of physicists is probably their tendency to describe phenomena from the "fundamental equations" (say, the Schrödinger equation from quantum mechanics). This approach works less well with complicated processes, such as chemical reactions and biological systems. This is expressed by Crick who said that the adjustment from the "elegance and deep simplicity" of physics to the "elaborate chemical mechanisms that natural selection had evolved over billions of years" could be compared "as if one had to be born again."

Another example where the propensity of physicists for modelling played an important role is economy. The first Nobel prize in Economy was awarded to the Dutchman Jan Tinbergen, who obtained his Ph.D. in physics with the title (in Dutch) "*Minimumproblemen in de natuurkunde en de economie*" (*natuurkunde* , the knowledge of nature, is the Dutch word for physics. Tinbergen developed the first macroeconomic model (mind the word model), which he first built for the Netherlands and subsequently for the United States and the United Kingdom. Note that Jan Tinbergen's brother Niko also won a Nobel prize for his studies of social behavior patterns in animal together with Karl von Frisch and Konrad Lorenz.

When we can also consider the length scales in the opposite direction. Most of you will be familiar with length scales from several kilometers to several thousands of kilometers. Our feeling for distance again becomes confused when we go beyond our usual experience, say the size of the Earth 10^7 m. For example, how far is the moon?

How to determine the distance from the earth to the moon.

STEP 1: How big is the earth?

The first thing that you need to know is the radius of the earth. This was determined by the Greek Eratosthenes (279-194 B.C.). He knew that on noon at the longest day of the year the sun was almost at its zenith in Syene. Therefore, no shadows were cast. However, Eratosthenes was in Alexandria and there a shadow was cast at noon on the summer solstice. Eratosthenes took a large obelisk and measured its shadow. Since he knew the height of the obelisk, he could determine that the Sun appeared at an angle of 7.2 degrees south of the zenith. This actually determines the curvature of the earth between Syene and Alexandria. What was left was to determine the distance between Syene and Alexandria. To this end (astronomers apparently had a bit more influence in those days), he ordered some soldiers to walk from Alexandria to Syene to determine the distance. It turn out to be 5000 stadia,

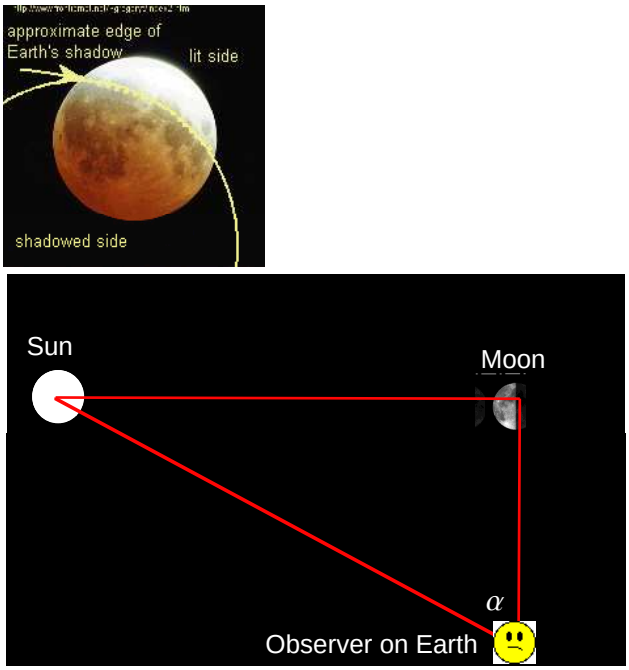


FIG. 3: The top part shows the Moon moving through the shadow of the Earth. The lower part shows how to determine the distance to the Sun when the Moon is in its third quarter.

roughly 750 km. The circumference of the Earth can then be determined

$$\text{circumference} = 750 \times \frac{360^\circ}{7.2^\circ} = 37,500 \text{ km}, \tag{1}$$

which is close enough to the real value of 40,000 km. The diameter is then 12,700 km.

Example Another way to do it is described in Giancoli Example 1-8. An alternative example is by watching the sunset. Suppose you are lying down, watching the Sun set. You start your stopwatch just after the Sun goes down. Then you stand up and watch the Sun disappear again. Suppose your height is h =1.7 m and the time elapsed is t =11.1 s. What is the radius of the Earth?
Solution: The idea is that if the Sun is setting the line connecting your eyes to the Sun is a tangent to the Earth's surface, see Fig. 4. Then you raise yourself by h and draw a new tangent. This is drawn with great exaggeration see Fig. 4. This forms a the red lines in the Figure form a triangle with 90° angle. We can therefore use Pythagoras theorem

$$d^2 + r^2 = (r+h)^2 \qquad \Rightarrow \qquad d^2 = 2rh + h^2 \cong 2rh. \tag{2}$$

We can do the last approximation since $h \ll r$. Now, $d = r\tan\phi$, where ϕ is given in Fig. 4 and we can also write

$$r^2 \tan^2\phi = 2rh \qquad \Rightarrow \qquad r = \frac{2h}{\tan^2\phi} \tag{3}$$

We have to determine ϕ. However, we now that 11 s has passed and that the Earth turn 360° in $24 \times 3600 = 86400$ s. The angle is then given by

$$\phi = \frac{11.1}{86400} \times 360^\circ = 0.04625^\circ. \tag{4}$$

This gives for the radius

$$r = \frac{2 \times 1.7}{\tan^2 0.04625} = 5.22 \times 10^6 \text{ m}. \tag{5}$$

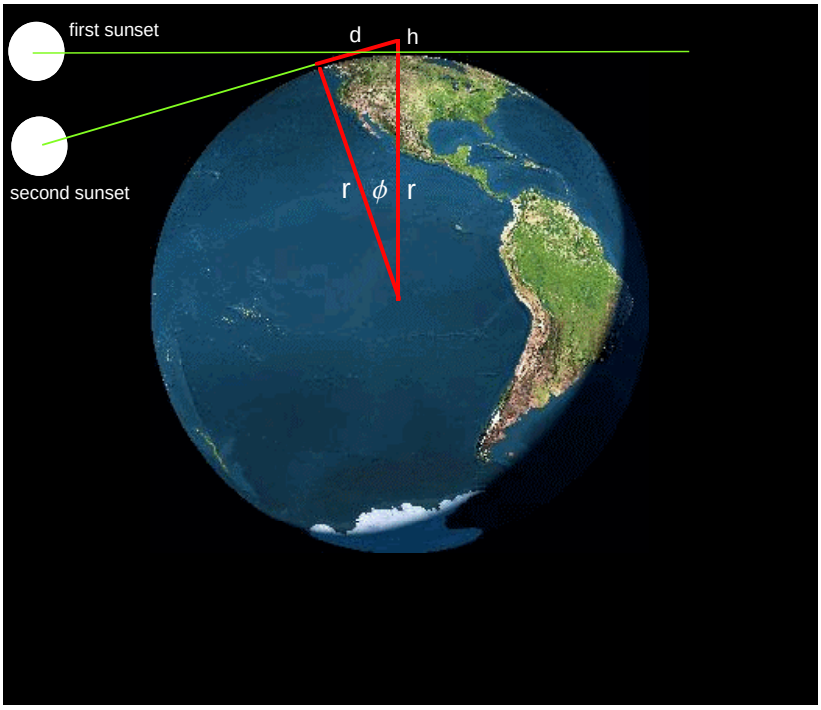


FIG. 4: Schematic diagram to calculate the size of the Earth by measuring to different sunsets. One lying down and one standing up increasing the distance to the center of the Earth by h. Note that h is greatly exaggerated.

Again not exact, but a reasonable estimate nevertheless.

STEP 2: What is the size of the Moon?
This idea is again thought to originate from Eratosthenes. However, it was first carried out by Aristarchus of Samos (310-230 B.C.). The idea is to determine the relative sizes of the Earth and the Moon. Since we know the size of the Earth, calculating the size of the Moon, should then be a piece of cake. The trick was to make use of a lunar eclips, when the Moon is passing through the shadow of the Earth. We want to compare the time it takes the Moon to travel to the shadow of the Earth (which is roughly equal to the size of the Earth is the Sun is sufficiently far away, so that its rays can be considered parallel) to the time it takes for the moon to traverse its own diameter. This can be done by measuring the difference between the time that a moon covers a bright star and the time that the star reappears. Doing this, Aristarchus found that the Moon's diameter is about 3/8 that of the Earth or 4700 km (it should be closer to 1/4). Okay, slightly off the 3476 km, but we are interested in the right order of magnitude.
STEP 3: What is the distance between the Earth and the Moon?
Now that we have the absolute diameter of the Moon, we can determine the distance by measuring the angular diameter on the sky. The angle occupied by the full moon is about 0.5°. or $\frac{0.5°2\pi}{360°} = 0.0087$ arcradians.

$$\text{distance Earth} - \text{Moon} = \frac{3476}{0.0087} \cong 400,000 \text{ km}. \tag{6}$$

This distance between the Earth and the Moon varies between 363,300-405,500 km, since the orbit of the Moon around the Earth is elliptical and not spherical.

Taking a cube of 100,000 km around the Earth (10^8 m) does not yet include the moon. Geosynchronous satellites (satellites that stay in a fixed position above the Earth's surface) circle at around 40,000 km above the equator. Taking one more step of ten (10^9 m), and we include the orbit of the moon which is about 400,000 km from the Earth. Skipping several steps of ten, we see that at 10^{12} m we start encompassing the sun in our cube. Can we determine the distance to the moon as well?
STEP 4: What is the distance between the Earth and the Sun?
This is a bit trickier to determine than the distance to the moon. Again, our good old friend Aristarchus found a method to determine the distance. The trick is to find a right angle. When we have a half Moon, the Sun-Moon-Earth angle is 90°. Since we know the distance from the earth to the moon, we can determine the distance Earth-Sun, when we know the angle α to the Sun, see Fig. 3. This is quite a difficult determination. There is the unhealthy aspect of looking directly at the Sun, but in addition there is the complication that the angle is almost 90°. In fact, it is 89.853°. The calculation of the distance would then be

$$\text{distance Earth} - \text{Sun} = \frac{\text{distance Earth} - \text{Moon}}{\cos\alpha} = \frac{400,000}{\cos 89.853} \cong 156,000,000 \text{ km}. \tag{7}$$

Aristarchus was off by a factor 20. Still, it means that he ended up with a number in the millions of kilometers. If this was all known more than two centuries before Christ, why did this knowledge get lost for many centuries, only to be rediscovered after the renaissance? One of the reasons is that people prefer ignore things that they do not find appealing. The idea of people walking upside down on the other side of the earth is terribly unappealing. A second factor is that people like to place themselves at the center of the universe. If the Sun is that far away, it is more logical that the Earth circles around it in 365 days, as opposed to the Sun circling around the Earth in 24 h. However, that makes the Earth less important than the Moon. Of course, we may find that all very silly, but still people like to stick to comfortable ideas and deny or ignore things that makes them feel less special (such as, having the same ancestors as apes, the age of the universe (a mere 10 billion years), or the incredible size of the universe). Note that this distance to the Sun means that we are flying through space at the relaxed speed of 100,000 km/h. While we are at it, we can also calculate the size of the Sun.

STEP 5: What is the size of the Sun?
From solar eclipses, we know that the angular diameter of the Sun is more or less that of the Moon, i.e. 0.5°. Using the distance from the Earth to the Sun, we can easily obtain a diameter of

$$\text{diameter} = 150 \times 10^6 \text{ km } \times \frac{0.5°}{180°} \times \pi \cong 1.3 \times 10^6 \text{ km} \tag{8}$$

The orbits of Mars, the Earth, Venus, and Mercury are all inside the square and the orbit of Jupiter is just at the edge. Going to 10^{13} m, we capture most of the solar system. The strong elliptical orbit is that of Pluto. The other four orbits are those, from outside in, Neptune, Uranus, Saturn, and Jupiter. Pluto is about 39.5 times further

away from the Sun and takes 248.5 years to orbit the Sun. Of course after that the distances become even more mindboggling. Around the solar system, we find several orders of magnitude of almost complete emptiness. How can we determine the distance to the nearest stars? For this we use parallaxes. This is a geometric effects that we obtain The nearest star (apart from the Sun) is Proxima Centauri, which is about 4.2 light years away. Now 1 lightyear is

$$1 \text{ lightyear} = 3 \times 10^8 \text{ m/s} \times 3600 \times 24 \times 365 \cong 9.5 \times 10^{15} \text{ m}. \tag{9}$$

Travelling at a speed of an regular jet airline (roughly 1000 km/h), this will take you about one million years. That is an unbearable amount of airline food. And nobody tells you that this solar system might be of any interest. Proxima Centauri is a rather dim red star which was discovered in 1915, see Fig. 5. More interesting would be a visit to Alpha Centauri, just 0.1 light year further away (only 10^{12} km), which is a double star, see Fig. 5 of consisting of two stars about the same size as the Sun. There closest distance is about 11.2 times the distance from the Earth to the Sun.

If we increase our cube by another order of magnitude (10^{18} m), we start to see more starts. We have to three orders of magnitude up before we start seeing the structure of what we call our Milky Way (obviously this picture is not our Milky-Way, but a different galaxy). Our galaxy is estimated to be about 100,000 lightyears across. It contains about 200 to 400 billion stars. Leaving our galaxy, we find again quite a stretch of emptiness before we start finding the nearest large galaxy, the Andromeda galaxy at 2.5 million lightyears away. There are an estimated hundred billion galaxies in the universe. This gives an estimated 7×10^{22} stars (give or take two orders of magnitude). At a cube of 10^{25} m, we start approaching the edge of our known Universe and scientist believe there are clear structures of galaxies to be seen. The Universe is estimated to be about 13-15 billion years old. The edge of the observable Universe is about 78 billion light years or 7.4×10^{26} m. Finally, what is beyond our universe. Well, your guess is at good as mine.

Obviously, the study of stars and galaxies is the field of astronomy. In the early days, a lot of the astronomy was descriptive. But, in the same way as chemistry on the small length scales, the desire developed to understand astronomical phenomena from the fundamental physics equations, and the field of astrophysics was born.

III. THE CONCEPTS OF DIFFERENTIATING AND INTEGRATING

A. Differentation

Did the car stop for the stop sign?
Let us suppose a car moves along at with a constant speed of 1 km/min (60 km per hour in layman terms, or something else in miles per hour in Western countries not conquered by Napoleon, such as Great Britain or the U.S.). At the time $t = 0$ min the car is passing a stop sign at 0 km. He is caught by the police and they pull the car over. The driver asks: "How do you know I did not stop at the stop sign? Do you have any prove for that?" "Sure," says the policeman. Now for educational purposes, the police have very special equipment and an enormous amount of camaras along the road, which happens to be filled with an enormous amount of markers (you might wonder why they do not have equipment to measure the speed of the car directly, but they don't...) The policeman takes out a photo and says: "Look here, this a picture of your car taking 1 min after you passed the stop sign and you are 1 km past the stop sign. This gives a speed of"

$$v = \frac{\Delta x}{\Delta t} = \frac{x(1 \text{ min}) - x(0 \text{ min})}{1 - 0 \text{ min}} = \frac{1 - 0 \text{ km}}{1 \text{ min}} = 1 \text{ km/min}, \tag{10}$$

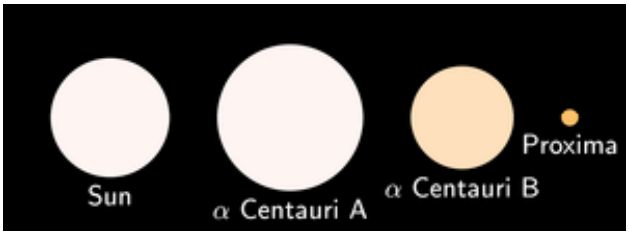


FIG. 5: The closest stars to our solar system compared with the Sun (source Wikipedia).

where the Δ indicates difference. "Sure," says the driver, "but that doesn't mean anything. My average speed between the stop sign and one kilometer down the road was 1 km/min, but I did really stop at the stop sign." The policeman is not that easily bluffed off and offers more evidence: "Here you are 0.1 min after crossing the stop sign and you passed the stop sign by 100 m." (admittedly, few policemen would talk about 0.1 min, 6 seconds, maybe...). The driver's speed is therefore:

$$v = \frac{\Delta x}{\Delta t} = \frac{x(0.1 \text{ min}) - x(0 \text{ min})}{0.1 - 0 \text{ min}} = \frac{0.1 - 0 \text{ km}}{0.1 \text{ min}} = 1 \text{ km/min}. \tag{11}$$

"You see, you're still driving at the same speed." "All right," says the driver I accelerated very quickly after the stop sign to 1 km/min, but I did stop." This discussion goes around in circles for quite some time until they end up at 0.0001 min and the speed is still

$$v = \frac{\Delta x}{\Delta t} = \frac{x(0.0001 \text{ min}) - x(0 \text{ min})}{0.0001 - 0 \text{ min}} = \frac{0.0001 - 0 \text{ km}}{0.0001 \text{ min}} = 1 \text{ km/min}. \tag{12}$$

To top it off, the policeman also has a picture 0.0001 min before the car passes the stop sign

$$v = \frac{\Delta x}{\Delta t} = \frac{x(0 \text{ min}) - x(0.0001 \text{ min})}{0 - 0.0001 \text{ min}} = \frac{0 - 0.0001 \text{ km}}{-0.0001 \text{ min}} = 1 \text{ km/min}, \tag{13}$$

which also shows that the car was going 1 km/min just before the stop sign. The driver finally decides to come up with the fine of \$75 dollars. However, in some sense the driver was correct in saying that is does not prove anything that the speed is 1 km/min some distance away from the stop sign. In principle, we should make the time difference Δt go to zero. However, this means we have to divide by zero, and that is very problematic as anyone knows who has tried to divide by zero on their calculator. Therefore, we should not take Δt exactly zero, but, as it is called, infinitesimally small. The concepts of infinitesimally small and also of infinite are conceptually rather complicated and took a long time to develop. The velocity in the limit that Δt goes to zero is called the instantaneous velocity.

A slightly more complicated example Now let us suppose that the car's position is given by t^2. Let us calculate the average speed between $t = 0$ and 2 min.

$$v = \frac{\Delta x}{\Delta t} = \frac{x(2 \text{ min}) - x(0 \text{ min})}{2 - 0 \text{ min}} = \frac{2^2 - 0 \text{ km}}{2 \text{ min}} = 2 \text{ km/min}. \tag{14}$$

However, in the first half, the average speed was

$$v = \frac{\Delta x}{\Delta t} = \frac{x(1 \text{ min}) - x(0 \text{ min})}{1 - 0 \text{ min}} = \frac{1^2 - 0 \text{ km}}{1 \text{ min}} = 1 \text{ km/min}. \tag{15}$$

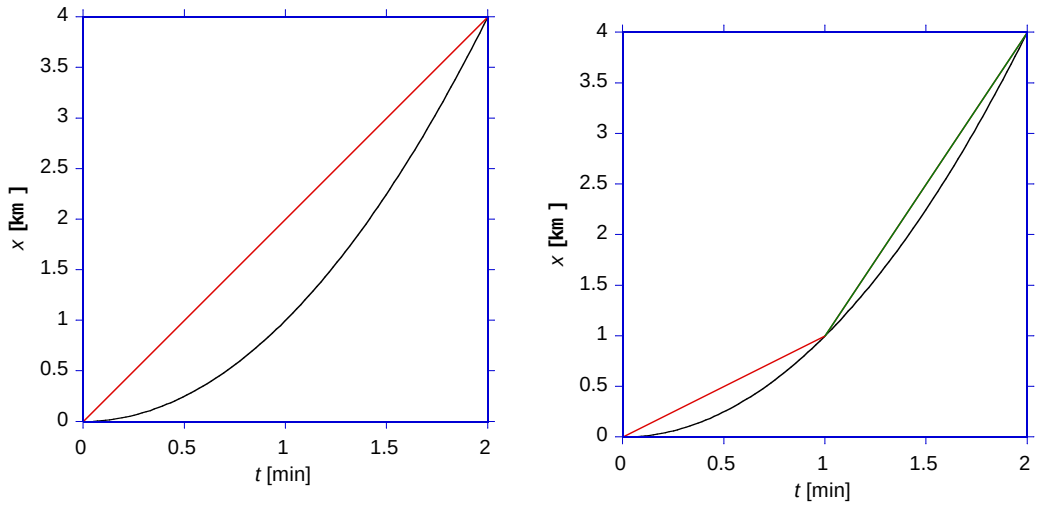


FIG. 6: The position x as a function of the time t. In the Figure on the left we see a line connecting the points at $t = 0$ and $t = 2$ min. This is the graph that we obtain when going from $x = 0$ to $x = 4$ km with a constant velocity of 2 km/min. In the right half, we approximated the parabola with two average speeds: 1 km/min from $x = 0$ to $x = 1$ km and 3 km/min from $x = 1$ to $x = 4$ km.

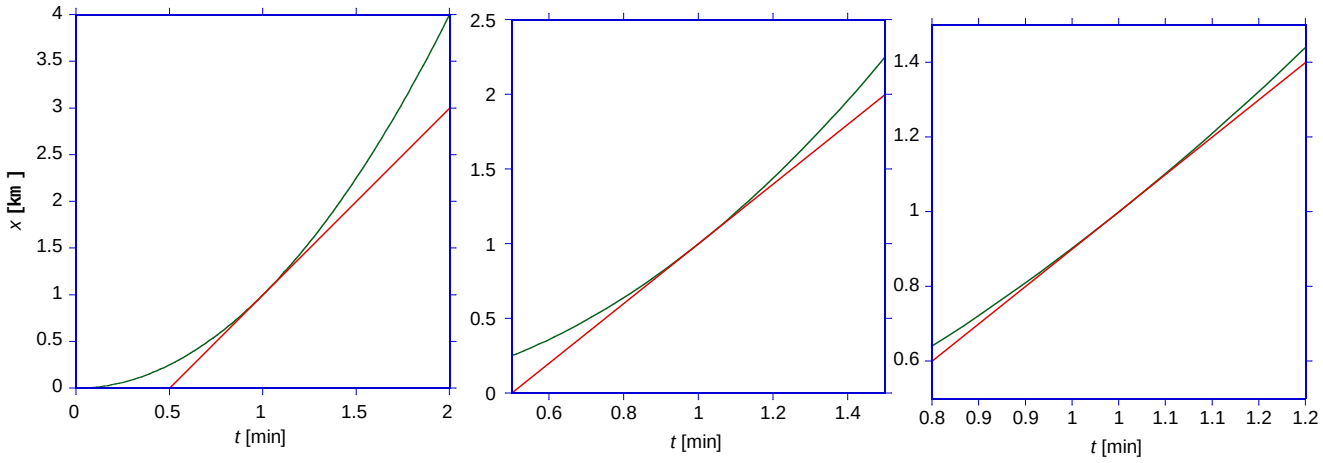


FIG. 7: The graphs show the position x as a function of time with $x(t) = t^2$. The line through $x = 1$ km shows the slope of the parabola at $t = 1$ min, i.e. 2 km/min. The two other graphs are for smaller intervals around $t = 1$ min, to show that the slope of the parabola and the line are the same for $t = 1$ min.

but in the second half the average speed was

$$v = \frac{\Delta x}{\Delta t} = \frac{x(2\text{ min}) - x(1\text{ min})}{1 - 0\text{ min}} = \frac{2^2 - 1^2\text{ km}}{1\text{ min}} = 3\text{ km/min}. \tag{16}$$

Apparently, the car's speed is increasing (Note that the average of 3 and 1 km/min is the 2 km/min, as obtained above. BTW, 3 km/min is 180 km/h, a pretty respectable speed). Now let us try to determine the instantaneous velocity at $t = 1$. Let us take a time difference of 0.1 min.

$$v = \frac{\Delta x}{\Delta t} = \frac{x(1.1\text{ min}) - x(1\text{ min})}{1.1 - 1\text{ min}} = \frac{(1.1)^2 - 1^2\text{ km}}{0.1\text{ min}} = 2.1\text{ km/min}. \tag{17}$$

However, as we saw in the "stop sign" example, the instantaneous velocity has only real mean if the time difference approaches zero. Let us try a somewhat smaller difference of $t = 0.01$ min.

$$v = \frac{\Delta x}{\Delta t} = \frac{x(1.01\text{ min}) - x(1\text{ min})}{1.01 - 1\text{ min}} = \frac{(1.01)^2 - 1^2\text{ km}}{0.01\text{ min}} = 2.01\text{ km/min}. \tag{18}$$

We see that this approaches 2 km/min. Doing some more calculations will give us 2.001 km/min for $t = 1.001$ min and 2.0001 km/min for $t = 1.0001$ min. We can now convince ourselves that the instantanuous velocity at $t = 1$ min is indeed 2 km/min. We can do this exercise for several times t an tabulate the results

t	$x(t)$	$v(t)$
0	0	0
0.5	0.25	1
1	1	2
1.5	2.25	3
2	4	4

(19)

By looking at the results for $v(t)$, we can easily see that the relationship between the instantaneous velocity and time is given by $v(t) = 2t$. Obviously, it would have been great if we could have determined the function for the velocity $v(t) = 2t$ directly from the function for the position $x(t) = t^2$. This process is known as differentiation and we will study it more thoroughly in the coming sections.

Graphical interpretation Another way to look at differentiation is graphically. Let us first consider average speeds, see Fig. 7. In our example the position varied as function of time as $x(t) = t^2$. This is drawn in the graph with time horizontally and position vertically. Note that although this is a two-dimensional graph, we are studying the motion in one direction as a function of time (if the axes represented x and y, we could plot motion in two direction, but then it would become complicated to include time). In Eqn. (14), we have seen that the average speed was 2 km/min. Let

us connect the points at times $t = 0$ and $t = 2$ min by a straight line, see the red line on the left side of Fig. 7. This line indicates how the position would vary as a function of time if the car was moving with a constant velocity of 2 km/min. However, this is an approximation to the real motion, because the car is not moving at a constant speed, but accelerating. We saw that from the two different average speeds between $t = 0$ and 1 min and that between $t = 1$ and $t = 2$ min. This is indicated by the red and green lines on the right side of Fig. 7. From Eqns. (15) and (16), we found that the average speeds are 1 and 3 km/min. This is reflected in the steeper slope of the second straight line.

It becomes slightly more complicated, when look at the instantaneous velocity, or the velocity averaged over an infinitesimally small time interval. In the previous section, We found that the instantaneous velocity for $t = 1$ min is 2 km/min (this happens to equal to the average speed between 0 and 2 min, which is a result of the fact that the speed linearly increases with time. Remember, we had established the relationship $v(t) = 2t$). We draw this by drawing a line through $x = 1$ km at $t = 1$ min with a slope of 2 km/min. You can do this by calculating another point of the line. For example, for $t = 2$ min (1 min later than $t = 1$ min), the car has moved another 2 km, so it should be at $x = 1 + 2 = 3$ km. Since we have two points now, we can draw the straight line, see the left side of Fig. 7. This looks slightly strange and you might wonder what this line means. Let us zoom in a bit. The two graphs on the right side of Fig. 7 show the same parabola and line but in a smaller time frame. We see that if we zoom in, the parabola appears flatter. This is comparable to our daily experience that the earth is flat because we are only focusing on a very small part of this large sphere. In the graph on the right, we see clearly that the line we have drawn has exactly the same slop as the parabola at $t = 1$ min.

Differentiation As we saw in the two previous examples, the idea of differentiation is that we want to directly obtain from the function for the position as a function of time the velocity as a function of time. This process is called differentiation with respect to t. Symbolically we are relating two functions with each other

$$x(t) \xrightarrow{\text{differentiation w.r.t. } t} v(t). \tag{20}$$

We will see during this course that the concept of differentiation is more general and not only related to position and velocity as a function of time.

B. Integration

After having introduced the concept of differentiation, it is natural to think about the inverse process. Suppose, we know the velocity as a function of time, can we derive the position as a function of time. This is indeed possible and the process is called integration. Or, symbolically,

$$v(t) \xrightarrow{\text{integration w.r.t. } t} x(t). \tag{21}$$

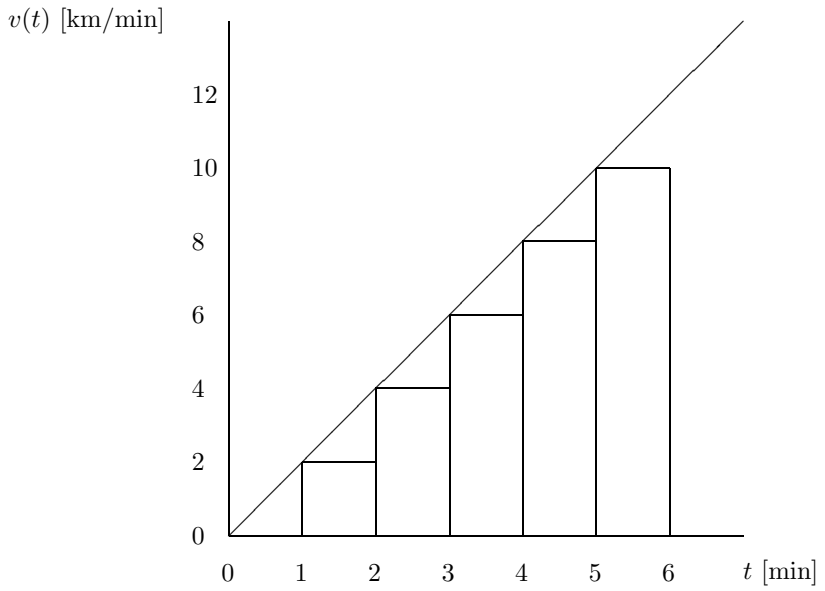
Constant velocity
This is relatively simple. Suppose, we are driving in a car that is going 1 km/min. Then when we are interested in the distance travelled, we just multiply the velocity times the time, i.e.

$$\Delta x = v\Delta t. \tag{22}$$

For example, after 10 min, we have travelled 10 min$\times$ 1 km/min= 10 km. Note that in the units the minutes cancel with each other and we are just left with distance.

Constantly increasing velocity. Now let us consider the example again where $v(t) = 2t$ and let us forget about the fact that we know already that the position is given by $x(t) = t^2$. Let us try to calculate the total distance travelled after six minutes and let us overlook the fact that the velocity at 6 min is 12 km/min or 720 km/h. We do not know (yet) how to deal with this problem, however we know how to treat the case for a constant velocity. We can approach this problem by looking at the speedometer every minute and assume that we can take the velocity constact for the

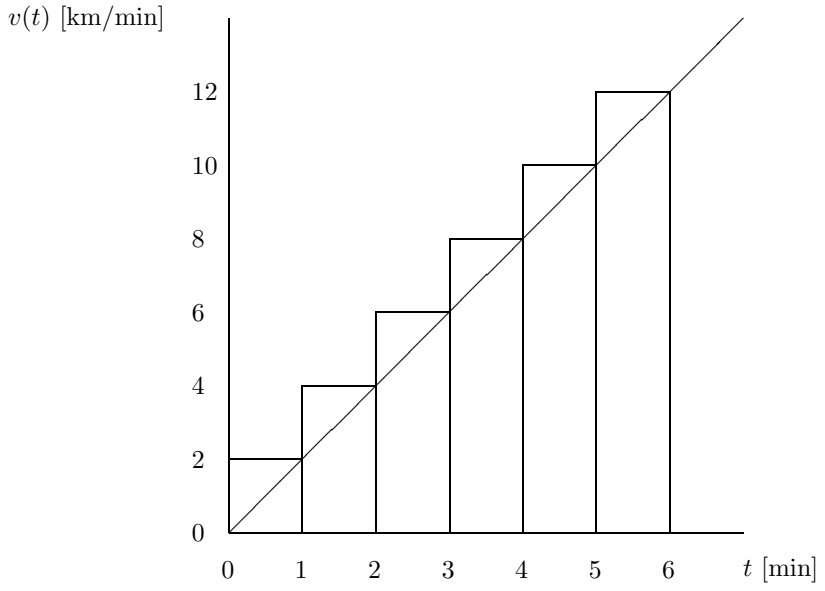
next minute. Effectively, we divide it into six constant velocity problems, as follows



Note that the straight line $v(t) = 2t$ is now appromximated by 6 sections where we assume the car travels with 0, 2, 4, 6, 8, and 10 km/min. The distance travelled is now given by

$$\begin{aligned} \Delta x &= v_0\Delta t + v_1\Delta t + v_2\Delta t + v_3\Delta t + v_4\Delta t + v_5\Delta t & (23) \\ &= 0\times 1 + 2\times 1 + 4\times 1 + 6\times 1 + 8\times 1 + 10\times 1 = 30 \text{ km}. & (24) \end{aligned}$$

However, we looked at the speedometer and then we drove for a minute. This is one way to do it, but we are underestimating the distance travelled since we are accelerating after we look at the speedometer. The alternative way of doing it is to look at the speedometer after we have travelled for one minute. Graphically, this look like



$$\Delta x = v_1\Delta t + v_2\Delta t + v_3\Delta t + v_4\Delta t + v_5\Delta t + v_6\Delta t \quad (25)$$
$$= 2\times 1 + 4\times 1 + 6\times 1 + 8\times 1 + 10\times 1 + 12\times 1 = 42 \text{ km}. \quad (26)$$

Clearly, now we have overestimated the distance travelled, since the velocity in the minute preceding the time we looked on the speedometer, we were actually travelling slower. However, we have clearly found that we traveled more than 30 but less than 42 km. In fact, the average 36 km is the right answer. We can check this since we know that $x(t) = t^2$, therefore $x(6 \text{ min}) = 36$ km. This is somewhat fortuitous and only works in the case of a linearly increasing velocity and not for more complicated situations. In the case for the instantaneous velocity, our result improved when we took smaller time steps. We can use the same approach here. For example, for Δt=0.5 min:

$$\Delta x = (v_0 + v_{0.5} + v_1 + v_{1.5} + v_2 + v_{2.5} + v_3 + v_{3.5} + v_4 + v_{4.5} + v_5 + v_{5.5})\Delta t \quad (27)$$
$$= (0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11)\times 0.5 = 66 \times 0.5 = 33 \text{ km}, \quad (28)$$

for the lower bound and

$$\Delta x = (v_{0.5} + v_1 + v_{1.5} + v_2 + v_{2.5} + v_3 + v_{3.5} + v_4 + v_{4.5} + v_5 + v_{5.5} + v_6)\Delta t \quad (29)$$
$$= (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12)\times 0.5 = 78 \times 0.5 = 39 \text{ km}, \quad (30)$$

for the upper bound (again the average is 36 km/min). As an aside, there are even better ways to calculate this. Integrating an area in this fashion is extremely suitable for numerical integration. However, there are methods that work better to a higher degree of precision. Simpson's rule says that we should not use $hf(x)$ for each step but $\frac{h}{6}[f(x) + 4f(x + h/2) + f(x + h)]$. Let us do the same thing again, but now with only one step ($h = 12$)according to the Simpson's method

$$\Delta x = \frac{h}{6}(v_0 + 4v_{h/2} + v_h) = \frac{6}{6}(0 + 4\times 6 + 12) = 36. \quad (31)$$

This gives directly the right answer. The reason for that is that it is set up in such a way that it will give the right answer when trying to integrate a straight line. If we were integrating a parabola or something else, the answer would not be perfect. Obviously, we can continue and devise even better methods that would directly give the right answer when integrating a parabola. Although it is known as Simpson's rule, it was not invented by Thomas Simpson (1710-1761), but Isaac Newton. As a way of compensation, Newton got the shared credit for the Newton-Raphson method for solving $f(x) = 0$, which was actually invented by Simpson. Simpson made his money as an itinerant lecturer who taught in the London coffee houses. These lectures were a cheap substitute for people who could not afford university.

However, what are we actually doing? We are dividing the straight line $v(t) = 2t$ into little section with a constant velocity. We then multiply this velocity with Δt. However, this is equivalent to the surface of one of the rectangles indicated in the figures. Adding all the rectangles amounts to estimating the surface area under the curve. Of course, we could have approached the problem differently then. Since we want to determine the area, we can calculate it directly

$$x(t) = \text{AREA} = \frac{1}{2}\text{basis}\times\text{height} = \frac{1}{2}t\times v(t) = \frac{1}{2}t\times 2t = t^2. \quad (32)$$

This gives for $t = 6$ min, $x = 36$ km. However, this of course only works because we took a constantly increasing velocity. If we had chosen $v(t) = t^2$ or something more complex, we could not have solved the problem in this fashion, whereas the method of subdividing it into smal rectangles still works.

In the preceding sections, we have demonstrated that it is possible to determine the velocity from the position, a process we call differentiation, and to determine the position from the velocity, a process we call integration. Effectively, we can easily go between between the two different function:

$$x(t) \underset{\text{integration w.r.t. } t}{\overset{\text{differentiation w.r.t. } t}{\rightleftarrows}} v(t). \quad (33)$$

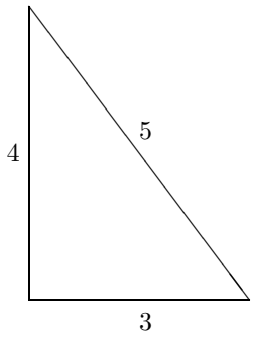
Knowing one means knowing the other (apart from a constant to which we will come back later). In the example that we have treated in the previous section, we saw that we could describe the position and velocity with two different function:

$$t^2 \underset{\text{integration w.r.t. } t}{\overset{\text{differentiation w.r.t. } t}{\rightleftarrows}} 2t. \quad (34)$$

The study of calculus tells us how to go from one function to the other.

C. A little history of calculus

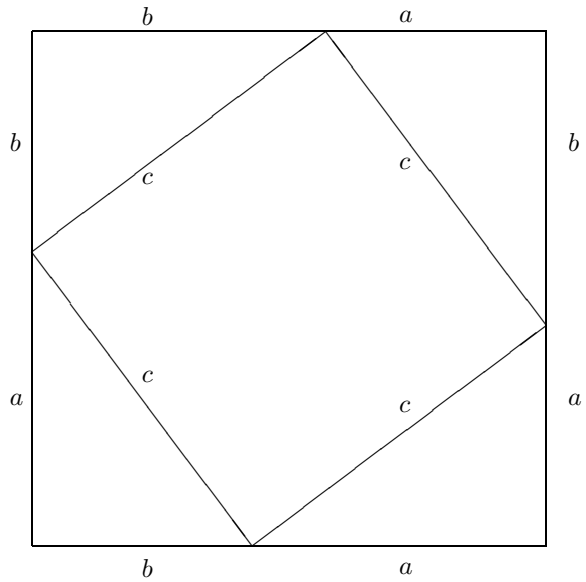
Theorem of Pythagoras. The rise of civilizations increased the need for mathematics. The invention of agriculture around 15,000-10,000 B.C. required that humans to some extent had to master some concepts of mathematics, such as multiplication and geometry. An increasingly complex civilization required more mathematics for a well-functioning of trade, architecture, and astronomy (of importance to agricultural societies, such as the Egyptians who needed to track the flooding of the Nile). It is known that around 2000 B.C., the Egyptians were aware of right angles.



where $3^2 + 4^2 = 9 + 16 = 25 = 5^2$

They used this by tying 12 pieces of ropes of equal length to each other and allowing workers to construct a perfect right angle. However, were the Egyptians aware of the mathematical concepts behind this of was this merely a luckily found insight? There is certainly no evidence that the Egyptians were aware that 5-12-13 and 65-72-97 also produced right angles. For this, we have to go to civilizations living in Mesopotamia, where clay tablets dating between 1900 and 1600 B.C. have been found proving that the Baylonians definitely understood this. The Babylonians where a great ancient civilization and their explorations into numerics can still be seen in the base-60 counting system that they adopted and that is still in use in our measurement of time (60 minutes in the hour) and angles ($6 \times 60 = 360$ degrees in a circle).

However, the Babylonians for all their achievements did not address the question why $a^2 + b^2 = c^2$. For this we have to go to the Aegean coasts of Greece and Asia minor, where one of the greatest ancient civilizations lived. The ancients unanimously attributed the proof to Pythagoras (around 572 B.C.-475 B.C.) although we have no record of his proof. We can, however, provide another one (actually, there are quite a few different ways to prove Pythagoras' theorem). Let us consider the following two squares:



(35)

We know that the area of the big square is given by $(a+b)^2$. However, we can see that the area of the big square can also be described by the small square plus four triangles:

$$\text{area} = c^2 + 4 \times \frac{1}{2} a \times b \tag{36}$$

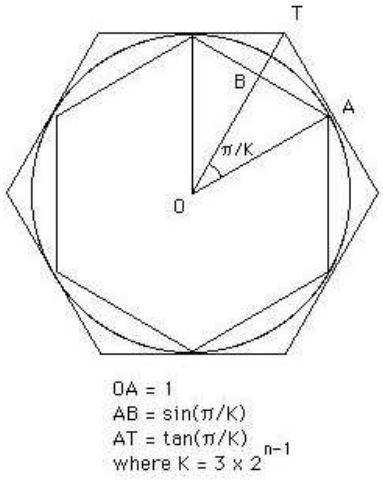


FIG. 8: Archimedes method for determining the area of a circle. Note that we can draw polygons inside and ouside the circle. In the figure this is done for $K = 12$. We see that we have a hexagon going through points A and B inside the circle and also a hexagon going through points A and T outside the circle. Note that the areas of the former and latter hexagons are less and greater than that of the circle, respectively.

Equating the two areas gives

$$(a+b)^2 = c^2 + 2ab \qquad \Rightarrow \qquad a^2 + 2ab + b^2 = c^2 + 2ab \qquad \Rightarrow \qquad a^2 + b^2 = c^2. \tag{37}$$

Note that this requires knowledge of the calculation of areas of squares and triangles and the evaluation of special products. Obviously, this has so far little to do calculus, but the developments have come to a point that things are approached in an abstract and theoretical fashion instead of the more practical approaches used by the Egyptians and Babylonians. Note that this already has taken about 1500 years. Pythagoras' theorem is also a typical example of the Greeks' love for geometry. Geometry was very important for the development of calculus. In fact, Isaac Newton, the most important figure in classical mechanics, developed his theory on geometry as opposed to the more standard approach using limits (which we will use in later section). This makes Newton's groundbreaking book the *Principia* hard to read for a modern scientist.

Zeno. Apart from geometry, another important aspect of calculus is the concept of inifinity and infinitesimally small. This is what we encountered in the example of the car stopping at the stop sign, where we needed to go to smaller and smaller time scales to prove that the car was indeed moving when it crossed the stop sign. The abstract nature of these concepts significantly slowed down the development of calculus. One of the first great philosophers to think about these problems was Zeno of Elea (about 490-425 B.C.). He constructed his arguments in terms of paradoxes. One of these paradoxes was:

There is no motion because that which is moved must arrive at the middle of its course before it arrives at the end.

In order to go from the beginning of a line to the end, we must pass the midpoint at $\frac{1}{2}$. But before we can pass that, we must past the half of that line, i.e. $\frac{1}{4}$. And before we can pass that, we must pass $\frac{1}{8}$, and so on, till an infinitesimally small number. Hence we never get started. This is somewhat puzzling and forces us to think about inifitesimally small numbers.

Archimedes. A very important figure in the development of calculus is Archimedes of Syracuse (287-212 B.C.). Let us follow his estimate of π. Archimedes' approach was to approximate the area of a circle by K triangles, see Fig. 8. He did that in two ways, one where the area of the K triangles is less than that of the circle and one where the area is greater than that of the circle. Let us the example for $K = 12$. Let us consider the triangle OAB and take the radius of the circle equal to one. Using trigonometry, we can find $AB = OA \sin \pi/12 = \sin 30^\circ = \frac{1}{2}$ and $OB = OA \cos \pi/12 = \cos 30^\circ = \frac{1}{2}\sqrt{3}$. The area of the triangle OAB is then $\frac{1}{2} AB \times OA = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\sqrt{3} = \frac{1}{8}\sqrt{3}$. We see that the hexagon described by triangles similar to OAB lied inside the circle. Therefore, its area should be less than that of the circle. A lower bound for the approximation of a circle is then $12 \times \frac{1}{8}\sqrt{3} = \frac{3}{2}\sqrt{3} \cong 2.59$. In a similar fashion, we can find an upper bound for the area of a circle. In Fig. 8, we see that we can draw a hexagon that touches the outside of the circle, going through points A and T. The length AT is given by $AT = OA \tan \frac{\pi}{12} = \frac{1}{3}\sqrt{3}$.

The area of the triangle OAT is then $\frac{1}{2}\frac{1}{3}\sqrt{3} = \frac{1}{6}\sqrt{3}$. The hexagon going through A and T is built up out of 12 of those triangles and its are is $12\frac{1}{6}\sqrt{3} = 2\sqrt{3} \cong 3.46$. Therefore the area of the circle is between 2.59 and 3.46. Now, of course we now that the area of a circle is given by πr^2 (note that, although π is a Greek letter, the did not use π in this context). Since the radius is one, the area is $\pi \cong 3.14$. Note that although this is a crude extimate, it provides a way to increase the accuracy of π. Archimedes managed to extend this procedure up to a 96-gon and was able to show that

$$3\frac{10}{71} = 3.1408 < \pi < 3.1428 = 3\frac{1}{7}. \tag{38}$$

This result is all the more remarkable considering that Archimedes did not have our standard knowledge on trigonometry, the decimal system, let along a calculator! In fact, Archimedes method was the standard method (at least in Europe) used to calculate π almost the next two millenia, up till Ludolph van Ceulen, who managed to calculate π up to 35 places using a polygon with 2^{62} sides. In fact, it was Newton who used series to calculate π, although he did not improve on Van Ceulen's calculation.

intermezzo We can obtain a somewhat better estimate for π by not looking a the area but at the radius. The radius of the hexagon through A and B is given by $12AB = 6$. The radius of the the outer hexagon is given by $12AT = 4\sqrt{3}$. Since we know now that the radius of a circle is given by $2\pi r$, we know that these number should be compared with 2π. Therefore, we find that $3 < \pi < 2\sqrt{3}$. Note that the lower bound is significantly closer to the real value of π.
Question: Why is the estimate of π when using the radius better than for the area?.

Obviously, you might wonder what this has to do with calculus. Archimedes tried to approach the area of the circle by approximating it by simpler area of which he knew how to calculate the area. This is comparable to our determination of the area under the curve $x(t)$ by approximating it by triangles. We are trying to integrate the area of the circle. This shows that differentiation and integration is not unique for the relationship between position and velocity, but can be applied for many different areas. For Archimedes case, we have the relationship between length (radius, in the case of a circle) and area:

$$\text{area} \underset{\text{integration w.r.t. } r}{\overset{\text{differentiation w.r.t. } r}{\rightleftarrows}} \text{length}. \tag{39}$$

or in equations

$$\pi r^2 \underset{\text{integration w.r.t. } r}{\overset{\text{differentiation w.r.t. } r}{\rightleftarrows}} 2\pi r. \tag{40}$$

Note that the relation is very similar as that between $2t$ and t^2.

We can also extend this area and volume:

$$\text{volume} \underset{\text{integration w.r.t. } r}{\overset{\text{differentiation w.r.t. } r}{\rightleftarrows}} \text{area}. \tag{41}$$

For example, the surface of a sphere is given by $4\pi r^2$ and its volume is $\frac{4}{3}\pi r^3$. We then have the relation:

$$\frac{4}{3}\pi r^3 \underset{\text{integration w.r.t. } r}{\overset{\text{differentiation w.r.t. } r}{\rightleftarrows}} 4\pi r^2. \tag{42}$$

If you look very carefully, you see that the relationship is again similar to that between t^2 and $2t$. It appears that we have to multiply by the power to obtain the right result, i.e. differentiating t^n gives nt^{n-1}.

Further progress No much progress was made until the 16th century. Then several contributions were made that lay a basis for mechanics and calculus. Of importance for mechanics was the determination of centers of gravity, as set out by Luca Valerio (1552-1618) in his *De centro gravitas*. He also used interesting ideas of the quotient of limits, which we will encounter later on. Johannes Kepler (1571-1630) studied planetary motion following the Copernican system with the sun as the center. In contrast to Copernicus, he proposed that the planets moved in ellipses. He discovered that when joining a line between the sun and a planet, the planets swept equal areas in equal times. To determine the areas, Kepler used a method of indivisibles. These concept are closely related to integration and were

developed further by the mathematicians Bonaventura Cavalieri (1598-1647). Cavalieri was able to derive that the integral of x^n from 0 to a was $\frac{1}{n+1}a^{n+1}$, which we will also prove in later sections. Gilles Personne de Roberval put these results on a more rigorous basis by subdividing the area under a curve into narrow strips and using series to evaluate the area. The famous French mathematician Pierre de Fermat (1601-1665) developed the ways of finding maxima and minima of a curve by determining when the derivative of a function is zero. This method is essentially as used today and we will use it later on. Both the Italian Evangelista Torricelli (1608-1647) and the Englishman Isaac Barrow (1630-1677) developed the method of tangents to a curve as we saw earlier in our geometrical interpretation of the derivative. They considered problems of variable velocities and developed the awareness that the derivative of the distance is the velocity and the inverse operation connect the velocity to the distance.

Newton and Leibniz The basis was now laid for the development of the theory of calculus. Isaac Newton (1643-1727) wrote developed his theory of "fluxions" in 1666. However, Newton's works took a long time to get published leading to a long controversy whether Leibniz or Newton should get credit for calculus. For example, Newton's work on *Analysis with infinite series* was written in 1669 and circulated in manuscript for many years before it finally got published in 1711. Newton's developed in his work series expansion for $\sin x$ and $\cos x$. Newton also laid the basis for obtaining derivatives with limits, which is the approach that we will use in the next sections. The other great figure in the development of calculus was Gottfried Wilhelm von Leibniz (1646-1716). Leibniz developed independently of Newton a theory of calculus. Leibniz is also responsible for developing a number of the modern notations, such as dx, dy, $\frac{dy}{dx}$, and $\int x dx = \frac{1}{2}x^2$.

After Newton and Leibniz, calculus was further developed by many people, such as the brothers Jacob (1654-1705) and Johann (1667-1748) Bernoulli, and Colin Maclaurin (1698-1746). However, a real rigorous treatment had to wait till the 19th century with the work of Augustin Louis Cauchy (1789-1857).

D. Formal approach to differentiating

In physics, we often encounter the mathematical concept of differentation. To introduce the concept of differentiation let us consider a road up a hill, i.e. a problem using x and y as opposed to t and x as we used earlier. Suppose we have a straight road. At the start of the road the elevation is 0 m. At the end of the road the elevation is $y = 10$ m. If the road is 100 m long, we can say that the road climbs with (10-0)/100=0.1 (or 10%). However, this is often a simplification, because the road might not climb continuously from 0 to 10 m. In fact, there could be a hill of 20 m high in between. If we want to understand how much the road climbs at a certain point x we do not want to compare with a different point far away from our point x, but only a very small distance. In fact, we want to compare with a point infinitesimally small away. Let us suppose that the road is described by a function $f(x)$ which gives the elevation at a certain distance x from the beginning of the road. The result above for the elevation over the whole road can now be written as

$$\frac{\Delta y}{\Delta x} = \frac{f(100) - f(0)}{100 - 0} = \frac{10 - 0}{100} = 0.1. \tag{43}$$

Note that we can define an angle α with $\tan\alpha = \frac{\Delta y}{\Delta x}$ that indicates at what angle the road is climbing. However, if we are interested in the change in elevation at a certain point x, we have to calculate

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}, \tag{44}$$

where $\frac{df(x)}{dx}$ (pronounce: d f x d x) is the derivative of the function $f(x)$. This is often also denoted as $f'(x)$. For brevity let us denote $h = \Delta x$. Furthermore, $\lim_{\Delta x \to 0}$ indicates that we want to take the limit that Δx goes to zero. The derivative of a function can therefore be calculated by calculating the differential quotient

$$f'(x) = \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}. \tag{45}$$

The evaluation of the differential quotient may not be always trivial, however let us start with a simple case. Let us take $f(x) = 0.1x$, i.e. the road climbs continuously from $y = 0$ m at the start of the road to $y = 10$ m at $x = 100$ m. The derivative is then

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{0.1(x + h) - 0.1x}{h} = \lim_{h \to 0} \frac{0.1h}{h} = 0.1. \tag{46}$$

Note that we did not even have to take the limit that h goes to zero because the h cancels. The reflects the fact that the road climbs in the same fashion no matter where we are on the road. Also, we could have added a constant,

$f(x) = 0.1x + a$. We would still have obtained $f'(x) = 0.1$. The derivative tells us something about the change in elevation and the derivative is the same whether we climbing to 0 to 10 m of from 133 to 143 m. This also shows that for a constant function $f(x) = a$ the derivative is zero, $f'(x) = 0$.

Let us now consider a somewhat more difficult function $f(x) = x^2$. We can calculate the derivative using the differential coefficient

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0}(2x + h) = 2x. \tag{47}$$

We see here that the derivative increases with increasing (the road analogy is less useful here).

1. *Differentiating x^n*

Obviously it would be nice to generalize this to x^n. Let us start with n is integer and greater or equal to zero. In this case, we need to expand

$$(x+h)^h = (x+h)\cdots(x+h) = \sum_k A_k x^{n-k} h^k. \tag{48}$$

$$\leftarrow n\text{terms} \rightarrow \tag{49}$$

This is called Newton's binomial expansion. What are now the coefficients A_k. Suppose we want a term with h^k. Term we need to select an h from k of the $x + h$ terms. From the $n - k$ remaining $x + h$ terms, we select the x giving x^{n-k}. For our first h, we have n possible $x + h$ terms to choose from. For the second, there are $n - 1$ terms left, because we cannot select an h from the same $x + h$ term twice. For the third, we have $n - 2$ possiblities, and so on. This gives $n(n-1)(n-2)\cdots(n-k+1) = n!/(n-k)!$. However, we are counting a bit too much, because, in the case that $k = 2$, choosing, for example first the third term and then the fifth is equivalent to choosing first the fifth term and then the third. Therefore, we have to divide by the $k!$ possible permutations. The coefficients are therefore

$$A_k = \frac{n!}{(n-k)!k!} = \left(\begin{array}{c} n \\ k \end{array}\right) \qquad \Rightarrow \qquad (x+h)^h = \sum_k \left(\begin{array}{c} n \\ k \end{array}\right) x^{n-k} h^k. \tag{50}$$

We can now calculate the derivative of $f(x) = x^n$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n - x^n}{h} \tag{51}$$

$$= \lim_{h\to 0}(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1}) = nx^{n-1}. \tag{52}$$

Note that the derivative is determined by the second term in the binomial expansion. This is an important result that we often use in approximations. For small h, we can write

$$(x+h)^n \cong x^n + nx^{n-1}h = x^n(1 + n\frac{h}{x}). \tag{53}$$

For example,

$$6.01^4 = 1304.661624 \tag{54}$$

$$6.01^4 = (6 + 0.01)^4 \cong 6^4(1 + 4\frac{0.01}{6}) = 6^4 \times 1.006666666 = 1304.64. \tag{55}$$

Note that $6^4 = 1296$, so this result is substantially better. We have shown that $(x^n)' = nx^{n-1}$, where $(f(x))'$ is a convenient shorthand notation for the derivative of the function $f(x)$. This is true for all real n positive and negative. However, we have only proved it for integer n greater or equal than zero. Later on, we shall prove it more generally which is somewhat more complicated.

E. Rules for differentiating

Obviously, there are lots more functions than x^n. We still do not know how to differentiate function like e^x or $\sin x$ or products of them, such as $x^2 \sin x$. It is convenient to look at some rules for differentiating functions before we

establish the derivative of some elementary functions:

$$(af(x)+bg(x))' = af'(x)+bg'(x) \tag{56}$$
$$(f(x)g(x))' = f'(x)g(x)+f(x)g'(x) \tag{57}$$
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}. \tag{58}$$

The first rule allows us, e.g., to differentiate polynomials:

$$(x^3+2x^2+5)' = 3x^2+4x \tag{59}$$
$$(30x^{30}+15x^5)' = 900x^{29}+75x^4. \tag{60}$$

Let us prove the second rule, known as the product rule:

$$\begin{aligned}(f(x)g(x))' &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x+h)+\lim_{h\to 0}f(x)\frac{g(x+h)-g(x)}{h} = f'(x)g(x)+f(x)g'(x).\end{aligned} \tag{61}$$

This rule alows us to differentiate products of functions. We shall determine late on that $(e^x)' = e^x$ and $(\sin x)' = \cos x$. Using that, we can derive with the product rule

$$(x^3\sin x)' = 3x^2\sin x + x^3\cos x \tag{62}$$
$$(4x^2e^x)' = 8xe^x + 4x^2e^x = 4(2x+x^2)e^x \tag{63}$$

The product can be generalized to many functions

$$(f_1f_2\cdots f_n)' = f_1'f_2\cdots f_n + f_1f_2'\cdots f_n + \cdots + f_1f_2\cdots f_n' \tag{64}$$

where f_i is a shorthand for $f_i(x)$. For example,

$$x^2e^x\sin x = 2xe^x\sin x + x^2e^x\sin x + x^2e^x\cos x. \tag{65}$$

An interesting example is

$$(x^n)' = (x\times x\times\cdots\times x)' = 1\times x\times\cdots\times x + x\times 1\times\cdots\times x + x\times x\times\cdots\times 1 = nx^{n-1}. \tag{66}$$
$$\leftarrow n\text{terms}\rightarrow \tag{67}$$

This rederives the result that we obtained before with the binomial expansion.

To derive the last rule, let us first have a look at the derivative of $1/f(x)$,

$$\left(\frac{1}{f(x)}\right)' = \lim_{h\to 0}\frac{\frac{1}{f(x+h)}-\frac{1}{f(x)}}{h} = \lim_{h\to 0}\frac{f(x)-f(x+h)}{f(x+h)f(x)h} = -\frac{1}{f(x)}\lim_{h\to 0}\frac{1}{f(x+h)}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = -\frac{f'(x)}{[f(x)]^2}. \tag{68}$$

This result we can use to differentiate

$$\left(\frac{1}{x^n}\right) = -\frac{nx^{n-1}}{(x^n)^2} = -\frac{n}{x^{n+1}} \tag{69}$$

This can be rewritten as $(x^{-n})' = -nx^{-n-1}$. Or substituting $n' = -n$, $(x^{n'})' = n'x^{n'-1}$. This is the same result as we derived before for positive integers n, but now we have shown that the result is also valid for negative integers. However, we still have to show that it is also valid for real values of n. Now let use the result from Eqn. (68), to prove the last rule:

$$\left(\frac{f(x)}{g(x)}\right)' = f'(x)\frac{1}{g(x)} + f(x)\left(\frac{1}{g(x)}\right)' = \frac{f'(x)}{g(x)} - f(x)\frac{g'(x)}{[g(x)]^2} = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}. \tag{70}$$

Some examples (using $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$)

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \tag{71}$$
$$\left(\frac{x^2}{e^x}\right)' = \frac{2xe^x - x^2e^x}{e^{2x}} = \frac{2x}{e^x} - \frac{x^2}{e^x}. \tag{72}$$

1. Chain Rule

So far, we have seen a number of rules for differentiating functions. However, although we know $(e^x)' = e^x$ (although we still have to prove it), what about e^{2x}? For this, we need to use the chain rule. The derivation is a bit tricky, but the result is easily seen in region where no mathematical complexities occur. We want to know the derivative of a function $f(g(x))$. For example, for e^{x^2}, $g(x) = x^2$ and $f(x) = e^x$. The derivative

$$(f(g(x)))' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h} = f'(g(x))g'(x). \tag{73}$$

where we have excluded mathematical difficulties that arise when $g(x+h) = g(x)$. Examples are

$$(e^{x^2})' = \frac{de^{x^2}}{d(x^2)} \frac{d(x^2)}{dx} = \frac{de^t}{dt} 2x = e^t 2x = 2xe^{x^2}, \tag{74}$$

where we have use the substitution $t = x^2$. However, some people may find it easier without substitution. Other examples,

$$(\sin 4x)' = (\cos 4x)4 = 4\cos 4x \tag{75}$$

$$\left((x^3 + 3x^2)^3\right)' = 3(x^3 + 3x^2)^2(3x^2 + 6x). \tag{76}$$

Obviously, it is possible to make life even more difficult with functions where we have to combine different rules:

$$\left((1 + 2x^3)(\cos 5x^2)^2\right)' = 6x^2(\cos(5x^2))^2 + (1 + 2x^3)2\cos(5x^2) \times \sin(5x^2) \times 10x \tag{77}$$

$$= \cos(5x^2)\left[6x^2\cos(5x^2) + (20x + 40x^4)\sin(5x^2)\right], \tag{78}$$

where we have to apply the chain rule twice.

F. Derivatives of some elementary functions

1. Exponential and logarithm

We already saw that $(e^x)' = e^x$. Although it looks simple, proving it is slightly more complicated. For those of you familiar with their calculator know that the inverse function of e^x is the natural logarithm, i.e., when $y = e^x$ then $x = \ln y$. The natural logarithm is defined as $\ln x = \int_1^x \frac{dt}{t}$. However, we have not treated integration in detail. But we can also note that the logarithm is the function whose derivative is given by $1/x$, i.e.

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \tag{79}$$

Note that, we had obtained $(x^n)' = nx^{n-1}$. Using this, we can obtain on the right hand side all integer power greater and equal than zero, e.g. $(x^2) = 2x$, $(x)' = 1$, and $(x^0)' = (1)' = 0$. Also we can obtain all negative powers less that -1, e.g. $(1/x)' = -1/x^2$ and $(1/x^2)' = -2/x^3$. However, we never seem to obtain $1/x$. This is indeed a special case and that is why in order to satisfy $f'(x) = 1/x$, we need a new function. Therefore, the derivative of $\ln x$ directly follows from its definition. Let us now calculate the derivative of e^x, by using the fact that e^x is the inverse of $\ln x$.

$$(e^x)' = \lim_{\Delta x\to 0} \frac{e^{x+\Delta x} - e^x}{x + \Delta x - x} = \lim_{\Delta y\to 0} \frac{y + \Delta y - y}{\ln(y + \Delta y) - \ln y} = \frac{1}{(\ln y)'} = \frac{1}{\frac{1}{y}} = y = e^x. \tag{80}$$

Showing indeed that $(e^x)' = e^x$.

2. Goniometric functions

Let us start with $\sin x$. The derivative is given by

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}. \tag{81}$$

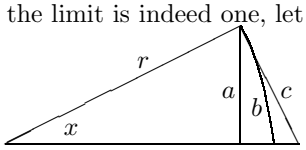
We can rewrite this using the identity

$$\sin\alpha - \sin\beta = 2\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha+\beta), \tag{82}$$

giving

$$(\sin x)' = \lim_{h\to 0} 2\frac{\cos(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h} = \lim_{h\to 0}\cos(x+\frac{1}{2}h)\lim_{h\to 0}\frac{\sin\frac{1}{2}h}{\frac{1}{2}h} = \cos x\lim_{h\to 0}\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}. \tag{83}$$

The limit on the right-hand side is not trivial. Note that for $h\to 0$, we have $0/0$. In general, when we have two equal numbers this division is 1, (e.g. 3/3=1), however, this is not necessarily true for functions. Suppose we have to calculate $\lim_{x\to 0} f(x)/g(x)$. This could be one is $f(x)=x$ and $g(x)=x$. However, it can also be zero, e.g., with $f(x)=x^2$ and $g(x)=x$, we have $\lim_{x\to 0} f(x)/g(x) = \lim_{x\to 0} x = 0$. On the other hand it can also be infinity, with $f(x)=x$ and $g(x)=x^2$, we have $\lim_{x\to 0} f(x)/g(x) = \lim_{x\to 0} 1/x = \infty$. The limit $\lim_{x\to 0}\frac{\sin x}{x}$ turns out to be 1. Basically, this means that $\sin x$ and x approach zero in the same fashion. For small x, we have $\sin x \cong x$. This is easily verified on a calculator, for example $\sin 0.1 = 0.0998$ (note, this is 0.1 in radians not degrees). Now that we have convinced ourselves that the limit is indeed one, let us prove it. Consider the following figure:



(84)

Clearly for the different lengths, we have the relationship $a<b<c$. The lengths are given by $a = r\sin x$; b is the length of the arc given by $b = rx$; and $c = r\tan x$. We therefore have the relation

$$a\le b\le c \quad\Rightarrow\quad r\sin x \le rx \le r\tan x \quad\Rightarrow\quad 1\le\frac{x}{\sin x}\le\frac{1}{\cos x} \quad\Rightarrow\quad \cos x\le\frac{\sin x}{x}\le 1. \tag{85}$$

If we now take the limit that x approaches zero, we obtain

$$\lim_{x\to 0}\cos x\le\lim_{x\to 0}\frac{\sin x}{x}\le\lim_{x\to 0}1 \quad\Rightarrow\quad 1\le\lim_{x\to 0}\frac{\sin x}{x}\le 1 \quad\text{or}\quad \lim_{x\to 0}\frac{\sin x}{x}=1 \tag{86}$$

Returning back to our differentiation of $\sin x$:

$$(\sin x)' = \cos x\lim_{h\to 0}\frac{\sin\frac{1}{2}h}{\frac{1}{2}h} = \cos x \tag{87}$$

In a similar fashion, we can find the derivative of $\cos x$.

$$(\cos x)' = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}. \tag{88}$$

Using the identity

$$\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta), \tag{89}$$

we obtain

$$(\cos x)' = -\lim_{h\to 0} 2\frac{\sin(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h} = -\lim_{h\to 0}\sin(x+\frac{1}{2}h)\lim_{h\to 0}\frac{\sin\frac{1}{2}h}{\frac{1}{2}h} = -\sin x. \tag{90}$$

We therefore have $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$.

G. Formal approach to Integrating

The integration of a function is the determination of the surface area below the curve. Let us give some simple examples. Let us take a constant function $f(x)=a$. The surface under the function between x_0 and x_1 is given by

$a(x_1 - x_0)$. In particular, for $x_0 = 0$ and $x_1 = x$, we have ax. Another important example is the surface under a straight line, for example $f(x) = bx$. The surface under the line from 0 to x is the surface of a triangle

$$\text{area} = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}x \times bx = \frac{1}{2}bx^2. \tag{91}$$

As with differentiation, this process becomes complicated is the function is not a straight line. However, we can subdivide the area into n little pieces were the curve is approximately straight and sum the areas of all these little pieces to obtain the total area

$$I = \sum_{i=1}^{n} f(\overline{x}_i)(x_i - x_{i-1}), \tag{92}$$

where $f(\overline{x}_i)$ with $x_{i-1} < \overline{x}_i < x_i$ represent well the value of the function in the region $[x_{i-1}, x_i]$. As with differentiation, we would like to take the limit $dx = x_i - x_{i-1} \to 0$. In this limit, the sum is replaced by an integral sign $\int$. We then have

$$I = \int_{x_0}^{x_n} f(x)dx. \tag{93}$$

It is important to realize that integration is the inverse process of differentiation. Suppose we have a function that satisfies $\frac{dF(x)}{dx} = f(x)$, we can then write

$$I = \int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_n} \frac{dF(x)}{dx}dx = \int_{x_0}^{x_n} dF(x) \tag{94}$$

$$= \sum_{i=1}^{n}(F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + \cdots + (F(x_n) - F(x_{n-1}) = F(x_n) - F(x_0). \tag{95}$$

Let us revisit our examples. We want to integrate the function $f(x) = a$. Now it is easy to verify that for $F(x) = ax+c$, we have $F'(x) = f(x)$. Note that we can always add a constant, since the derivative of a constant is zero. We can therefore perform the integration as

$$\int_{x_0}^{x_1} bdx = [bx + c]_{x_0}^{x_1} = b(x_1 - x_0), \tag{96}$$

where $[F(x)]_{x_0}^{x_1} = F(x_n) - F(x_0)$. Similarly, for $F(x) = \frac{1}{2}bx^2 + c$, we have $F'(x) = bx$. Integration gives

$$\int_{x_0}^{x_1} bxdx = [\frac{1}{2}bx^2 + c]_{x_0}^{x_1} = \frac{1}{2}b(x_1^2 - x_0^2). \tag{97}$$

We can generalize this to any power of n since we know that $(x^n)' = nx^{n-1}$. Or if we substitute $m = n - 1$, $(x^{m+1})' = (m + 1)x^m$. Therefore, we can write

$$\frac{d}{dx}\left(\frac{1}{m+1}x^{m+1} + c\right) = x^m \qquad \Rightarrow \qquad \int x^m dx = \frac{1}{m+1}x^{m+1} + c, \tag{98}$$

where we have added an integration constant c.
Examples 1. Integrate $f(x) = 4x^2 + 6x + 12$.
Solution:

$$\int f(x)dx = \frac{4}{3}x^3 + 3x^2 + 12x + c. \tag{99}$$

2. Integrate $f(x) = e^{3x}$.
Solution:

$$\int f(x)dx = \frac{1}{3}e^{3x}. \tag{100}$$

3. Integrate $f(x) = \frac{x}{1+x^2}$.
Solution:

$$\int f(x)dx = \frac{1}{2}\ln(1+x^2). \tag{101}$$

4. Integrate $f(x) = \frac{1}{x^2+2x-3}$.
Solution: This is more complicated. The chain rule would give a $2x+2$ and we do not have that in the numerator. The trick here is to split the denominator.

$$f(x) = \frac{1}{x^2+2x-3} = \frac{1}{(x+3)(x-1)} = \frac{1}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x+3}. \tag{102}$$

This can be integrated

$$\int f(x)dx = \int \frac{1}{4}\left[\frac{1}{x-1} - \frac{1}{x+3}\right]dx = \frac{1}{4}\left[\ln(x-1) - \ln(x+3)\right] \tag{103}$$

5. Integrate $\int \cos^2 x dx$.
Solution: Again, we need a trick to do this one

$$\int \cos^2 x dx = \int \left\{\frac{1}{2}\cos 2x + \frac{1}{2}\right\}dx = \frac{x}{2} + \frac{1}{4}\sin 2x \tag{104}$$

6. Integrate $\int \sin x \cos x dx$.
Solution: Here we make use of the fact that we can write $\sin x dx = -\frac{d\cos x}{dx}dx = -d\cos x$,

$$\int \sin x \cos x dx = -\int \cos x d\cos x. \tag{105}$$

It we now substitute $t = \cos x$, we find a familiar integral

$$-\int t dt = -\frac{1}{2}t^2 = -\frac{1}{2}\cos^2 x \tag{106}$$

IV. MECHANICS: WHAT CAME BEFORE

It is interesting to look what the ideas were on mechanics before the "modern" classical mechanics was developed. As usual, we have to return to the Greeks, where the early notions were developed. Empedocles (490-430 B.C.) developed the notion that everything consisted of water, earth, air, and fire. The idea was that objects have their "natural" place. Why does a rock fall down? Because it is in its "nature". However, before dismissing these ideas as simplistic, we have to realize that this was the standard theory for more than two millenea, before it was rejected by classical mechanics. So let us discuss some of the aspects that make the notion of a natural place so appealing. For example, it explains why stones fall faster than feathers. Feathers are more air-like than stone, hence the former likes to stay in the air and the latter will fall faster. Also, a stone will fall more slowly in water than in air, because water is closer to its own nature. Of course, the Greeks were smart people and they could produce undoubtedly many counter examples. However, many of them were considered less relevant. Experimental proof was inferior to logical reasoning and geometric beauty (although, Aristotle and Archimedes might be exceptions to the Greeks' aversion to experimental proof. Although, they were often more accurate observers than experimentalists in the modern sense). For example, the Greeks were able to calculate π to a higher accuracy than would be possible by experiment. In addition, many experimental tools, such as clocks, were not developed or inaccurate. A major inconsistency is of course the celestial bodies. Clearly, they are pretty solid and would come crashing down on earth. To account for that, Aristotle (384-322 B.C.) introduced a fifth element aether of which all of the heavens were composed. Therefore, heaven and earth have different natural laws (an idea that would be sympathetic to a large number of people even today). The Greeks were also responsible for the notion that the earth was the center of the universe. Although it should be noted that Aristarchus proposed that the sun was at the center, but those ideas were rejected in favor of the geocentric viewpoint. This ideas were gathered and organized by Claudius Ptolemy, working in Alexandria around 150 A.D. This led to the Ptolemiac theory that the universe was a closed space bounded by a spherical envelope beyond which there was nothing. At the center was the earth, fixed and immovable, and all the celestial bodies were

moving around it. A very appealing theory indeed, in good agreement with our daily observations and with the earth at the center of the universe, what more could you want?

So, this was the world view up to approximately the fifteenth century. In 1530, the Pole Copernicus completed his work *De Revolutionibus*. In it, Copernicus proposed that the earth revolved around its axis about every 24 hours and that it rotated around the sun every year. This was revolutionary (despite the fact that the Greeks has already proposed it) and even heretical to propose that Man made by God in His image were not at the center of it all and maybe just part of the natural world just as any other creature (Note, that this thought is still difficult to accept, judging by the resistance to evolution). Also, do not think that it was the vast amount of empirical evidence that helped Copernicus. In fact, Copernicus proposed that the planets moved in circles, which made hist theory highly inaccurate in predicting celestial motion. After 1.5 millenium of perfection, the Ptolomiac system extended with epicircles was much better at predicting the complex motion of the planets.

However, the ideas of Copernicus were embraced by two Italians Galileo Galilei (1564-1642) and Giordano Bruno (1548-1600). And they suffered dearly for it at the hands of the inquisitors. Bruno's insights that space was boundless and that there were other solar systems (implicating that there might even be intelligent or superior beings elsewhere) was considered so blasphemous that Bruno was burned at the stake in 1600. Galileo at trial in 1633 did not push the heliocentric point of view that far and was put under house arrest for the rest of his life.

Galileo did other great things. He constructed several telescopes and made some remarkable astronomical discoveries: he observed mountains on the moon, was able to see that the Milky Way really consisted of separate stars,observed four moons of Jupiter, the rings around Saturn, sunspotsm and the phases of Venus. Many of these observations were strong indications of the validity of the Copernican system. Other important astronomical observations at the time were done by Tycho Brahe (1546-1601) and Johannes Kepler. Using Brahe's and his own extensive observations, Kepler stated several laws to describe the motion of the planets around the Sun: Law 1. the orbit of a planet about the Sun is an ellipse with the Sun at one focus. Law 2. A line joining a planet and the Sun sweeps out equal areas in equal intervals of time. Law 3. The squares of the periods of the planets are proportional to the cubes of their semimajor (or mean distance) axes: $T_a^2/T_b^2 = R_a^3/R_b^3$.

Galileo also started attacking Aristotle's views that heavy objects fall more quickly than light objects. On the one hand, he did this experimentally (although the famous picture of Galileo dropping objects from the tower of Pisa is probably incorrect). What he did use were inclined planes that severely slowed down the motion. He was able to demonstrate that the distance that a body moves from rest under a uniform acceleration is proportional to the time squared. He also showed that the projectiles follow parabolic paths. On the other hand, he approached view Aristotle's view in a more philosophical way and demonstrated that it leads to contradictions. Let us assume that we have two objects with a different mass. According to Aristotle, they will fall to the earth in different fashions (according to their "nature"). Suppose now that we combine the two objects (put them in one bag or bind them together). How will the combined body fall. One might argue, that one object likes to fall faster to earth and that the other object would prefer to go slower, therefore the combined object should reached the ground in a time somewhere betweent that of the light and of the heavy object. However, one might also argue, that the mass of the combined object is greater that each of the objects separate so it should reach the ground more quickly. Of course, neither answer is correct, because the Aristotle's views are incorrect.

It was up to a genius to combine these aspects into one theory. This person was Isaac Newton (1642-1727), who in 1687 published his *Philosophiae Naturalis Principia Mathematica*, shortly known at the *Principia*. Newton had a profound influence on several disciplines. As described in previous sections, together with Leibniz, he is credited with the development of calculus. Note that this is closely related to his work on mechanics. His famous work translates as "mathematical principles of natural philosophy". (Note that science was known as natural philosophy in those days and the subdivision into fields like physics only came later). He developed the ideas for gravitation and classical mechanics. He did extensive work in optics and argued that it was made of particles. The particle nature has been revived in quantum mechanics, although in Newton's days the arguments were more in favor of Christiaan Huygens (1629-1695) proposal that light consists of waves.

V. MOTION IN ONE DIMENSION

When discussing the concepts of differentiating and integrating, we considered the relationship between position and velocity in the situation of a constant velocity. Often, we encounter the situation were the velocity changes as a function of time.

$$x(t) \underset{\text{integration w.r.t. } t}{\overset{\text{differentiation w.r.t. } t}{\rightleftarrows}} v(t) \underset{\text{integration w.r.t. } t}{\overset{\text{differentiation w.r.t. } t}{\rightleftarrows}} a(t). \tag{107}$$

A. Constant acceleration

Let us consider the case of a constant acceleration a, which is given by the change in velocity over time

$$a = \frac{\Delta v}{\Delta t}. \tag{108}$$

On the other hand, the velocity continuously over time and not linearly increasing. This is comparable to what we saw earlier, where the position as a function of time changed continuously and we no longer could define an average velocity, but had to introduce an instantaneous velocity. The analogue of this for acceleration is

$$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}\left(\frac{dx(t)}{dt}\right) = \frac{d^2x(t)}{dt^2}. \tag{109}$$

Can we convince ourselves that our calculus also works for the second derivative. Let us consider the limit function

$$a(t) = \lim_{\Delta t\to 0}\frac{v(t+\Delta t)-v(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{x(t+2\Delta t)-x(t+\Delta t)-(x(t+\Delta t)-x(t))}{(\Delta t)^2} \tag{110}$$

$$= \lim_{\Delta t\to 0}\frac{x(t+2\Delta t)+x(t)-2x(t+\Delta t)}{(\Delta t)^2}. \tag{111}$$

Constant acceleration. Let us check this when the position as a function of time is given by $x(t) = \frac{1}{2}at^2$,

$$a(t) = \lim_{\Delta t\to 0}\frac{\frac{1}{2}a[(t+2\Delta t)^2+t^2-2(t+\Delta t)^2]}{(\Delta t)^2} = \lim_{\Delta t\to 0}\frac{\frac{1}{2}a[t^2+4t\Delta+4(\Delta t)^2+t^2-2(t^2+2t\Delta t+(\Delta t)^2)]}{(\Delta t)^2} \tag{112}$$

$$= \lim_{\Delta t\to 0}\frac{\frac{1}{2}a\ 2(\Delta t)^2}{(\Delta t)^2} = a. \tag{113}$$

Therefore, if the position is given by $x(t) = \frac{1}{2}at^2$, the acceleration is a. Obviously, we could have obtained the result also by differentiating twice:

$$x(t) = \frac{1}{2}at^2 \qquad \Rightarrow \qquad v(t) = \frac{dx(t)}{dt} = at \qquad \Rightarrow \qquad a(t) = \frac{dv(t)}{dt} = a. \tag{114}$$

Often the problem is stated in a different fashion. Suppose an object is feeling a constant acceleration a. The velocity is then given by

$$\frac{dv(t)}{dt} = a \qquad \Rightarrow \qquad v(t) = \int a dt = at + v_0. \tag{115}$$

Note that we have to add an integration constant. At $t = 0$, the velocity is $v(0) = v_0$. To know the position as a function of time, we have to integrate another time

$$\frac{dx(t)}{dt} = v(t) \qquad \Rightarrow \qquad x(t) = \int v(t)dt = \int (at+v_0)dt = \frac{1}{2}at^2 + v_0t + x_0. \tag{116}$$

Again, we obtain an integration constant that defines the position of the object at $t = 0$, $x(0) = x_0$. If is also possible to fix the velocity at a time t_0. In that case, we have

$$\frac{dv(t)}{dt} = a \qquad \Rightarrow \qquad v(t) = a(t-t_0) + v_0. \tag{117}$$

with $v(t_0) = v_0$. And the position is

$$x(t) = \frac{1}{2}a(t-t_0)^2 + v_0(t-t_0) + x_0, \tag{118}$$

where now $x(t_0) = x_0$.

Example Some people decided that it is a good idea to go down Niagara falls in a barrel. The height is 48 m. This problem is not entirely one-dimensional, since someone going over the edge must have some velocity in the horizontal

direction. However, let us neglect that and assume he falls straight down. We will neglect the effect of the air.
(a) How long does someone fall before reaching the water surface.
Solution: We can consider this as a free fall under a constant acceleration (gravity). We can therefore use Eqns. (117) and (118). Let us take y as variable. Let us take the water surface at $y = 0$. The barrel therefore starts falling at $y = 48$ m. The gravitational acceleration is -9.8 m/s^2. This gives

$$y = \frac{1}{2}gt^2 + y_0 = -\frac{1}{2}9.8t^2 + 48 \tag{119}$$

For reaching the surface, we need $y = 0$. Therefore

$$-4.9t^2 + 48 = 0 \qquad \Rightarrow \qquad t = \sqrt{\frac{48}{4.9}} = 3.1 \text{ s}. \tag{120}$$

The person that was in the barrel could therefore count to three (barely) before she/he hit the water (the "she" is not only political correctness. The first person to conquer the falls in a barrel was Annie Taylor in 1901. She did not get the expected fame and fortune, but died in poverty. Others included Barry Leach who managed to break both kneecaps and his jaw in his jump in 1911. Ironically, Bobby died years later from gangrene contracted from complications due to slipping on an orange peel. In 1920, the Englishman Charles Stephens tied himself to an anvil in his wooden barrel for ballast. The only item left in the barrel was Charles' right arm attached to the anvil. In 1995, Robert Overcracker jetskied over the edge to promote awareness of the homeless. His intended descent with a parachute turned into a promotion for better parachute safety since it never opened. Again, do not try this. If you do survive it, you will be fined several thousands of dollars and could be banned from entering Canada for life).

(b) Determine the position at each second.
Solution: We can use the same expression as before, but substitute now the different values of t.

$$t = -4.91^2 + 48 = 43.1 \text{ m} \qquad t = -4.92^2 + 48 = 28.4 \text{ m} \qquad t = -4.91^2 + 48 = 3.9 \text{ m} \tag{121}$$

(c) What is the velocity of the barrel (plus person) when it hits the surface of the water.
Solution: We can use the expression for the velocity in Eqn. (117). Using that the initial velocity $v_0 = 0$ m/s.

$$v = -9.8t. \tag{122}$$

At 3.1 s, the velocity is $v = -9.8 \times 3.1 = -31$ m/s or $v = -110$ km/h or roughly 68 miles/h. This is definitely an unhealthy velocity to hit the water.

(d) What was the velocity at each count.
Solution: Again, we can simply substitute the times finding $v = -9.8, -19.6, -29.4$ m/s for $t = 1, 2$, and 3 s, respectively.

B. Gravity

Let us consider the movement of an object, say a ball, under the acceleration due to gravity. Its magnitude is approximately 9.80 m/s^2. Later on, we will see how we can determine this value. We also assume (which we will also derive later) that it is pointing towards the center of the Earth or straight down from a human's perspective. We also take it constant. This is a rather good approximation since the force depends on the distance to the center of the Earth as $1/r^2$. For example, 100 m above the Earth's surface (which is 6350,000 m from the center of the Earth), the force is 0.99997 times the force on the Earth's surface. For simplicity, let us neglect the effects of friction between the ball and the air. Acceleration gives the change in the velocity as a function of time:

$$\frac{dv(t)}{dt} = -g, \tag{123}$$

where the sign is negative because the acceleration is towards the ground, i.e. in the negative y direction. Integrating gives

$$v(t) = -gt + v_0, \tag{124}$$

where v_0 is the velocity at time zero. We describe the motion from time $t = 0$. At this time, the object can have a certain velocity v_0. Since we are only considering the motion in one direction, this velocity can be positive (throwing the object upwards) or negative (throwing it downwards). From the velocity, we can determine the height y, since

$$\frac{dy(t)}{dt} = v(t) = -gt + v_0 \qquad \Rightarrow \qquad y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0, \tag{125}$$

where y_0 is the height at $t = 0$. Let us now calculate some properties of interest. First, we would like to know when the object hits the ground. For this we need to solve

$$-\frac{1}{2}gt^2 + v_0 t + y_0 = 0, \tag{126}$$

whose solutions are given by

$$t = \frac{1}{g}\left(v_0 \pm \sqrt{v_0^2 + 2gy_0}\right) \tag{127}$$

Let us first have a look at the situation with $v_0 = 0$, i.e. we just drop the object. In that case we can write (let us take the positive value)

$$t = \sqrt{\frac{2y_0}{g}}. \tag{128}$$

The velocity when the object reaches the ground is

$$v(\sqrt{\frac{2y_0}{g}}) = -g\sqrt{\frac{2y_0}{g}} = \sqrt{2gy_0}. \tag{129}$$

Note that the velocity is negative. We can also square this result:

$$v^2 = 2gy_0 \tag{130}$$

Rewriting this gives

$$\frac{1}{2}mv^2 = mgy_0. \tag{131}$$

We will encounted this result later on as the law of conservation of energy where the left-hand side is known as kinetic energy (the energy resulting from the velocity) and the right-hand side is known as potential energy (the energy contained in the height, which is transferred into velocity when we drop the object).

Fig. 9 shows the example for $v_0 = 7$ m/s and $y_0 = 10$ m. Although this is a two-dimensional plot, it only describes the motion in one direction, since the lower axis indicates t and not the horizontal position x. Inserting gives

$$t = \frac{1}{9.8}(7 \pm \sqrt{7^2 + 2 \times 9.8 \times 10}) = \frac{1}{9.8}(7 \pm \sqrt{7^2 + 2 \times 9.8 \times 10}) = \frac{1}{9.8}(7 \pm \sqrt{245}) = \begin{cases} 2.31 \text{ s} \\ -0.88 \text{ s} \end{cases} \tag{132}$$

Note that we get two values for the time when the object hits the ground, i.e. $y = 0$. One that directly makes sense, we throw the object into the air and 2.31 s later it hits the ground. However, why should it hit the ground for a negative time? Fig. 9 also shows the y values for negative t. We indeed see that it nicely continues to zero. What does this mean? We had only said that at $t = 0$, the object is at a height $y_0 = 10$ m and going upwards with a velocity $v_0 = 7$ m/s. It is natural to assume that, say, somebody is standing at a height of 10 m throwing the object upwards with a velocity of 7 m/s. However, there is another possibility: the object could have been shot from the ground with a high velocity (in a moment, we will see that this has to be 15.6 m/s). We let it go upwards for 0.88 s. We then start our stopwatch ($t = 0$ s) and notice that the object is at $y = 10$ m and has a velocity of 7 m/s upwards (it has slowed down because of gravity).

What is now the velocity when the object hits the ground? The velocity is given by

$$v(2.31) = -9.8 \times 2.31 + 7 = -15.6 \text{ m/s}, \tag{133}$$

where the velocity is less than zero, since the object is going down. Let us also look at the other time when it was on the ground in the situation where we shot the object upwards from the ground.

$$v(-0.88) = -9.8 \times (-0.88) + 7 = 15.6 \text{ m/s}. \tag{134}$$

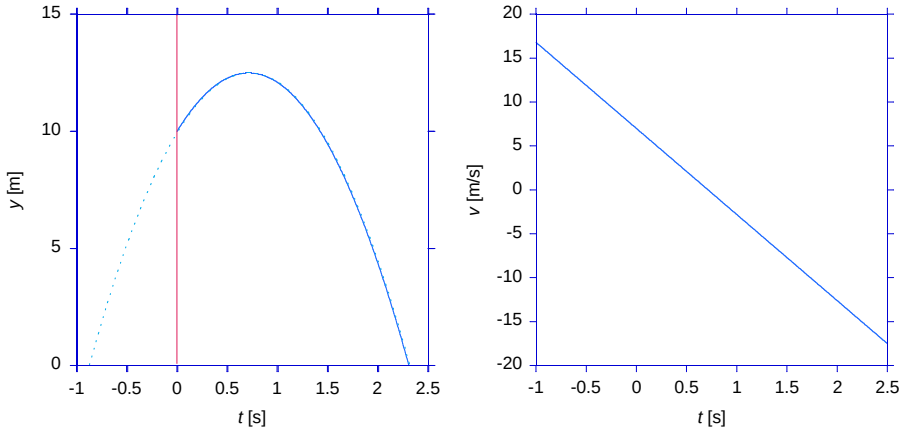


FIG. 9: The motion of a falling object due to gravity. The object is at $t = 0$ s at a height of $y_0 = 10$ m moving upwards with a velocity of $v_0 = 7$ m/s. The left and right sides show the position y and velocity v as a function of time, respectively.

This is exactly the same velocity, but now going upwards. So if you throw something in the air with a certain speed, it comes back at you with the same speed!

There is another aspect to the observation that the object that we threw upwards is coming down again, namely at a certain point it must have reached a maximum height. What characterizes this maximum? At the maximum, the object is not sure whether it wants to go up or down. Since the velocity cannot be both positive and negative, it must be zero. The time dependence of the velocity therefore needs to satisfy

$$v(t) = -9.8t + 7 = 0 \qquad \Rightarrow \qquad t_{\max} = 7/9.8 = 0.71 \ \text{ s.} \tag{135}$$

The maximum height it reached is then given by

$$x_{\max} = -\frac{1}{2}gt_{\max}^2 + v_0 t_{\max} + x_0 = -\frac{1}{2}9.8(0.71)^2 + 7 \times 0.71 + 10 = 12.5 \ \text{ m.} \tag{136}$$

Example. A ball is seen to pass upward by a window 25 m above the street with a vertical speed of 14 m/s. If the ball was thrown from the street,
(a) What was the initial velocity?
Solution: The height as a function of time is given by

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0 = -4.9t^2 + 14t + 25 \tag{137}$$

From this it follows that $y = 0$ for

$$y = \frac{1}{g}\left\{v_0 \pm \sqrt{v_0^2 + 2gy_0}\right\} = \frac{1}{9.8}(14 \pm \sqrt{14^2 + 2 \times 9.8 \times 25}) = \begin{cases} -1.24 \\ 4.10 \end{cases} \tag{138}$$

The initial velocity was therefore

$$v(t) = -gt + v_0 = -9.8(-1.24) + 14 = 26.2 \text{ m/s} \tag{139}$$

(b) What altitude does it reach?
Solution: At the maximum altitude, the velocity is zero

$$v(t) = -gt + v_0 = 0 \qquad \Rightarrow \qquad t = \frac{v_0}{g} = \frac{14}{9.8} = 1.43 \ \text{ s} \tag{140}$$

(or the average of -1.24 and 4.10). The height is then

$$y(t) = -\frac{1}{2}t^2 + v_0 t + y_0 = -\frac{1}{2}9.8(1.43)^2 + 14 \times 1.43 + 25 = 35 \text{ m} \tag{141}$$

C. Braking distance

Another variation on the same theme of constant acceleration is the calculation of the breaking distance. In this case, we want to calculate the acceleration needed to bring a car from a certain velocity v_0 to a complete stop. What we require therefore is that the velocity is zero. The velocity as a function of time is given by

$$v(t) = at + v_0. \tag{142}$$

Therefore, for $v(t)$ to be zero, we have

$$at + v_0 = 0 \qquad \Rightarrow \qquad t = -\frac{v_0}{a}. \tag{143}$$

Note that the time is positive therefore the acceleration is negative, which is what we would expect because we are slowing down. The distance travelled is now

$$x(t) = \frac{1}{2}at^2 + v_0 t, \tag{144}$$

where we take the position at the time we begin applying the brake to be zero. For $t = -\frac{v_0}{a}$, we find

$$x_{\text{decel}} = \frac{1}{2}a\left(\frac{v_0}{a}\right)^2 - v_0\frac{v_0}{a} = -\frac{1}{2}\frac{v_0^2}{a}. \tag{145}$$

Another factor that can contribute to the braking distance is that the person driving the car need some time to react t_{react} before applying the brakes. In this time, the car will move at the constant initial velocity and the distance travelled is

$$x_{\text{react}} = v_0 t_{\text{react}}. \tag{146}$$

The total distance travelled is therefore

$$x_{\text{total}} = x_{\text{react}} + x_{\text{decel}} = v_0 t_{\text{react}} - \frac{1}{2}\frac{v_0^2}{a}. \tag{147}$$

Suppose that we need to know the acceleration to design the brakes and the requirement is that we stay within a certain total distance:

$$a = -\frac{v_0^2}{2(x_{\text{total}} - v_0 t_{\text{react}})}. \tag{148}$$

Does everything make sense here. a is negative, which is correct since we are decelerating; if v_0 increaes the absolute value of a increases, makes sense too; if we allow for a longer braking distance x_{tot}, we need a smaller absolute value of a; if the reaction time increases, we need a larger absolute value of a. This all makes sense.

D. Time-dependent acceleration

More complicated accelerations. Let us treat an example where the acceleration is not a constant.

Example. A particle's position on the x axis is given by

$$x = 4 - 27t + t^3. \tag{149}$$

(a) Find the velocity and acceleration as a function of time.
Solution: We can obtain the velocity by differentiating the position with respect to time. Here we find

$$v(t) = \frac{dx}{dt} = -27 + 3t^2. \tag{150}$$

The acceleration is obtained by differentiating the velocity

$$a(t) = \frac{dv}{dt} = 6t. \tag{151}$$

(b) Is there ever a time when $v = 0$?
Solution: Setting $v = 0$ yields

$$0 = -27 + 3t^2 \qquad \Rightarrow \qquad t^2 = 9 \qquad \Rightarrow \qquad t = \pm 3 \text{ s}. \tag{152}$$

Thus the velocity is zero 3 second before and after the clock read 0.
(c) Describe the particles motion for $t \geq 0$.
Solution: At $t = 0$, the particle is at $x(0) = 4$ m and is moving at $v(0) = -27$ m/s, i.e. in the negative x direction. We know that the particle stops at $t = 3$ s, since its velocity is zero at that time. Note that in the interval $0 < t < 3$ s, the velocity is negative and the acceleration positive. Since they are in opposite directions, the particle must be slowing down. At $t = 3$ s the particle is at $x = -50$ m. So somewhere in the interval $0 < t < 3$ s it must have crossed $x = 0$. This can be found by putting $x(t) = 0$. Using a computer, we find $t = 0.14826\ldots$ (and also $t = 5.12043$ and $t = -5.2687$). For $t > 3$ s, the particle is moving to the right, since $v > 0$. Since the acceleration is also larger than zero, the particle is accelerating. From the solutions for $x(t) = 0$, we see that the particle crosses $x = 0$ again at $t = 5.12043$.

Since we know how to do calculus, we can attack more complicated problems, where the acceleration depends on time. An important example is where the acceleration is given by

$$a(t) = -A \sin \omega t. \tag{153}$$

We see that the acceleration oscillates in time. We can integrate this to obtain the velocity. Taking the velocity at $t = 0$ equal to zero gives

$$v(t) = \frac{A}{\omega} \cos \omega t \tag{154}$$

Integrating once more gives the position

$$x(t) = \frac{A}{\omega^2} \sin \omega t. \tag{155}$$

Note that we can write $a(t) = \omega^2 x(t)$. This is known as a harmonic oscillator, which describes the motion of an object that is bound to a certain position (in our case $x = 0$) by an acceleration that is directly proportional to the distance. A good approximation of something producing such an acceleration is a spring. Also, the velocity moves 90° out of phase with the position, i.e. the velocity is maximum at $x = 0$. And when x is maximum for $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots = (n + \frac{1}{2})\pi$ where n is an integer, the velocity is zero. This is equivalent to what we saw before since at a maximum the derivative of the position $\frac{dx(t)}{dt} = 0$, hence $v(t) = 0$.

VI. KINEMATICS IN TWO DIMENSIONS

A. Vectors

So far we have been only been considering velocity and position in one dimension. In that case, the can express them as a scalar, i.e. a number. Some physical parameters only make sense as a scalar, such as temperature, volume. However, we already saw that there was a directional component to velocity. As opposed to speed, which is always positive, we noticed that velocity could have a direction. We could be driving in the positive or negative directions. In classical mechanics, we are often dealing with problems in more than one dimension. In that case, the position, velocity, and acceleration are given by vectors. Vectors not only have a size but also a direction. In the lecture notes, the notation $\mathbf{a}$ is used for a vector. Other books use $\vec{a}$.

An important operations is the addition of two vectors $\mathbf{a}+\mathbf{b}$. We can solve this problem graphically. One approach is the parallelogram, see Fig. 10, where the vectors $\mathbf{a}$ and $\mathbf{b}$ form one side of the parallelogram. It is now straightforward to construct the parallelogram. The vector $\mathbf{a} + \mathbf{b}$ is the diagonal of the parallelogram. The author method, yielding the same result, is to connect the vectors head-to-tail, see Fig. 10(b). We would also like to subtract vectors $\mathbf{a} - \mathbf{b}$. Vector subtraction is done by turning it into vector addition:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}). \tag{156}$$

So we find $\mathbf{a} - \mathbf{b}$ by adding $-\mathbf{b}$ to $\mathbf{a}$. The vector $-\mathbf{b}$ has the same magnitude (length) as $\mathbf{b}$ but is pointing in the opposite direction, see Fig. 10(c). Once we have $-\mathbf{b}$, we can find $\mathbf{a} - \mathbf{b}$ by, e.g. the parallelogram method, see Fig. 10(c). Note that the vector $\mathbf{a} - \mathbf{b}$ is a vector that connects the head of $\mathbf{b}$ to the head of $\mathbf{a}$.

There are several operations we can perform with vectors, which are described in more detail in Giancoli, Chapter 3. Vector operations follow very similar rules as those with regular numbers. We have the following laws

$$\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u} \qquad \text{commutative law} \tag{157}$$
$$\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w} \qquad \text{associative law} \tag{158}$$
$$\tag{159}$$

For the substraction of two vectors, we use the addition of the negative vector, where the negative vector is given by the same vector but pointing in the opposite direction. Therefore,

$$\mathbf{u}-\mathbf{v} = \mathbf{u}+(-\mathbf{v}). \tag{160}$$

Although graphical method are a nice way to look at vectors, a more powerful method is to express it in components. The idea is that any vector can be buit up from several independent comporents. The number of components depends on the dimensionality. In two dimensions, the number of independent vectors is two, whereas in three dimensions, the number of independent vectors is, not surprisingly, three. Any other vector can be built up of those independent components. A very typical choice is to make use of the Cartesian coordinate system, i.e. we define to axes perpendicular to each other. We now choose two vectors of length one (also known as unit vectors), one lying along one axis and the other along the other. Conventionally, these are called the x and y axis. In three dimensions, we choose three perpendicular axes known as the x, y, and z axis. Let us first look at the two-dimensional case. Any vector in two dimension can be written as a combination of a vector lying along the x-axis $\mathbf{v}_x$ and a vector lying along the y axis $\mathbf{v}_x$,

$$\mathbf{v} = \mathbf{v}_x + \mathbf{v}_y. \tag{161}$$

Each of these components can be written as the unit vector multiplied by a scalar

$$\mathbf{v}_x = v_x\hat{\mathbf{i}} \qquad \text{and} \qquad \mathbf{v}_y = v_y\hat{\mathbf{j}} \tag{162}$$

Or for the total vector

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}. \tag{163}$$

The unit vectors are denotes in many different ways $\mathbf{x}$ and $\mathbf{y}$, $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$, etc. So do not let this confuse you. In principal it is not necessary to choose two basis vectors that are perpendicular. In general it works for any two vectors

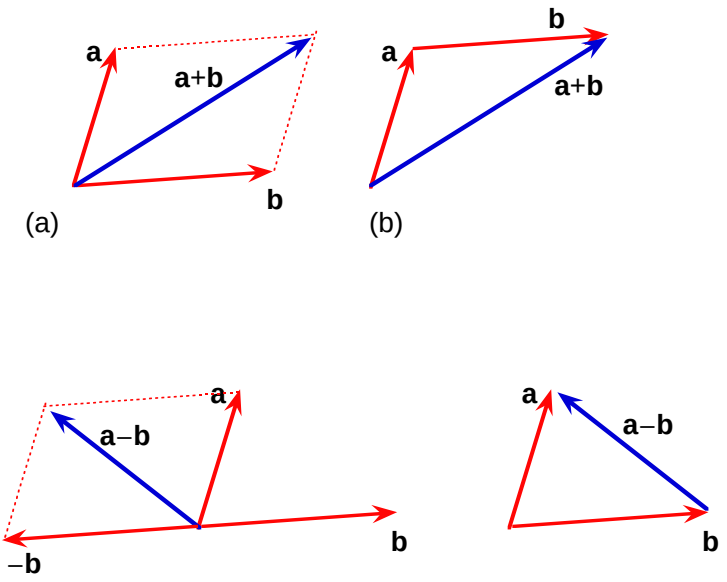


FIG. 10: (a) Addition of $\mathbf{a}$ and $\mathbf{b}$ using the parallelogram method. (b) Adding $\mathbf{a}$ and $\mathbf{b}$ by adding their heads and tails. (c) Subtraction of $\mathbf{b}$ from $\mathbf{a}$ by addition of $-\mathbf{b}$ using the parallelogram method. (d) The vector $\mathbf{a}-\mathbf{b}$ is a vector connecting the head of $\mathbf{b}$ with the head of $\mathbf{a}$.

as long as there are not parallel. However, choosing such a system is bound to lead to problems, so do not do it. Also, we do not have to choose unit vectors along the x and y axis, there are many different coordinate system possible, such as the polar system in two dimensions and the cylindrical and spherical system in three dimensions.

We can now easily calculate the norm of the vector. Since we know that the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ have length one, the length of a vector is given by

$$v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2}. \tag{164}$$

We can now also define the angle of the vector with the x-axis with

$$\tan\theta = \frac{v_y}{v_x}. \tag{165}$$

Alternatively, we can express

$$v_x = v\cos\theta \qquad \text{and} \qquad v_y = v\sin\theta. \tag{166}$$

Vector addition is now given by

$$\mathbf{u} + \mathbf{v} = u_x\hat{\mathbf{i}} + u_y\hat{\mathbf{j}} + v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} = (u_x + v_x)\hat{\mathbf{i}} + (u_y + v_y)\hat{\mathbf{j}}. \tag{167}$$

As is clear vector addition simply implies addition of the components of the vector. For this it is easy to see the why the different rules for vector operations look so similar to those for normal scalars,

$$\mathbf{u} + \mathbf{v} = (u_x + v_x)\hat{\mathbf{i}} + (u_y + v_y)\hat{\mathbf{j}} = (v_x + u_x)\hat{\mathbf{i}} + (v_y + u_y)\hat{\mathbf{j}} = \mathbf{v} + \mathbf{u} \qquad \text{commutative law} \tag{168}$$

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = \{(u_x + v_x) + w_x\}\hat{\mathbf{i}} + \{(u_y + v_y) + w_y\}\hat{\mathbf{j}} \tag{169}$$

$$= \{u_x + (v_x + w_x)\}\hat{\mathbf{i}} + \{u_y + (v_y + w_y)\}\hat{\mathbf{j}} = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \qquad \text{associative law.} \tag{170}$$

because the commutative and associative law are valid for scalars. Also, we can directly see why substraction is equivalent to addition of the negative vector

$$\mathbf{u} - \mathbf{v} = (u_x - v_x)\hat{\mathbf{i}} + (u_y - v_y)\hat{\mathbf{j}} = \{u_x + (-v_x)\}\hat{\mathbf{i}} + \{u_y + (-v_y)\}\hat{\mathbf{j}} = \mathbf{u} + (-\mathbf{v}) \tag{171}$$

Instead of using the notation with the unit vectors, the vectors are often denoted by only their coordinates

$$\mathbf{v} = (u_x, u_y, u_z) \qquad \text{or} \qquad \mathbf{v} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}. \tag{172}$$

Example. An airplane leaves an airport and is later sighted 215 km away, in a direction making an angle θ =22° east of north. How far east and north is the airplane when sighted.
Solution: We are given here the magnitude and direction of the vector. What we are asked to calculate are the components in the x and y directions. Note that the angle is given with the y axis (often the angle is given with the x axis as in the polar coordinate system. Do not mix up your sines and cosines). We therefore have

$$x = 215 \times \sin\theta = 81 \text{ km} \tag{173}$$

$$y = 215 \times \cos\theta = 199 \text{ km} \tag{174}$$

Example A group has to go through several checkpoints before reaching their final destination. The checkpoints are given by the following displacements:
(1) **a** to checkpoint Able, magnitude 36 km, due east.
(2) **b** to checkpoint Baker, due north.
(3) **c** to checkpoint Charlie, magnitude 25 km, at an angle of 135° of the axis from west to east.
The magnitude of the net displacement **d** is 62 km. What is the magnitude of **b**?
Solution: The total displacement is given by:

$$\mathbf{d} = \mathbf{a} + \mathbf{b} + \mathbf{c}. \tag{175}$$

There are two unknowns in this problem. Of vector **b**, we know the direction (north) but not the magnitude. Of the displacement **d** we know the magnitude but not the direction. The approach is that the west-east axis is the x axis and the south-north axis is the y axis. Splitting things in components gives

$$d\cos\theta = a + c\cos 135^\circ \tag{176}$$
$$d\sin\theta = b + c\sin 135^\circ. \tag{177}$$

The second equation has two unknowns b and θ. From the first equation, we can obtain θ

$$\cos\theta = \frac{a + c\cos 135^\circ}{d} \qquad \Rightarrow \qquad \theta = \arccos\frac{36 + 25\cos 135^\circ}{62} = 72.81^\circ. \tag{178}$$

This allows us to obtain the magnitude of b:

$$b = d\sin\theta - c\sin 135^\circ = 62\sin 72.81^\circ - 25\sin 135^\circ = 42 \text{ km}. \tag{179}$$

B. Three dimensions

The situation in three dimensions is very similar, except that we need a third component. The position vector **r** is given by

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}. \tag{180}$$

Now we would like to take the derivative in a similar way that we took the derivative in one dimension. For this, we need the displacement vector that an object travel in the time from t_1 to t_2,

$$\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \tag{181}$$

where the vectors

$$\mathbf{r}_1 = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}} \qquad \text{and} \qquad \mathbf{r}_2 = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}} \tag{182}$$

are the position at times t_1 and t_2, respectively. This gives

$$\Delta\mathbf{r} = (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}. \tag{183}$$

We would now like to define the derivative in a way similar to that for scalar functions. Let us therefore consider the average velocity for vectors

$$\overline{\mathbf{v}} = \frac{\Delta\mathbf{r}}{\Delta t}. \tag{184}$$

In order to find the instantaneous velocity, we again take the limit that the time difference goes to zero

$$\mathbf{v} = \lim_{\Delta t\to 0}\frac{\Delta\mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}. \tag{185}$$

Taking the derivative of a vector with respect to time is equivalent to taking the derivative of its components with respect to time

$$\mathbf{v} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}. \tag{186}$$

Constant acceleration Just as in one dimension, we can derive the velocity and position of an object feeling a constant acceleration **a** in more than one dimension by simply integrating.

$$\frac{d\mathbf{v}}{dt} = \mathbf{a} \qquad \Rightarrow \qquad \mathbf{v} = \int \mathbf{a}dt = \mathbf{a}t + \mathbf{v}_0, \tag{187}$$

where $\mathbf{v}_0$ is an integration constant that gives the velocity at $t = 0$, since the acceleration only gives the change in velocity. If we want to know the position $\mathbf{r}$, we can integrate the velocity

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}(t) \qquad \Rightarrow \qquad \mathbf{r} = \int \mathbf{v}(t)dt = \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}_0 t + \mathbf{r}_0, \tag{188}$$

where $\mathbf{r}_0$ gives the position of the object at $t = 0$. Obviously, we can separate this result again in coordinates. For the velocity, we have, in two dimensions,

$$v_x = a_x t + v_{0x} \qquad \text{and} \qquad v_y = a_y t + v_{0y} \tag{189}$$

and for the position

$$x = \frac{1}{2}a_x t^2 + v_{0x}t + x_0 \qquad \text{and} \qquad y = \frac{1}{2}a_y t^2 + v_{0y}t + y_0. \tag{190}$$

Example Giancoli 3.4
Example Giancoli 3.5
Example Giancoli 3.6
Example Giancoli 3.7

Example. A skier is accelerating down a 30° hill at 3.8 m/s^2.
(a) What is the vertical component of her acceleration?
Solution: The vertical component is

$$a_y = a \sin 30° = 3.8 \times \frac{1}{2} = 1.9 \text{ m/s}^2 \tag{191}$$

(b) How long will it take her to reach the bottom of the hill, assuming she starts from rest and accelerates uniformly, if the elevation change is 250 m?
Solution: The elevation is 250 m. Therefore the distance on the slope is 250/sin 30°=500 m. The acceleration is along the slope. Therefore

$$\frac{1}{2}at^2 = 500 \qquad \Rightarrow \qquad t = \sqrt{\frac{2 \times 500}{3.8}} = 16.2 \ s \tag{192}$$

Example: Cannon problems. We can now approach the typical artillery problems describing the motion of a projectile under the influence of gravity. This is an approximation since we are neglecting air resistance all together. Suppose we should something from the ground $y_0 = 0$ with a certain velocity $\mathbf{v}$. We would like to know the trajectory of the projectile, i.e. the height as a function of the distance. Under the acceleration of gravity, the position as a function of time is given by

$$x = v_{0x}t \qquad \text{and} \qquad y = -\frac{1}{2}gt^2 + v_{0y}t. \tag{193}$$

From this we can easily express the time as a function of the position

$$t = \frac{x}{v_{0x}}. \tag{194}$$

Which we can use to express the height as a function of the distance

$$y(t) = y(\frac{x}{v_{0x}}) = -\frac{1}{2}g\left(\frac{x}{v_{0x}}\right)^2 + v_{0y}\frac{x}{v_{0x}}. \tag{195}$$

We can clearly see that the trajectory is parabolic. Obviously, we would like to know when the projectile hits the ground, i.e. $y = 0$. For this we need to solve

$$-\frac{1}{2}g\left(\frac{x}{v_{0x}}\right)^2 + v_{0y}\frac{x}{v_{0x}} = \frac{x}{v_{0x}}\left(-\frac{1}{2}g\frac{x}{v_{0x}} + v_{0y}\right) = 0. \tag{196}$$

One solution is obvious, $x = 0$ which is where we are standing with our cannon. The other solution is

$$x = \frac{2v_{0x}v_{0y}}{g}. \tag{197}$$

Now we would like to find out what is the maximum range. We are shooting our projectile from the cannon with a certain velocity v under an angle θ with the ground. We can therefore express $v_{0x} = v\cos\theta$ and $v_{0y} = v\sin\theta$. Therefore the range is given by

$$x = \frac{2v^2\sin\theta\cos\theta}{g} = x = \frac{v^2\sin 2\theta}{g}. \tag{198}$$

The maximum range is given by making the derivative of x with respect to θ equal to zero

$$\frac{dx}{d\theta} = \frac{2v^2\cos 2\theta}{g}. \tag{199}$$

$\cos 2\theta = 0$ for $2\theta = \frac{\pi}{2}$ (we only want solution in the first quadrant). Therefore $\theta = \frac{\pi}{4} = 45°$.

Example. A fort is trying to attack a pirate ship 560 m away by shooting a cannon located at sea level. The cannon ball has an initial speed of v=82 m/s. Neglect air resistance.
(a) At what angle θ from the horizontal must the cannon be fired?
To solve this, we use Eqn. (198)

$$x = \frac{v^2\sin 2\theta}{g} \qquad \Rightarrow \qquad \theta = \frac{1}{2}\arcsin\frac{gx}{v^2} = \frac{1}{2}\arcsin\frac{9.8\times 560}{82^2} = \begin{cases} 27° \\ 63° \end{cases} \tag{200}$$

Note that there are two different angles less than 90° at which the cannon can be shot.
(b) How far should be pirate ship be away to prevent it getting hit by cannon balls.
Solution: The maximum distance is obtained for an angle of 45° (note that this is the angle where we do not have two possible trajectories). The distance at this angle is

$$x = \frac{v^2}{g}\sin 2\theta = \frac{82^2}{9.8}\sin 90° \cong 690 \text{ m}. \tag{201}$$

Example. An object is thrown horizontally with a velocity v from a height y_0. A second object is dropped at a distance Δx from a height $y_0 + \Delta y$.
(a) What is the condition for Δy in order for the two objects to hit each other.
Solution: For object 1, the trajectories are given by

$$x_1 = vt \qquad \text{and} \qquad y_1 = -\frac{1}{2}gt^2 + y_0 \tag{202}$$

For object 2, the trajectories are given by

$$x_2 = \Delta x \qquad \text{and} \qquad y_2 = -\frac{1}{2}gt^2 + y_0 + \Delta y. \tag{203}$$

For the two objects to hit each other, we need $x_1(t_{\text{hit}}) = x_2(t_{\text{hit}})$ and $y_1(t_{\text{hit}}) = y_2(t_{\text{hit}})$. From the second equality, we directly see the $y_0 = y_0 + \Delta y$. Therefore, $\Delta y = 0$.
(b) Find the expression for v, such that the two objects hit each other at $y_0/2$.
Solution: The two objects hit each other at the time $vt_{\text{hit}} = \Delta x$, therefore $t_{\text{hit}} = \Delta x/v$. The objects are then at a height

$$y = -\frac{1}{2}gt_{\text{hit}}^2 + y_0, \tag{204}$$

which should equal $y_0/2$. Therefore,

$$\frac{1}{2}gt_{\text{hit}}^2 = \frac{1}{2}y_0 \qquad \Rightarrow \qquad \frac{1}{2}g\left(\frac{\Delta x}{v}\right)^2 = \frac{1}{2}y_0 \qquad \Rightarrow \qquad v = \sqrt{\frac{g}{y_0}}\Delta x \tag{205}$$

Hitting a moving object Let us now try to make our life a bit more difficult by trying to hit an object which is moving away from us. Suppose at $t = 0$ the object is moving away from us with a velocity v from a position x_1, whereas we are standing at a position $x = 0$. Let us assume that the velocity in the x direction of our projectile is $2v$. What is now the relation between the velocity in the y direction of the projectile v_{0y} and v and x_1.

First, we would like to know when the two objects are at the same position is the x direction. Therefore we need

$$2vt_{\rm hit} = vt_{\rm hit} + x_1 \qquad \Rightarrow \qquad t_{\rm hit} = \frac{x_1}{v}. \tag{206}$$

The other thing that we require is that our projectile hits the ground at exactly the same time. As before, the motion of our projectile is given by

$$y(x) \;=\; -\frac{1}{2}gt_{\rm hit}^2 + v_{0y}t_{\rm hit} = -\frac{1}{2}g\left(\frac{x_1}{v}\right)^2 + v_{0y}\frac{x_1}{v} = 0 \qquad \Rightarrow \qquad v_{0y} = \frac{v}{x_1}\frac{1}{2}g\left(\frac{x_1}{v}\right)^2 = \frac{1}{2}g\frac{x_1}{v}. \tag{207}$$

VII. CIRCULAR MOTION

(Note that this is treated in Giancoli 3-9). Let us consider an object moving in a circle at a constant speed. Note that although the absolute value of the velocity does not change, its direction does. Therefore, a force must be exerted on the object to keep it in that circular motion. The acceleration is then given by

$$\mathbf{a} = \lim_{\Delta t\to 0}\frac{\Delta \mathbf{v}}{\Delta t}. \tag{208}$$

Let us consider the change when the object moves for a very small distance along its circular motion. The change in angle with respect to the origin is then $\Delta\theta$, see Giancoli Fig. 3.32. The velocity has now changed from $\mathbf{v}_1$ to $\mathbf{v}_2$. Let us take $\mathbf{v}_1$ in the x-direction. Therefore $\mathbf{v}_1 = v\mathbf{i}$. Δt later, the direction of $\mathbf{v}$ has changed slightly

$$\mathbf{v}_2 = v\cos\Delta\theta\mathbf{i} - v\sin\Delta\theta\mathbf{j} = v\mathbf{i} - v\Delta\theta\mathbf{j} \tag{209}$$

The acceleration is therefore

$$\mathbf{a} = \lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t} = \lim_{\Delta t\to 0}\frac{\mathbf{v}_2 - \mathbf{v}_1}{\Delta t} = -\lim_{\Delta t\to 0}\frac{v\Delta\theta}{\Delta t}\mathbf{j}. \tag{210}$$

Now we know that the distance travelled along the circle can be expressed as $\Delta l = r\Delta\theta$. Inserting this, gives

$$\mathbf{a} = -\lim_{\Delta t\to 0}\frac{v\Delta l}{r\Delta t}\mathbf{j} = -\frac{vdl}{rdt}\mathbf{j} = -\frac{v^2}{r}\mathbf{j}. \tag{211}$$

Note that the acceleration is pointing in the negative y direction, i.e. to the center of the circle.

A different way of looking at circular motion. We can describe the motion in a circle by

$$\mathbf{r} = r\cos\omega t\;\hat{\mathbf{i}} + r\sin\omega t\;\hat{\mathbf{j}}, \tag{212}$$

where ω is the angular velocity. This angular velocity is given by

$$\omega = \frac{d\varphi}{dt}, \tag{213}$$

where φ is the angle between $\mathbf{r}$ and the positive x-axis. For example, $\omega = 2\pi$ rad/s implies that the angle of $\mathbf{r}$ with the x-axis changes by 2π each second, i.e. it goes in a circle once every second. If the length of $\mathbf{r}$

$$|\mathbf{r}| = \sqrt{r^2\cos^2\omega t + r^2\sin^2\omega t} = r \tag{214}$$

remains constant, the head of $\mathbf{r}$ will describe a circle. An angular velocity of $\omega = 4\pi$ rad/s means that $\mathbf{r}$ makes two circles per second.

By taking the derivative of $\mathbf{r}$, we can find

$$\mathbf{v} = -r\omega\sin\omega t\;\hat{\mathbf{i}} + r\omega\cos\omega t\;\hat{\mathbf{j}}, \tag{215}$$

$$|\mathbf{v}| = \sqrt{r^2\omega^2 \sin^2 \omega t + r^2\omega^2 \cos^2 \omega t} = r\omega. \tag{216}$$

Note that this makes perfect sense. If $\omega = 2\pi$ rad/s, the $v = \omega r = 2\pi r$ m/s. If $\mathbf{r}$ goes in a circle once every second than the distance travelled is the circumference of the circle or $2\pi r$. The acceleration is now

$$\mathbf{a} = -r\omega^2 \cos \omega t \, \hat{\mathbf{i}} - r\omega^2 \sin \omega t \, \hat{\mathbf{j}}, \tag{217}$$

with a magnitude

$$|\mathbf{a}| = \sqrt{r^2\omega^4 \cos^2 \omega t + r^2\omega^4 \sin^2 \omega t} = \omega^2 r = \frac{v^2}{r}. \tag{218}$$

Example Giancoli 5.10
Example Giancoli 5.13

VIII. NEWTON'S LAWS

The first law of Newton says

Every body continues in its state of rest or uniform velocity in a straight line as long as not net force acts on it.

This law states refutes clearly the common perception that rest is the natural state of an object and that a moving body would be inclined to come to a halt. However, this is a result of friction, which is not present in an ideal system. The second law of Newton states

The acceleration of an object is directly proportional to the net force acting on it and is inversely proportional to its mass. The direction of the acceleration is in the direction of the net force acting on the object.

In equation form, this reads

$$\mathbf{a} = \frac{1}{m} \sum_i \mathbf{F}_i, \tag{219}$$

where the sum goes over the different forces acting on the object. We can also write

$$\sum_i \mathbf{F}_i = m\mathbf{a}. \tag{220}$$

Note that the first law of Newton is contained in the second. When there is no net force

$$m\mathbf{a} = 0 \qquad \Rightarrow \qquad m\frac{d\mathbf{v}}{dt} = 0 \qquad \Rightarrow \qquad \mathbf{v} = \mathbf{v}_0, \tag{221}$$

where $\mathbf{v}_0$ is a constant vector. As we saw in the section on the fundamental forces, we can imagine a force as the number of exchange particles arriving on some object per second. We also noticed that the essential "currency" for particles is not velocity but momentum. A more fundamental way of writing Newton's second law is therefore

$$\sum_i \mathbf{F}_i = \frac{d\mathbf{p}}{dt}, \tag{222}$$

where $\mathbf{p} = m\mathbf{v}$. Note, that we can now write this as

$$\sum_i \mathbf{F}_i = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt}. \tag{223}$$

We see that the first term on the right-hand side is equivalent to that in Newton's second law. The second term is related to a change in mass. Newton assumed that the mass does not change. For most purposes this is a reasonable

assumption. Obviously, the equations have to be adapted when the mass changes, for example an object is losing mass in its trajectory. Certainly of interest when launching a rocket which looses fuel very quickly during the launch. However, this can all be done and that is not really fundamental. Things start fundamentally going wrong when the velocity v of an object starts approaching the speed of light c. As briefly mentioned in the section on fundamental forces, the relativistic mass is given by $m = m_0/\sqrt{1-\frac{v^2}{c^2}}$, where m_0 is the mass at rest. However, we will not be dealing with objects moving at the speed of light (apart from photons) and we can ignore relativistic effects.

The third law of Newton states:

Whenever one object exerts of force on a second object, the second exerts an equal and opposite force on the first.

Often this is also rephrased as "for every action there is an equal but opposite reaction." Correct, but not entirely satisfactory. Note, that this seems to imply that one body is doing the action and the other body is on the receiving end, whereas their roles are actually similar. The third law often leads to conceptual problems. Consider, for example, this statement:

According to Newton, every force has an equal and opposite force. So, how does anything move? If I am going forward with the force of 10 Newtons, there is a force of 10 Newtons pushing me back. Wouldn't the forces cancel each other out?

There are two conceptual problems here. First, it says that somebody is going forward with a certain force. However, when you are going forward you are actually exerting a force backwards (this is probably even more obvious with swimming). When you are walking you are pushing yourself forward by exerting a force backwards on the the ground. The second conceptual problem is that the assumption is made that all the forces work on the person that is walking. However, the force exerted by the person on the ground is not working on the person and will not change his velocity. However, the force exerted by the ground works on the person. Since this force is opposite to the force that the person is exerting on the ground it is in the direction that the person is walking.

In the Section on Fundamental Forces, we had a different way of looking at interactions. Here we saw that interactions were a result of the exchange of particles. One of the most important things that was exchanged was momentum. The idea was that momentum is the currency of particles. The force between two particles is nothing but the total exchange of momentum per second. However, remember how it is with money. If I give a dollar to you, I become a dollar poorer but you become a dollar richer. It is somewhat more complicated with momentum, because momentum is a vector. If particle 1 is transferring a certain amount of momentum to particle 2 then particle 2 is gaining that momentum, but particle 1 is loosing it, see Fig. 27. This looks a bit like for every action there is an equal but opposite reaction. We are lacking the symmetry here because we are transferring the momentum from 1 to 2. However, at the same time particle 2 is transferring momentum to particle 1. However, there is no way to avoid the fact that if one particle gains momentum the other particle looses it. For a repulsive force, you can visualize this by people in two boats or carts making each other move by throwing things at each other. From Eqn. (222), we see that a force is just related to the change of momentum in time. Imagine that you are throwing very small things at each other very fast. In that case, you can think of a stream of particles going between the two boats or carts, changing each others momentum by a certain amount per second. This is basically a force. However, if one gain momentum, the other has to loose it, so if one exerts a force on the other, the other will respond with an equal and opposite force.
Example Giancoli 4.4
Example Giancoli 4.9

Example Pulling a tire
Three persons (A, B, and C) a pulling horizontally a tire. The tire remains stationary, despite the fact that three people are pulling on it. We want to know the force that person B is using given the fact that at his left at an angle of 137° on the tire, person A is pulling with a force of magnitude $F_A = 220$ N. On his right, person C is pulling with a force of mangnitude 170 N. However, we do not know the exact angle φ between B and C.

Solution: The key here is that the tire is stationary, therefore

$$\mathbf{F}_A + \mathbf{F}_B + \mathbf{F}_C = m\mathbf{a} = 0, \tag{224}$$

since the acceleration $\mathbf{a} = 0$. Let us evaluate the components. Let us take person B pulling in the positive x direction.

$$F_A \cos 137^\circ + F_B + F_C \cos\varphi = 0 \tag{225}$$
$$F_A \sin 137^\circ + F_C \sin\varphi = 0. \tag{226}$$

From the second equation (with only one unknown), we can determine the angle φ,

$$\sin\varphi = -\frac{F_A \sin 137^\circ}{F_C} = -\frac{220 \sin 137^\circ}{170} = -0.8825. \tag{227}$$

This gives in principle two solutions $\varphi = -61.96^\circ$ and $\varphi = -118.04^\circ$. The forces F_B corresponding with the two different angles are

$$F_B = -F_C \cos\varphi - F_A \cos 137^\circ = \begin{cases} 170 \times \cos(-61.96^\circ) - 220 \times \cos 137^\circ = 81 \text{ N} \\ -170 \times \cos(-118.04^\circ) - 220 \times \cos 137^\circ = 240 \text{ N} \end{cases} \tag{228}$$

A. The normal force

When putting a box on the ground, we know that there is a gravitational force working on the box. However, despite of that the box is not moving towards the center of the Earth. Obviously, the floor is preventing this. The floor therefore exerts a force on the box equal and opposite to the gravitational force $\mathbf{F}_g$. This is called the normal force $\mathbf{F}_N$. Newton's second law gives

$$\mathbf{F}_g + \mathbf{F}_N = m\mathbf{a} = 0 \tag{229}$$

Since the forces only work in the y direction we can write

$$mg - F_N = 0 \qquad \Rightarrow \qquad F_N = mg. \tag{230}$$

The normal force is a contact force. It is mainly a result of Coulomb forces preventing the box to enter the floor. Note that according to Newton's third law, the box is exerting an equal but opposite force on the floor.

Example Giancoli 4.11
Example Giancoli 4.13

B. Tension

If a person pulls a cord to pull a box, he/she is exerting a force on that cord. If the box is moving at a constant velocity, the cord will exert the same force on you. The force is transferred to the other side of the cord. The force in the cord is called a tension force. At the other side of the cord, the cord is pulling at the box. Again, according to Newton's third law, the box will be pulling with the same force on the cord. In the end there are two equal forces pulling the cord (the person pulling it on one side and the box pulling it on the other side in the opposite direction). Since both forces are equal, the cord will not accelerate. A cord is often assumed massless and unstretchable. Basically, it is only a connection between two different bodies.

Example. A block with mass $M = 15$ kg hangs by a cord from a knot K of mass m_K, see Fig. 11. The cords have a negligible mass, and $m_K \ll M$. What are the tensions in the three cords?
Solution: The easiest starting point is the block which has one cord attached to it. The block is at rest, so for the force in the y direction, one has

$$F_{T3} - F_g = F_{T3} - Mg = Ma_y = 0. \tag{231}$$

This directly gives $F_{T3} = Mg = 15 \times 9.8 = 147$ N. Let us now consider the forces on the knot:

$$\mathbf{F}_{T1} + \mathbf{F}_{T2} + \mathbf{F}_{T3} = m_K \mathbf{a}_k = 0. \tag{232}$$

Since we know the directions of all the forces and the magnitude F_{T3}, the unknown quantities are F_{T1} and F_{T2}. Although Eqn. (232) looks like one equation, the are in fact two equations, since we can separate the x and y coordinates

$$-F_{T1} \cos 28^\circ + F_{T2} \cos 47^\circ = 0 \tag{233}$$
$$F_{T1} \sin 28^\circ + F_{T2} \sin 47^\circ - F_{T3} = 0 \tag{234}$$

Two equations, two unknowns, that should be doable. The first equation gives

$$F_{T1} = \frac{\cos 47^\circ}{\cos 27^\circ} F_{T2} = 0.7724 F_{T2}. \tag{235}$$

The second equation then becomes

$$0.4694 \times 0.7724 F_{T2} + 0.7313 F_{T2} - 147 = 0 \qquad \Rightarrow \qquad F_{T2} = \frac{147}{+0.7313} = 134 \text{ N}. \tag{236}$$

Which then gives for the tension force in cord 1, $F_{T1} = 0.7724 \times 134 = 104$ N.

Example. A cord is holding a block with a mass $m = 15$ kg, see Fig. 12 on a frictionless plane that is inclined at an angle $\theta = 27^\circ$.
(a) What are the magnitudes of the tension force $\mathbf{F}_T$ from the cord on the block and the normal force $\mathbf{F}_N$ from the slope on the block?
Solution: The block is at rest so the sum of the forces on the block is zero

$$\mathbf{F}_N + \mathbf{F}_T + \mathbf{F}_g = m\mathbf{a} = 0. \tag{237}$$

The normal force is normal to the surface of the slope, the tension force is along the slope, whereas gravity is vertical with respect to the ground. We now want to choose a coordinate system. There are two obvious choices. Taking horizontal and vertical as the axes or taking the x and y axis along and perpendicular to the slope, respectively. The latter is slightly preferable since only gravity is not parallel or perpendicular to the slope. The other choice does give the same answer. The answer should never depend on your choice of axis system. The choice of axis system is a human choice and the physics should be independent of those choices. We then obtain, using $F_g = mg$,

$$-mg \sin\theta + F_T = 0 \tag{238}$$
$$-mg \cos\theta + F_N = 0. \tag{239}$$

This is easily solved and gvies $F_T = mg \sin\theta = 15 \times 9.8 \times \sin 27^\circ = 67$ N and $F_N = mg \cos\theta = 15 \times 9.8 \times \cos 27^\circ = 130$ N.
(b) We now cut the cord. What is the acceleration for the block down the slope?
Solution: In the direction perpendicular to the slope the normal force still cancels gravity and the acceleration is zero. However, in the direction parallel to the slope, we find

$$-mg \sin\theta = ma_x \qquad \Rightarrow \qquad a_x = -g \sin\theta = -9.8 \sin 27^\circ = -4.4 \text{ m/s}^2. \tag{240}$$

This is less than the acceleration for a free fall.

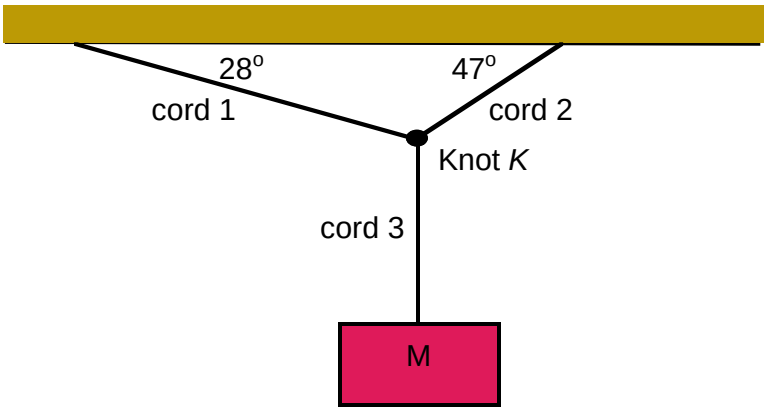


FIG. 11: A mass M hangs from three cords connect by a knot.

C. Friction

The standard theory of friction is a great simplification of a very complicated problem. The basic assumption is that the friction is proportional to the force exerted of the object on the surface. This equals the force exerted by the surface on the object, i.e. the normal force. Therefore, we can write

$$F_{\text{fr}} = \mu F_N. \tag{241}$$

However, we have to distinguish to different friction constants. The friction constant for a static object is greater than that for a moving object,

$$F_{\text{fr}} = \mu_s F_N \qquad \text{for an object at rest} \tag{242}$$
$$F_{\text{fr}} = \mu_k F_N \qquad \text{for an object in motion.} \tag{243}$$

This corresponds to the well-known experience the force to pull a moving object is smaller than that needed to set it in motion. The direct linearity between the normal force and the friction is based on certain assumptions

- The frictional force is proportional to the normal force. There are exceptions to the assumption. For example, different tires can have seriously different friction in snow even though their weight (and therefore the normal force) is the same. Narrow tires tend to compact the snow more compared to braod tires. The compact snow has actually a smaller coefficient of friction changing the frictional force.
- The frictional force is independent of the area of contact. The area of contact is much smaller than the area of coverage. Because of microscopic irregularities on the surface the block is actually gliding on a small fraction of the coverage area. The points of contact are deformable and an increase in the normal force will increase the actual contact area. The assumption that frictional force is independent of the coverage area fails when the coverage area becomes so narrow that it digs into the surface.
- The frictional force is independent of the velocity. Although this is approximately true for a wide range of materials, it is not true with air friction. Air friction depends on the square of the speed and for higher velocities even on higher powers. Another example is when the contact area contains viscous fluids, such as treacle and lubricants.

Figure 13 shows how the friction force depends on the applied force when we are pushing against a block. At first we have static friction. The acceleration is zero and the friction force matches the applied force. Microscopically what seems to happen is that areas on the surface actually coldweld together. When we bring very smooth metal surfaces together in vacuum (to keep the surfaces clean), then it becomes very difficult to separate the two. Many atoms on the surface bind together and almost form one piece of metal. This is an extreme version of what happens with friction. Usually surfaces are not very clean or are oxidized and the actual contact area can be several orders of magnitude smaller than the surface area. However, still parts of the surface are colwelded together. When we increase the force at a certain point the coldwelds break (the threshold of motion) and the block starts to move. When the block is in motion, the friction force remains more or less constant.

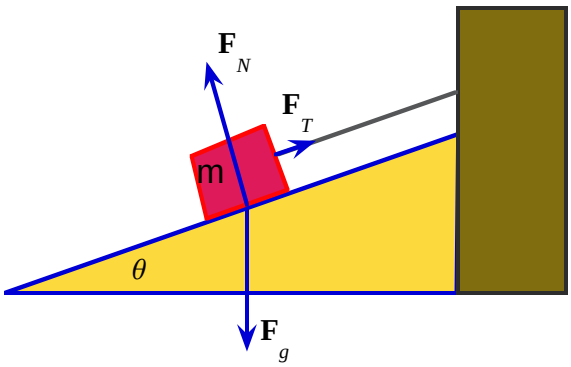


FIG. 12: A mass m is held stationary on a slope by a cord.

Example Giancoli 5.1
Example Giancoli 5.2

Example. Let us consider someone pulling a block with a force F_A under an angle θ with the surface. What is the relation between the optimal angle and the kinetic friction coefficient. The equations of motion of the block is as follows

$$\mathbf{F}_A + \mathbf{F}_N + \mathbf{F}_g + \mathbf{F}_{\rm fr} = m\mathbf{a}, \tag{244}$$

where $\mathbf{F}_N$, $\mathbf{F}_g$, and $\mathbf{F}_{\rm fr}$ are the normal force, the gravitation force and the friction, respectively. The can be divided into x and y coordinates

$$x: \qquad F_A\cos\theta - F_{fr} = ma_x \tag{245}$$
$$y: \qquad -mg + F_N + F_A\sin\theta = 0, \tag{246}$$

where there is no acceleration in the y direction. For the second equation it follows that the normal force is

$$F_N = mg - F_A\sin\theta. \tag{247}$$

The friction is related to the normal force by $F_N = \mu_k F_N$. We can substitute the expression for the normal force into the equation of motion in the x direction,

$$F_A\cos\theta - \mu_k(mg - F_A\sin\theta) = ma_x \qquad \Rightarrow \qquad a = \frac{1}{m}(\cos\theta + \mu_k\sin\theta)F_A - \mu_k g. \tag{248}$$

Now we want to maximize the acceleration as a function of the angle. Therefore we that the derivative with respect to θ:

$$\frac{da_x}{d\theta} = \frac{1}{m}(-\sin\theta + \mu_k\cos\theta)F_A = 0 \qquad \Rightarrow \qquad \tan\theta = \mu_k. \tag{249}$$

Note that if $\mu_k \to 0$, i.e. no friction, the optimal angle goes to zero and the best effect is obtained by pushing it horizontally. When the friction coefficient increases we want to increase the angle to reduce the friction.

Example Giancoli 5.5
Example Giancoli 5.6
Example Giancoli 5.8

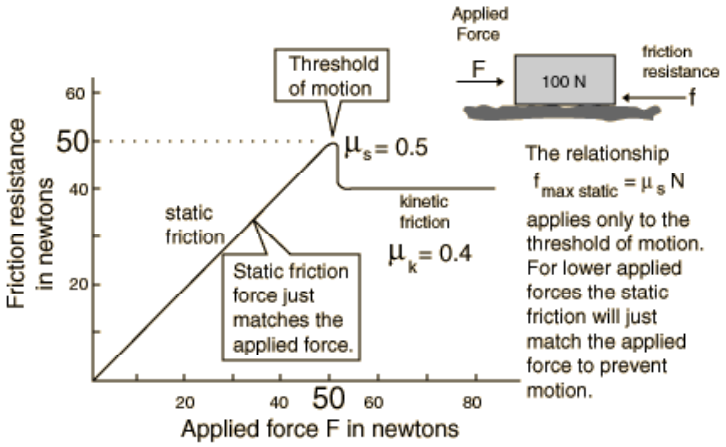


FIG. 13: The friction force as a function of the applied force (from hyperphysics).

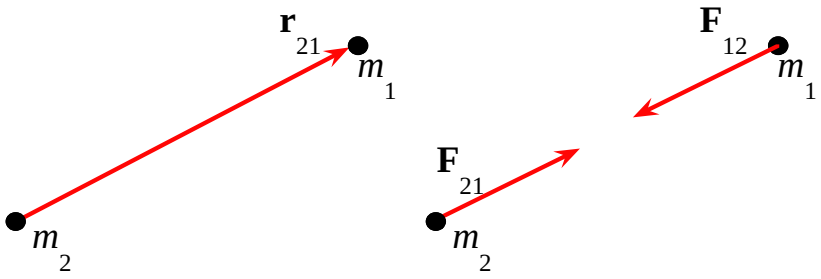


FIG. 14: One the left side, the vector $\mathbf{r}_{21} = \mathbf{r}_1 - \mathbf{r}_2$ connects the position $\mathbf{r}_2$ to $\mathbf{r}_1$. On the right side, according to Newton's third law, the gravitation force $\mathbf{F}_{12}$ on particle 1 by particle 2 is equal in size but opposite to the force $\mathbf{F}_{21}$ on particle 2 by particle 1.

IX. GRAVITY

In the Section on fundamental forces, we saw that the gravitational force is given by

$$F = G\frac{m_1 m_2}{r^2}. \tag{250}$$

In vector notation this is written as

$$\mathbf{F}_{12} = -G\frac{m_1 m_2}{r_{21}^2}\hat{\mathbf{r}}_{21}, \tag{251}$$

where $\mathbf{F}_{12}$ is the force on particle 1 with mass m_1 exerted by particle 2 with mass m_2. The vector $\mathbf{r}_{21} = \mathbf{r}_1 - \mathbf{r}_2$ is a vector pointing from 2 towards 1, which has a length r_{21}. We can now define the unit vector $\hat{\mathbf{r}}_{21} = \mathbf{r}_{21}/r_{21}$. The force on 2 by 1 is, according to Newton's third law, equal but opposite to $\mathbf{F}_{12}$

$$\mathbf{F}_{21} = -\mathbf{F}_{12} = G\frac{m_1 m_2}{r_{21}^2}\hat{\mathbf{r}}_{21} = -G\frac{m_1 m_2}{r_{21}^2}\hat{\mathbf{r}}_{12}. \tag{252}$$

Example Giancoli 3.12
The Moon has an orbit which is close to circular. The radius is about 384,000 km, which we estimated in the section on length scales. The Moon circles around the earth in approximately 27.3 days. We would now like to determine the acceleration of the Moon toward the Earth.
Solution: Since the know the radius and the period, we can determine the velocity. The period in seconds is $T = 27.3 \times 24 \times 3600 = 2.36 \times 10^6$ s. The total length of the orbit is $l = 2\pi \times 384 \times 10^6 = 2.41 \times 10^9$ m. The velocity is therefore $v = l/T = 1022$ m/s (or 3680 km/h). The acceleration is then

$$a_{\text{Moon}} = \frac{v^2}{r} = \frac{(1022)^2}{384 \times 10^6} = 0.00272\ \frac{\text{m}}{\text{s}^2}. \tag{253}$$

We also determined the radius of the Earth, which is about 6350 km. The ratio of the accelerations due to the Earth on the Moon relative to that on the Earth's surface are related to the ratio of the forces

$$\frac{a_{\text{Earth}}}{a_{\text{Moon}}} = \frac{\frac{GM}{R_{\text{Moon}}^2}}{\frac{GM}{R_{\text{Earth}}^2}} = \frac{R_{\text{Earth}}^2}{R_{\text{Moon}}^2} = \frac{(6.35 \times 10^6)^2}{(3.84 \times 10^8)^2} \cong \frac{1}{3600}. \tag{254}$$

where M is the mass of the Earth. This gives for the acceleratation on the Earth's surface $a_{\text{Earth}} = 3600 \times a_{\text{Moon}} = 3600 \times 0.00272 = 9.9$ m/s^2. This is slightly off the commonly used value of 9.8 m/s^2, probably due to some simplifactions in the Moon's orbit.

Therefore Newton determined the acceleration due to gravity. Since this can also be written as

$$g = G\frac{M}{R_{\text{Earth}}^2}. \tag{255}$$

Since Newton knew the Earth's radius, he had basically determined the factor GM. This is very important for people on Earth, but it is not the fundamental constant G. Determining G was done by Henry Cavendish (1731-1810).

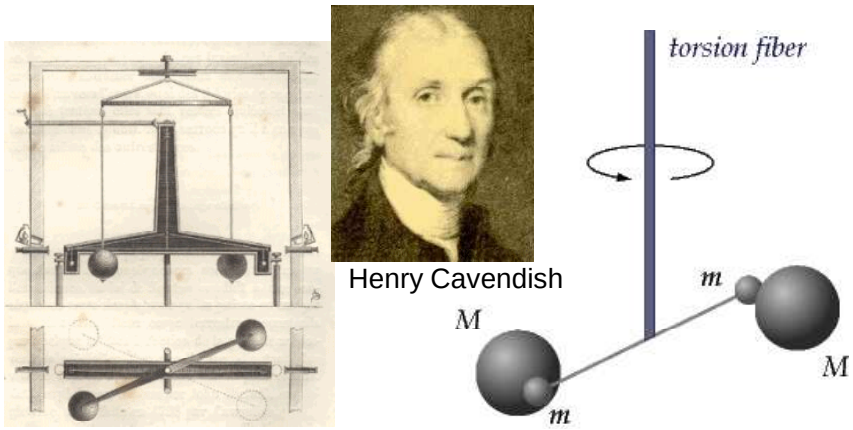


FIG. 15: Henry Cavendish determined the gravitation constant G using a torsion tod, see the left side. A more schematic picture is shown on the right.

Cavendish was one of the great "amateurs" in science. He was provided for by his noble family being the elder son of Lord Charles Cavendish, son of the second Duke of Devonshire. Cavendish inherited the family fortune later in life. He was a great experimentalist who worked on gravity, electricity, and chemistry. When Maxwell edited unpublished work by Cavendish he realized that many of the results pre-dated the important work and conclusions made by Faraday and Coulomb. He also determined that air consists of 79.167% phlogisticated air and 20.8333% dephlogisticated air. The latter is now known to be oxygen and modern measurements put the percentage at 20.95%. This is impressive indeed. BTW, the phlogiston theory is completely obsolete now. The idea was that flammable materials contained phlogiston. During burning phlogiston is released from materials. The presence of a dephlogisticated gas helps the burning. Unfortunately, this is rather opposite to what happens. Flammable materials are in fact oxygenated when burned, for example, magnesium becomes heavier when burned. However, a lot of materials disappear when burned (they turn into CO_2 and H_2O), so it is understandable that people thought that burning makes the materials lighter. Cavendish is best known for his determination of the gravitatinal constant. He did this using a torsion experiment, see Fig. 15. A rod with two light balls is attached to a torsion rod. When hung between the two large masses the rod will turn. The amount of torsion can then be related to G via Newton's expression for gravity. Henry Cavendish was one of the eccentrics in science. He rarely spoke and his only social events seemed to be the Royal Society Club (a kind of science club). Not that he spoke to there very often. He was incredibly shy of women (does it need to be mentioned that he never married?) and communicated to his servants through notes. He saw his principle heir for only a few minutes each year (who inherited 700,000 pound plus an estate worth about 8,000 pound a year. For comparison, Mr. Darcy, the rich admirer of Elisabeth Bennet in Jane Austen's "Pride and Prejudice" had an income of about 10,000 pound.) Note that the famous Cavendish Laboratory was founded with an endowment by William Cavendish, seventh Duke of Devonshire in 1874.

A. Density of the earth

Once Cavendish measured the gravitational constant, he was able to determine the density of the earth:

$$G\frac{mM}{r^2} = mg \qquad \Rightarrow \qquad M = \frac{gr^2}{G} = \frac{9.8 \times (6 \times 10^6)^2}{6.67 \times 10^{-11}} = 5.2 \times 10^{24} \ \text{kg}. \tag{256}$$

The density is then

$$\rho = \frac{M}{\frac{4}{3}\pi r^3} = \frac{5.2 \times 10^{24}}{\frac{4}{3}\pi (6 \times 10^6)^3} = 5700 \ \text{kg/m}^3 = 5.7 \ \text{kg/dm}^3. \tag{257}$$

Note that for water the density is 1 kg/dm^3 (note that 1 dm^3 is a liter). The earth is also made out of 34.1% iron (density 7.8), 17.2 % silicon (density 2.3). Also quite a bit of oxygen (28.2%), however this is not in the form of solid oxygen, but as part of compounds.

Example Giancoli 6.6 Kepler's laws for a circular orbit, Giancoli page 144.

X. WORK AND ENERGY

A. one dimension

The concept of energy is intimately related to Newton's laws. For simplicity let us consider the situation in one dimension

$$m\frac{dv}{dt} = \sum_i F_i \tag{258}$$

Since $v = dx/dt$ we can write $dt = dx/v$.

$$mv\frac{dv}{dx} = \sum_i F_i \qquad \text{or} \qquad mvdv = (\sum_i F_i)dx \tag{259}$$

We can integrate this from a point 0 to a point 1.

$$\int_0^1 mvdv = \sum_i \int_0^1 F_i dx, \tag{260}$$

or

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = \sum_i \int_0^1 F_i dx, \tag{261}$$

The expression on the right-hand side is called work:

$$\sum_i \int_0^1 F_i dx. \tag{262}$$

Work is the energy transferred to or from an object by means of a force acting on this object. Energy transferred to the object is positive work, and energy transferred from the object is negative work. This greatly simplifies if we assume that the forces are constant

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = \sum_i F_i(x_1 - x_0) \tag{263}$$

where the subscript indicates the beginning (0) and end (1) of the time that we are considering. We see that the difference in $\frac{1}{2}mv^2$, which is known as the kinetic energy, equals the sum of the forces times the distance, i.e. the work done by the forces of the distance $x_1 - x_0$.

Example: Dropping an object. We can obviously solve this problem by starting from the equations of motion, derived from $m\frac{d^2y}{dt^2} = -mg$ (choosing the positive direction away from the surface of the Earth). We obtain for the height

$$y = -\frac{1}{2}gt^2 + h. \tag{264}$$

When the object hits the ground, $y = 0$ and therefore $t = \sqrt{\frac{2h}{g}}$. The velocity is given by

$$v = -gt = -g\sqrt{\frac{2h}{g}} = -\sqrt{2gh}. \tag{265}$$

We can solve the same problem, starting from the concept of work, see Eqn. (263), which gives

$$\frac{1}{2}mv^2 = -mg(-h) = mgh \qquad \Rightarrow \qquad |v| = \sqrt{2gh}, \tag{266}$$

giving the (almost) same answer. Note that there is one point that you have to be careful about. We see that the force is negetive, because gravity is working in the negative y direction. However, the displacement is also in the negative y

direction (the object is falling down). Effectively the work done is positive. Why does it say almost the same answer? What we obtain is a value of the magnitude of the velocity, we do not obtain any information about the direction. Of course, you might say, that $|v| = \sqrt{2gh}$ has two possible solution $v = \pm\sqrt{2gh}$ and the right one is among them. This is true, but that is not so easily done when we generalize the concepts to three dimensions. In one dimension, there are only to possibilities for the direction (the positive and negative direction). With three dimensions, there is an infinite number of possibilities for the direction of the velocity.

Let us have a look again at the expression for the work that is done by gravity

$$W = mg(y_1 - y_0). \tag{267}$$

This expression only depend on the initial and final heights and is independent on whatever we did in between. This is counterintuitive. Our impression is that if we have an object, move it down and back up again, we have done work. That is true, but that is because our body is very inefficient. A better comparison would be a trampoline. We drop the object, the object bounces, and, if the trampoline is perfect, it will end up at the same height with zero velocity. When the object hits the trampoline, the trampoline is performing negative work and energy is transferred from the object to the trampoline. When the oject bounces back the trampoline is transferring the energy back again to the object, thereby performing positive work. This way, all the kinetic energy is put back in putting the object back at the same height. However, most of the time the trampoline is not perfect and some energy is lost, usually in the form of heat and the ball does not bounce back to the same height. If a human being holds the object in their hands and brings it down and up again to the same, most of the kinetic energy that the object might gain is lost. Unlike the trampoline, humans do not have a good way of transferring the kinetic energy into something useful. If we catch an object most of the kinetic energy is absorbed by our body and transformed into heat. We are even using energy when we hold our arms straight. This is because we are not really holding it straight. Our arms fall just a little bit (barely noticeable), we stop the fall, but loose the kinetic energy, and move our arm just a little bit up again, costing energy.

Example. An elevator is cab of mass $m = 500$ kg is descending with speed $v_i = 4.0$ m/s when its cable starts to slip allowing it to fall with an acceleration of $\mathbf{a} = \frac{1}{5}\mathbf{g}$.
(a) It falls for a distance of $d = 12$ m, what is the work W_g done on the cab by the gravitational force $\mathbf{F}_g$?
Solution: Note that the cab and acceleration are pointing in the same direction, gravity is the cause that the cab is accelerating. The work done

$$W_g = F_g d = mgd = 500 \times 9.8 \times 12 = 5.88 \times 10^4 \text{ J} = 59 \text{ kJ} \tag{268}$$

(b) During the fall, what is the work done by the cord on the cab?
Solution: At first, we need to find the tension force $\mathbf{F}_T$. This can be obtained from Newton's second law

$$\mathbf{F}_T + \mathbf{F}_g = m\mathbf{a} \qquad \Rightarrow \qquad F_T - mg = -\frac{1}{5}mg \qquad \Rightarrow \qquad F_T = \frac{4}{5}mg. \tag{269}$$

Care should be taken with the signs of the different forces. Gravity and the acceleration are both in the negative y direction, whereas the tension force is in the positive y direction. The work done by the tension force is

$$W_T = -\frac{4}{5}mgd = -\frac{4}{5}500 \times 9.8 \times 12 = 4.70 \times 10^4 \text{ J} = -47 \text{ kJ} \tag{270}$$

Note the negative sign. The tension force is opposite to the acceleration (there is an angle of 180°) between them).
(c) What is the total work done on the cab during the fall.
Solution: The total work done is the sum of the work done by all the forces on the cab:

$$W = W_g + W_T = -4.70 + 5.88 = 12 \text{ kJ}. \tag{271}$$

(d) What is the cab's kinetic energy and velocity at the end of the fall?
Solution: Since there is positive work done on the cab, its kinetic energy increases from its initial value $E_{k,i}$. The kinetic energy after the fall $E_{k,f}$ is therefore

$$E_{k,f} = E_{k,i} + W = \frac{1}{2}mv_i^2 + W = \frac{1}{2}500 \times 4^2 + 1.18 \times 10^4 = 1.58 \times 10^4 \text{ J} = 16 \text{ kJ}. \tag{272}$$

The velocity is then

$$v_f = \sqrt{\frac{2E_{k,f}}{m}} = \sqrt{\frac{2 \times 16,000}{500}} = 8 \text{ m/s} \tag{273}$$

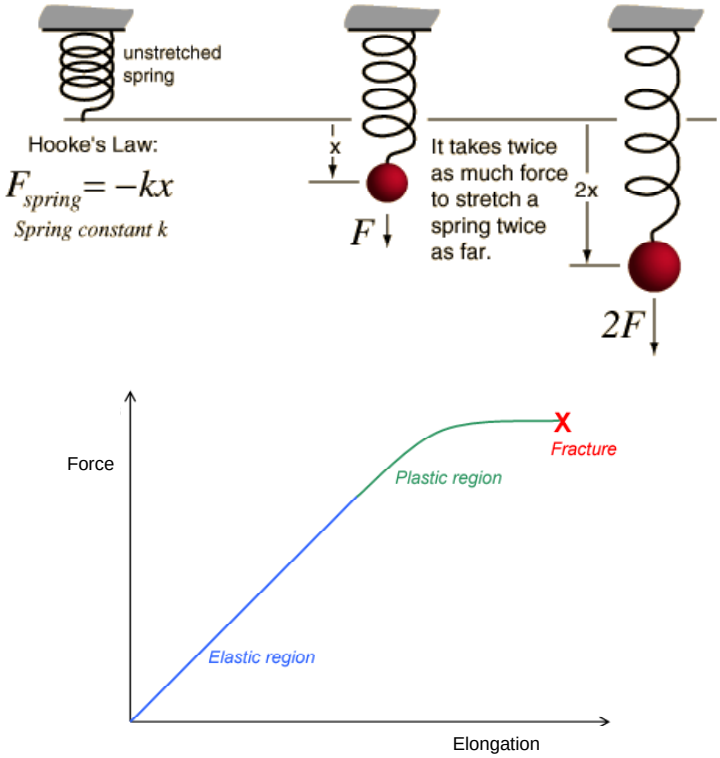


FIG. 16: Hooke's law says that the force need to displace a spring from its equilibrium position is directly proportional to its displacement. The lower half shows that this is only true if the applied force is not too strong. When the force is too strong, the materials enters the plastic region, where a materials does not return to its original length but remains deformed. When too much force is applied the material breaks.

B. The work done by a spring force.

In many of the previous problems so far, we have been dealing with constant forces that do not depend on the position. One very known example of a variable force is the spring force. Let us start from a spring in a relaxed position, see Fig. 16. A force is needed to extend the spring. If you now extend the spring twice as far, then, as you probably know from personal experience, a larger force is needed. It was demonstrated by Robert Hooke in 1660, that the force needed is twice as large: the force is directly proportional to the displacement. We can also compress the spring. Again, the force **F** is directly proportional to the displacement **d** from the equilibrium position. Note, that we are now talking about the force that the person is applying to the spring. Often, we are interested in the force exerted by the spring, which is exactly in the opposite direction. We can write

$$\mathbf{F} = -K\mathbf{d}, \tag{274}$$

which is known as Hooke's law. The constant K is known as the spring constant which is material dependent. We can also write it in a scalar form

$$F = -Kx, \tag{275}$$

The force is therefore opposite to the displacement or, in other words, the spring force likes to put the spring back in its equilibrium position. The expression for the spring force is a phenemenological expression, which works very well in many occasions. It is related to the elasticity of materials. Hooke's law starts to fail when too much force is applied. It then enters what is called the plastic region. The material will not return to its original length after the force is removed. Therefore, materials that might look very elastic (rubber and chewing gum) are in fact less elastic (from a physics/engineering perspective) than for example a piano wire, which will return to its equilibrium position many times. They are just easier to stretch. When the force is increased even more the materials will break.

We can now calculate the work done by a spring. We cannot simply multiply the force times the distance anymore since the force is a function of the displacement and we need to use Eqn. (262). The work done by the spring W_s

when compressing/extending it from its initial position x_i to its final position x_f is

$$W_s = \int_{x_i}^{x_f} F dx = \int_{x_i}^{x_f} (-Kx)dx = [-\frac{1}{2}Kx^2]_{x_i}^{x_f} = \frac{1}{2}Kx_i^2 - \frac{1}{2}Kx_f^2 \tag{276}$$

Note that the work done by an object/person pulling the sping is just the opposite $W_o = -W_s$.

Example. We are sliding an object with a mass of $m = 0.40$ kg on a frictionless table towards a spring with a speed of $v_i = 0.5$ m/s. When the object hits the spring it compresses it until it finally stops. The spring constant is $K = 750$ N/m. How much is the spring compressed, when the object stops.
Solution: One way to look at is that the difference in final and initial kinetic energies equals the work done by the spring:

$$E_{k,f} - E_{k,i} = -\frac{1}{2}mv_i^2 = W_s = \frac{1}{2}Kx_i^2 - \frac{1}{2}Kx_f^2 = -\frac{1}{2}Kd^2 \tag{277}$$

using $x_i = 0$ and $x_f = d$. The alternative way to look at it, is that the work done by the object equals the difference in kinetic energy before and after the compression

$$W_o = -W_s = \frac{1}{2}Kx_f^2 - \frac{1}{2}Kx_i^2 = \frac{1}{2}Kd^2 = E_{k,i} - E_{k,f} = \frac{1}{2}mv_i^2. \tag{278}$$

The final result is the same $\frac{1}{2}mv_i^2 = \frac{1}{2}Kd^2$, apart from the minus sign. The object feels it is doing work (its putting energy in the spring, positive work), the spring feels it is decreasing the energy of the object (negative work). Solving for d gives

$$d = v_i\sqrt{\frac{m}{K}} = 0.5\sqrt{\frac{0.40}{750}} = 1.2 \times 10^{-2} \text{ m} = 1.2 \text{ cm}. \tag{279}$$

C. Work in three dimensions

We can generalize the expression for one dimension to three dimensions, by adding a number of vector signs and inner products:

$$m\frac{d\mathbf{v}}{dt} = \sum_i \mathbf{F_i}. \tag{280}$$

Multiplying with $d\mathbf{r}$

$$m\frac{d\mathbf{v}}{dt} \cdot d\mathbf{r} = \sum_i \mathbf{F_i} \cdot d\mathbf{r} \tag{281}$$

Note the inner product. This is important, since

$$\mathbf{F}_i \cdot d\mathbf{r} = F_i \cos\varphi dr \tag{282}$$

where φ is the angle between the displacement and the force. For example, if $\varphi = 0$, the force and the displacement are in the same direction. This is like pushing some in the direction in which it is moving. This will make it go faster. If $\varphi = 180°$ on the other hand, the force and displacement are in opposite directions. This should be compared with pushing something against the direction in which it is moving. This will make it slow down. We can integrate this

$$m\int_0^1 \frac{d\mathbf{v}}{dt} \cdot d\mathbf{r} = \int_0^1 \frac{d\mathbf{r}}{dt} \cdot d\mathbf{v} = \int_0^1 \mathbf{v} \cdot d\mathbf{v} = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = \sum_i \int_0^1 \mathbf{F_i} \cdot d\mathbf{r} \tag{283}$$

Example. The force $\mathbf{F} = 3x^2\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ acts on a particle changing its kinetic energy. How much work is done if the particle moves from the coordinates (2,3) to (3,0)? Does the speed of the particle increase, decrease, or remain the same?
Solution: The work can be calculated using Eqn. (283)

$$W = \int_{(2,3)}^{(3,0)} \mathbf{F} \cdot d\mathbf{r} = \int_2^3 F_x dx + \int_3^0 F_y dy = \int_2^3 3x^2 dx + \int_3^0 4dy \tag{284}$$

$$= [x^3]_2^3 + [4x]_3^0 = (27 - 8) + 4(0 - 3) = 19 - 12 = 7 \text{ J}. \tag{285}$$

The positive sign indicates that energy is transferred to the particle by the force $\mathbf{F}$. Thus, the kinetic energy increases and so must the speed.

XI. POTENTIALS AND CONSERVATION OF ENERGY

In this section, we demonstrate that there are different types of forces. Let us make a comparison between gravitation and friction. Let us make go around in a square of size a. For gravity, we make an object go around in a square in the vertical plane. (Let us for the moment not worry about the experimental difficulties of making the object go in a square in the vertical plane) The work done in this square is

$$W = mga + 0 \times a - mga + 0 \times a = 0 \tag{286}$$

In the first stretch, positive work is done on the object by gravity and kinetic energy is gained. In the horizontal stretch, no work is done by gravity since the displacement and force are perpendicular ($\mathbf{F} \cdot \mathbf{r} = 0$). When the object moves upward again, gravity is doing negative work, and the object will loos kinetic energy. This followed by another horizontal stretch, where no work is done. The total amount of work done by gravity is zero. Let us now compare this to the work done by a constant friction force. We take a horizontal plane (in principle we could also take a vertical plane, but then we have to press the object against a surface). The total work done is

$$W = -F_{\rm fr}a - F_{\rm fr}a - F_{\rm fr}a - F_{\rm fr}a = -4F_{\rm fr}a. \tag{287}$$

The total amount of work is negative. Friction tries to slow down the object all along its path.

We see that there is a big difference between the two forces. For gravity, we find that when we return at the same position, the work done is zero. This is general and does not dependent on the particular path that we have chosen here. For friction, we find a finite (negative) value for the work done. It is easy to see that the work done depends on the path chosen (in particular, the total length of the path). Many forces in nature satisfy the criterium that the net work done when moving a object in a closed path is zero or in equation form

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0, \tag{288}$$

if $\mathbf{F}$ is a conservative force. The integral symbol $\oint$ means the integral over a closed loop. These forces are called conservative forces. A related statement is that the work done by a conservative force on a particle object moving it between two points does not depend on the path taken by the object. This can be proven as follows.

$$W_{ab,1} = \int_{\mathbf{r}_a,1}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r} \tag{289}$$

where $\int_{\mathbf{r}_a,1}^{\mathbf{r}_b}$ indicates the integral from a to b over path 1. Let us now look at the work done along path 2.

$$W_{ab,2} = \int_{\mathbf{r}_a,2}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r} = -\int_{\mathbf{r}_b,2}^{\mathbf{r}_a} \mathbf{F} \cdot d\mathbf{r} = -W_{ba,2}, \tag{290}$$

where we have used the fact that changing the integration constants gives a minus sign, this is easily seen from

$$\int_{x_a}^{x_b} g(x)dx = G(x_b) - G(x_a) = -[G(x_a) - G(x_b)] = -\int_{x_b}^{x_a} g(x)dx. \tag{291}$$

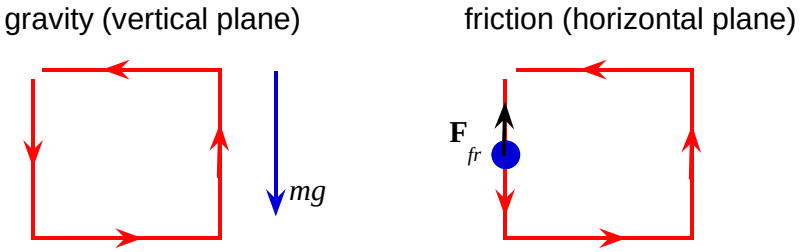


FIG. 17: An object is moved around in a square in the vertical plane for gravity and in a horizontal plane for friction.

where the function G satisfies $G(x) = \frac{dg}{dx}$. However, combining the work $W_{ab,1}$ and $W_{ba,2}$ gives a complete loop from a via b back to a. Since we had taken $\mathbf{F}$ as a conservative force, we have

$$W_{ab,1} + W_{ba,2} = 0 \qquad \Rightarrow \qquad W_{ab,1} = -W_{ba,2} = W_{ab,2}, \tag{292}$$

demonstrating indeed that the work done on an object by a conservative force between a and b does not depend on the path taken. Since it does not depend on the path, it should only depend on the begin and end points. We can therefore write

$$W_{ab} = -[U(\mathbf{r}_b) - U(\mathbf{r}_a)], \tag{293}$$

where the minus sign is the conventional choice for the sign of the function V. The function V is known as the potential energy. We can also write this as an integral

$$\int_{\mathbf{r}_a}^{\mathbf{r}_b} dU(\mathbf{r}) = U(\mathbf{r}_b) - U(\mathbf{r}_a). \tag{294}$$

The following is a bit more complicated, so let us first do it in one dimension. The work from a point a to a point b in one dimension can be written as

$$W_{ab} = \int_{x_a}^{x_b} F(x)dx = -[U(x_b) - U(x_b)] = -\int_{x_a}^{x_b} dU(x). \tag{295}$$

From this it implies that

$$F(x)dx = -dU(x) \qquad \Rightarrow \qquad F(x) = -\frac{dU}{dx}. \tag{296}$$

A. Gravitational potential energy

The difference in potential energy can be determined from the work done in moving an object from a height y_0 to y_1. This gives

$$\Delta U = U(y_1) - U(y_0) = -\int_{y_0}^{y_1} (-mg)dy = \int_{y_0}^{y_1} mgdy = [mgh]_{y_0}^{y_1} = mg(y_1 - y_0) = mg\Delta y. \tag{297}$$

Pay attention to the minus signs: one minus sign from the definition of the potential energy and another minus sign because the gravitational force is in the negative y direction. Making two mistakes to arrive at the right result is not going to cut it.... Another way of obtaining the same result is to look at the relation

$$F = mg = -\frac{dU}{dy} \qquad \Rightarrow \qquad U = \int mgdy = mgy + \text{constant}. \tag{298}$$

Note that we obtain an integration constant here. When we are considering the work done, we are only interested in the difference in potential energy

$$W_{01} = -[U(y_1) - U(y_0)] = -mgy_1 + \text{constant} + mgy_0 + \text{constant} = -mg\Delta(y_1 - y_0) = -mg\Delta y. \tag{299}$$

It is up to our discretion to choose the integration constant. A convenient choice is to choose the potential energy zero for $y = 0$, implying constant $= 0$. This is not necessary. You might be doing experiments at the second floor of a building and it might be more convenient to choose the second floor level zero. For constant $= 0$, the potential energy is

$$U = mgy. \tag{300}$$

Here we can also start to understand the origin of the minus sign and the name "potential energy". The gravitational potential energy is the energy an object can gain by the work done by the gravity. Once we start putting that energy in the object, it looses its potential to gain more energy. At the Earth's surface the potential energy to be gained is zero (unless we dig a hole underneath our object. Note that, although the potential energy at the Earth's surface is zero, this does not mean we cannot gain any more energy. $\Delta U = 0 - mgy = -mgy > 0$ if $y < 0$. Therefore, the Earth's surface is still at a higher potential energy compared to a point below the Earth's surface. This again shows that it is only the relative and not the absolute potential energy which is relevant.)

B. Elastic potential energy

Let us consider again a block at the end of a spring. The potential energy can be derived in a similar fashion as the gravitational potential energy:

$$\Delta U = -\int_{x_0}^{x_1} (-Kx)dx = K\int_{x_0}^{x_1} xdx = K[\frac{1}{2}x^2]_{x_0}^{x_1} = \frac{1}{2}Kx_1^2 - \frac{1}{2}Kx_0^2. \tag{301}$$

Again, we have the two signs cancelling each other. We can also look at the relation between the force and the potential energy:

$$F = -Kx = -\frac{dU}{dx} \qquad \Rightarrow \qquad U = \int Kxdx = \frac{1}{2}Kx^2 + \text{constant}. \tag{302}$$

Again we have an integration constant which we can choose for our convenience. Let us take constant = 0, giving

$$U = \frac{1}{2}Kx^2 \tag{303}$$

C. Conservation of energy

So far we have been dealing with work. When considering an object, we found that the change in kinetic energy was related to the amount of work done on the system

$$\Delta E_k = W. \tag{304}$$

In the case of a conservative force, we can write relate the work to a change in potential energy $W = -\Delta U$. Note that this is not possible with nonconservative forces, such as friction. For conservative forces, we have

$$\Delta E_k = -\Delta U \qquad \Rightarrow \qquad E_{k,2} - E_{k,1} = U_1 - U_2, \tag{305}$$

where $E_{k,1}$ and $E_{k,2}$ are the kinetic energies at positions 1 and 2, respectively. We then arrive at the important result

$$\Delta E_k = -\Delta U \qquad \Rightarrow \qquad E_{k,2} + U_2 = E_{k,1} + U_1. \tag{306}$$

This is known as the law of conservation of (mechanical) energy. It shows that the sum of potential and kinetic energy at postition 2 equals that at position 1. Although potential energy and kinetic energy are not conserved separately, only their sum remains constant. This is only true in the absence of nonconservative forces.

Example Using conservation of energy, we can derive the velocity of an object being dropped from a certain height h (we have done this already several times).

$$E_{k,2} + U_2 = E_{k,1} + U_1 \qquad \Rightarrow \qquad \frac{1}{2}mv_2^2 + 0 = 0 + mgh \qquad \Rightarrow \qquad v_2 = \sqrt{2gh}. \tag{307}$$

Note that this is essentially equivalent to the Example *Dropping an object* in Section X A. It is certainly a faster way to obtain it than from the equations of motion in Section V B. However, although simpler, we loose information about the time dependence of the motion.

Example Harmonic oscillation. An interesting example of conservation of energy is the oscillation of a spring. Newton's law is

$$-Kx = m\frac{d^2x}{dt^2}. \tag{308}$$

In fact, we looked at the solutions in Section V D. The solutions of the equation can be written as $x(t) = C\sin\omega t$ with C the amplitude of the oscillation. Differentiating gives $v(t) = \frac{dx(t)}{dt} = C\omega\cos\omega t$ and $a(t) = -C\omega^2\sin\omega t$. Inserting this in the equation of motion gives

$$-KC\sin\omega t = -m\omega^2 C\sin\omega t. \tag{309}$$

The left- and right-hand sides are equal is the oscillation frequency is given by $\omega = \sqrt{\frac{K}{m}}$. The potential energy for a spring is given by $\frac{1}{2}Kx^2 = \frac{1}{2}m\omega^2x^2$. The sum of the kinetic and potential energies is

$$E_k + U = \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2x^2 = \frac{1}{2}m(\omega C\cos\omega t)^2 + \frac{1}{2}m\omega^2(C\sin\omega t)^2 = \frac{1}{2}m\omega^2C^2(\cos^2\omega t + \sin^2\omega t) = \frac{1}{2}m\omega^2C^2. \quad (310)$$

Since the far right-hand side only contains constants, we can directly see that the sum of kinetic and potential energies remains constant during an oscillation.

Example. A bungee jumper weighing 61.0 kg jumps from a bridge $H = 45.0$ m above a river. The elastic bungee cord has a length $L = 25.0$ m. Assume that the cord obeys Hooke's law with a spring constant of $K = 160$ N/m. If the jumper stops before reaching the water, what is the height h or the jumper's feet above the water.
Solution: We have three energies involved in this problem: the kinetic energy and the potentials of two conservative forces, gravity and elastic energy. Before the jump there is only potential energy (no velocity and the bungee cord is not stretched)

$$E_0 = mgH. \quad (311)$$

After the jump, the bungee cord is stretched by a distance $d = H - L - h$ (the total height minus the length of the cord and the distance above the water). Again the kinetic energy is zero. The energy is therefore given by

$$E_1 = mgh + \frac{1}{2}Kd^2. \quad (312)$$

Since energy is conserved

$$mgH = mgh + \frac{1}{2}Kd^2. \quad (313)$$

Since $h = H - L - d$, we can rewrite this as

$$mgH = mg(H - L - d) + \frac{1}{2}Kd^2 \qquad \Rightarrow \qquad mgL + mgd = \frac{1}{2}Kd^2 \quad \Rightarrow \quad d^2 - \frac{2mg}{K}d - \frac{2mg}{K}L = 0. \quad (314)$$

We therefore have a quadratic equation

$$d^2 - ad - aL = 0 \qquad \Rightarrow \qquad d = \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 + 4aL} \quad (315)$$

with $a = \frac{2mg}{K} = \frac{2\times61.0\times9.8}{160} = 7.47$ m giving

$$d = \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 + 4aL} = \frac{7.47}{2} \pm \frac{1}{2}\sqrt{(7.47)^2 + 4\times7.47\times25} = 3.74 \pm 14.17 = \begin{cases} -10.43 \\ 17.909 \end{cases} \quad (316)$$

The negative number corrsponds to a solution when the cord is compressed. This is not the solution we are looking for, so we take $d = 17.9$ m. The total distance is therefore $L + d = 17.9 + 25 = 42.9$ m, which means 2.1 m above the water.

D. Potential energy in three dimensions

For three dimensions, we can write

$$W_{ab} = \int_a^b \mathbf{F}(\mathbf{r})d\mathbf{r} = -[V(\mathbf{r}_b) - V(\mathbf{r}_a)] = -\int_a^b dV(\mathbf{r}). \quad (317)$$

As above, we can now write

$$F(\mathbf{r})d\mathbf{r} = -dV(\mathbf{r}). \quad (318)$$

We can split the left-hand side into components

$$F(\mathbf{r})d\mathbf{r} = F_x(x)dx + F_y(x)dy + F_z(x)dz. \quad (319)$$

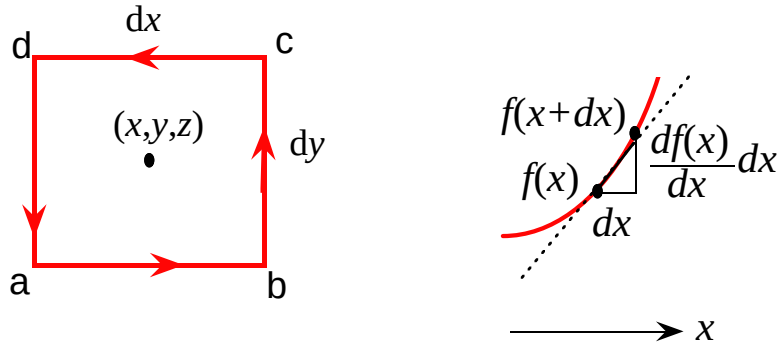


FIG. 18: The loop taken in calculating the work done by a conservative force on a particle when going from $a \to b \to c \to d \to a$.

Combining this with Eqn. (318), gives

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad F_y = -\frac{\partial V}{\partial z}. \tag{320}$$

You will often find this written differently. In vector notation, we can write

$$\mathbf{F} = -\frac{\partial U}{\partial x}\hat{\mathbf{i}} - \frac{\partial U}{\partial y}\hat{\mathbf{j}} - \frac{\partial U}{\partial z}\hat{\mathbf{k}}. \tag{321}$$

So far, we have not seen treated partial differentiation. Often function depend on more than one variable, such as $f(x, y, z)$. The partial derivative $\frac{\partial}{\partial x}$ means that we take the derivative with respect to x while maintaining the other variables constant. For example, let us consider the function $f(x, y, z) = x^2 \cos y$. The partial derivatives are given by

$$\frac{\partial f}{\partial x} = 2x \cos y \qquad , \qquad \frac{\partial f}{\partial y} = -x^2 \sin y \qquad , \qquad \frac{\partial f}{\partial z} = 0. \tag{322}$$

We can rewrite Eqn. (321) in a shorter form by introducing the nabla operator

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}, \tag{323}$$

We can then write it in shorthand

$$\mathbf{F} = -\nabla U(\mathbf{r}). \tag{324}$$

This expression has the additional advantage that it is also true in different coordinate systems, such as spherical and cylindrical coordinates. Obviously, the definition of ∇ will depend on the coordinate system. However, the relationship between a conservative and the derivative of a scalar function with respect to the different coordinates (x, y, and z, or r, θ, and φ for spherical coordinates) remains valid.

E. Rotation

The theory of conservative forces is generally presented in a somewhat more complicated fashion. Let us consider an infinitesimally small square loop. For a conservative force, the work done going around that loop is zero. Let us take the position $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ at the center of the square. The sides of the square have lengths dx and dy. The force is pointing in a certain direction $\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$. Let us denote the corners by a, b, c, and d, see Fig. 18.

Let us first consider the section ab going in the x direction. The position of a is $(x-\frac{dx}{2})\hat{\mathbf{i}}+(y-\frac{dy}{2})\hat{\mathbf{j}}+z\hat{\mathbf{k}}$. We then go to point b at $(x+\frac{dx}{2})\hat{\mathbf{i}}+(y-\frac{dy}{2})\hat{\mathbf{j}}+z\hat{\mathbf{k}}$. Note that z does not change, the x-position is on average equal to x, and the y-position is $y-\frac{dy}{2}$. The work done is now

$$\int_a^b \mathbf{F}\cdot d\mathbf{r} = F_x(x, y-\frac{dy}{2}, z)dx. \tag{325}$$

We want to approximate this with a Taylor expansion, which is an important approximation in many areas of physics. This starts from the idea that a function can be linearized over a small distance, for example for a function $f(x)$

$$f(x+dx) \cong f(x) + \frac{df(x)}{dx}dx. \tag{326}$$

In Fig. 18, we see what this entails. The function $f(x+dx)$ is assumed to be almost equal to $f(x)$ the difference is described by $\frac{df(x)}{dx}dx$, where $\frac{df(x)}{dx}$ is the tangent of angle of the tangent with the function. Multiplying this by the change in distance in the x direction dx gives the change in y direction. The principle is the same in three dimensions. Some important examples of Taylor expansions are those of elementary functions around $x=0$. In the case that x is small:

$$\sin x = \sin(x=0) + \left.\frac{d\sin x}{dx}\right|_{x=0} x = 0 + \cos(x=0) = x \tag{327}$$

$$e^x = e^0 + \left.\frac{d(e^x)}{dx}\right|_{x=0} x = 1 + e^0 x = 1 + x \tag{328}$$

$$\tag{329}$$

Obviously, in calculating the work, we know have to keep track of three coordinates and we have to take partial derivatives since the function depends on three variables.

$$F_x(x, y-\frac{dy}{2}, z)dx = \left(F_x(x,y,z) - \frac{dy}{2}\frac{\partial F_x}{\partial y}\right)dx \tag{330}$$

We can do the same exercise for the opposite side from c to d:

$$\int_c^d \mathbf{F}\cdot d\mathbf{r} = -F_x(x, y+\frac{dy}{2}, z)dx = -\left(F_x(x,y,z) + \frac{dy}{2}\frac{\partial F_x}{\partial y}\right)dx. \tag{331}$$

Adding the two together gives

$$\int_a^b \mathbf{F}\cdot d\mathbf{r} + \int_c^d \mathbf{F}\cdot d\mathbf{r} = -\frac{\partial F_x}{\partial y}dxdy \tag{332}$$

Analogously, we can write when we are going in the y direction

$$\int_b^c \mathbf{F}\cdot d\mathbf{r} = F_y(x+\frac{dx}{2}, y, z)dy = \left(F_y(x,y,z) + \frac{dx}{2}\frac{\partial F_y}{\partial x}\right)dy. \tag{333}$$

and

$$\int_d^a \mathbf{F}\cdot d\mathbf{r} = -F_y(x-\frac{dx}{2}, y, z)dy = -\left(F_y(x,y,z) - \frac{dx}{2}\frac{\partial F_y}{\partial x}\right)dy. \tag{334}$$

Adding the two together gives

$$\int_b^c \mathbf{F}\cdot d\mathbf{r} + \int_d^a \mathbf{F}\cdot d\mathbf{r} = \frac{\partial F_y}{\partial x}dxdy \tag{335}$$

Adding all the different sides together gives

$$\oint \mathbf{F}\cdot d\mathbf{r} = (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})dxdy. \tag{336}$$

Now we want that to be zero. The way to do this is to relate the forces to a potential U,

$$\mathbf{F} = -\frac{\partial U}{\partial x}\hat{\mathbf{i}} - \frac{\partial U}{\partial y}\hat{\mathbf{j}} - \frac{\partial U}{\partial z}\hat{\mathbf{k}}. \tag{337}$$

Let us insert this into the expression for the work done over the little square

$$\oint \mathbf{F}\cdot d\mathbf{r} = \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right] dxdy = \left[\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right)\right] dxdy = \left(\frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 U}{\partial y \partial x}\right) dxdy = 0, \tag{338}$$

since the two partial derivatives are zero. To make life even more difficult. Note that we are only consider a very particular loop. The loop is defined in the xy plane, therefore the surface normal of the loop is in the z-direction. I fact we can write the expression of the partial derivatives of the force as the z component of

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = (\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z})\hat{\mathbf{i}} + (\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x})\hat{\mathbf{k}} + (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})\hat{\mathbf{k}}. \tag{339}$$

Using this, we can rewrite the result

$$\oint \mathbf{F}\cdot d\mathbf{r} = \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right] dxdy = (\nabla \times \mathbf{F})_z dxdy.$$

Now the surface normal of the square is $\mathbf{e}_n = \mathbf{k}$. We can then write

$$\oint \mathbf{F}\cdot d\mathbf{r} = (\nabla \times \mathbf{F})\cdot \mathbf{e}_n dxdy. \tag{341}$$

We can define a vector $d\mathbf{a} = \mathbf{e}_n dxdy$, i.e. a vector whose direction is given by the surface normal and whose size is determined by its area. We can generalize the expression for an arbitrary infinitesimally small loop, giving

$$\oint \mathbf{F}\cdot d\mathbf{r} = \nabla \times \mathbf{F}\cdot d\mathbf{a}. \tag{342}$$

This result can be generalized to arbitrary loops.

F. Gravitation

Let us start with the expression

$$\mathbf{F} = -\nabla U(\mathbf{r}). \tag{343}$$

The potential follows directly from the work done on an object in going from a point 1 to a point 2

$$W = \int_1^2 \mathbf{F}\cdot d\mathbf{l} = -\int_1^2 \frac{GmM}{r^2}\hat{\mathbf{r}}\cdot d\mathbf{l}. \tag{344}$$

Using the fact that $\hat{\mathbf{r}}\cdot d\mathbf{l} = dr$, we can write

$$W = -\int_1^2 \frac{GmM}{r^2} dr = \left[\frac{GmM}{r}\right]_1^2 = \frac{GmM}{r_2} - \frac{GmM}{r_1}. \tag{345}$$

Now the work done is opposite to the change in potential energy. For example, we need to do a positive work to decrease the potential energy.

$$W = -(U_2 - U_1) \qquad \Rightarrow \qquad U = -\frac{GmM}{r}. \tag{346}$$

Let us now see if we can deduce the force from the potential energy. Since we need to take the partial derivatives of $1/r$ with respect to x, y, and z, we need to know

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2x = -\frac{x}{r}. \tag{347}$$

TABLE I: Fundamental forces in nature

Name	Acts between	Strength	Mediator
Strong (nuclear)	Quark and/or gluons	1	Gluons
Electromagnetic	All charged particles	$\cong 10^{-2}$	Photon
Weak (nuclear)	Quarks and leptons	$\cong 10^{-13}$	Vector bosons
Gravitational	All particles	$\cong 10^{-38}$	Gravitons (?)

We then find for the force

$$\mathbf{F} = -\frac{\partial U}{\partial x}\hat{\mathbf{i}} - \frac{\partial U}{\partial y}\hat{\mathbf{j}} - \frac{\partial U}{\partial z}\hat{\mathbf{k}} = -GmM\left(\frac{x}{r^3}\hat{\mathbf{i}} + \frac{y}{r^3}\hat{\mathbf{j}} + \frac{z}{r^3}\hat{\mathbf{k}}\right) = -\frac{GmM\mathbf{r}}{r^3} = -\frac{GmM}{r^2}\hat{\mathbf{r}}. \tag{348}$$

The Cartesian coordinate system is not the most obvious choice for the gravitational field of the earth. A more obvious choice would be a spherical coordinate system. Many books give the nabla operator in spherical coordinates:

$$\nabla = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\hat{\varphi}. \tag{349}$$

Our potential for the gravity fortunately only depends on r, and we have

$$\mathbf{F} = -\nabla U = \frac{\partial}{\partial r}\frac{GmM}{r}\hat{\mathbf{r}} = -\frac{GmM}{r^2}\hat{\mathbf{r}}, \tag{350}$$

which reproduces the above result.

Example Giancoli 8.8
Example Giancoli 8.10

XII. THE FUNDAMENTAL FORCES

Physics has been enormously successful in establishing the bricks and mortar of nature. In nature, we know four fundamental forces

- Strong force
- Electromagnetic (EM) force
- Weak force
- Gravity

During most of the course, we shall be dealing with only two of these four forces: the gravitational and the electromagnetic forces. In this introductory part, we will get a bit of a flavor of all these interactions and also get a feeling why we are not going to treat the strong and weak forces. Hopefully, you will understand at the end why the strong force, although being about 10^{38} times stronger that gravity, is not part of your daily experience. Most of you know are familiar with gravity and electromagnetic forces. You might say that you know more forces. Most people are familiar with gravity which pulls you towards the center of the earth. At the same time the earth is pushing at you. However, this is not a new force. The atoms that make up the earth repel the atoms that make up your body through electromagnetic interactions.

A. Gravitational and electric forces

How do particles interact? In the remainder of the course, we will treat the forces as continuous and infinitely divisible. However, more advanced theories show that physics is in fact discrete and made up of indivisible units. What does this mean? Consider a TV. For most practical purposes this looks continuous both in time and space. Let us look at in space. We can divide the screen into four parts (that is more easily done in our minds than in reality)

and it would still look continuous. However, we know that this cannot be done forever, because our TV screen is made up of the order of a million red, green and blue dots. Also in time it is not continuous, the picture only changes about 50-60 times per second. This is fast enough in general for the cone cells that allow us to see color. However, if we look from the corner of our eyes we can often see the TV flashing. This is because the rod cells (which are relatively more present in the corner of our eyes) that allow us to see in the dark (though only in black and white) have a faster reaction time. The successful theories from the eighteenth and nineteenth century, such as classical mechanics, thermodynamics, and electricity and magnetism. Of course, already early on, it was expected that there were indivisible units. Obviously, ancient Greeks in the late fifth century B.C., such as Democritus and Leucippus, dabbled with the ideas. The word atom is derived from the Greek *átomos* meaning indivisible. However, the ideas were rejected by powerful Greeks such as Plato and Artistotle. First of all, they thought the ideas were pure speculation devoid of any experimental evidence. The other important objection was the mechanistic and deterministic nature of such a theory. How can such a beautiful world be created by indivisible particles just crashing into each other, according to Plato. A thought that still has many adherents. Also Newton was not adverse to the idea of atoms: "It seems probably to me that God in the beginning form'd matter in solid, massy, hard, inpenetrable, movable particles."

During the nineteenth century, atomistic ideas were starting to get hold again. A chemical reaction, such as $2H_2+O_2 \rightarrow H_2O$ seems to imply indivisible quantities. It not 6.43 H+3.89 O$\rightarrow$ $H_{6.43}O_{3.89}$. Probably, the first theory to be dsecribed from a discrete point of view was thermodynamics. This met at first with a lot of resistance. The laws of thermodynamics were developed from the late eigteenth century. They describe the relationships between pressure, volume, heat, tempeerature etc. Of course, in those days, of great technological importance, with the advent of steam engines. Note that these laws are enormously powerful and still valid. So when, after some pioneering thoughts by other, Ludwig Boltzmann and James Clerk Maxwell started formulating thermodynamics in terms of indivisible particles (known now as statistical thermodynamics, they were not met with great enthusiasm). Consider, you have a beautiful theory that works extremely well, why on earth would you consider, for example, gases to be composed of indivisible particles. Although, everybody knows nowadays that atoms exist, that is not at all that obvious. Let us look at it from the point of view from a nineteenth century scientist: Take a liquid. Show me that it consists of atoms. I can just keep dividing it as many times I like. A liquid is continuous for almost any practical purpose. The same is true for solids. So a gas is a continuous medium that I can divide in as many pieces as I like. Gas is a real thing, you can feel the wind blowing, it moves the trees. You can see the pressure exerted by the gas on the balloon. Considering that a gas is made up of indivisible entities is maybe an interesting "Gedanken experiment" (thought experiment), but not at all based on any observable fact. Why would you want to describe pressure as indivisible objects colliding against the wall? How big are these indvisble things then. And it is true, neither Maxwell nor Boltzmann had any answer to that question. They went ahead anyway and showed that describing pressure, heat, etc. in terms of indivisible particles in fact leads to the same results as the conventional thermodynamics. However,

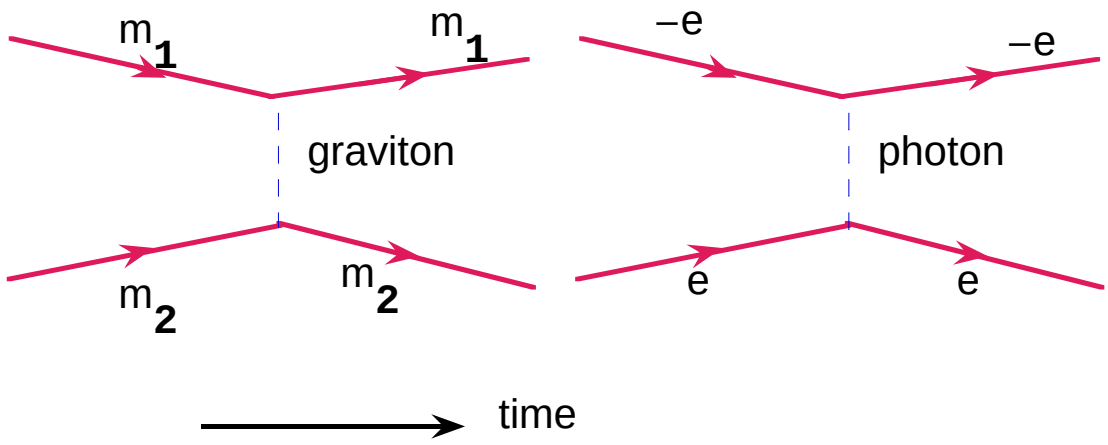


FIG. 19: Schematic pictures of the interaction between particles by exchange of other particles. The left side shows the interaction between to masses m_1 and m_2 through the exchange of a graviton. The right side shows the interaction the Coulomb interaction between a proton (e) and an electron ($-e$) through the exchange of a photon. The horizontal axis shows the time. Before and after the interaction nothing happens to the particles, but something has changed after the interaction. Compare this to two skaters gliding in straight lines. One throws something to the other skater which is heavy enough to change the skater's direction. However, when the other skater catches the item, his direction will also be changed. After this interaction the skater keep gliding in their new direction, until another interaction takes place.

it also led to new insights, such as the dependence of pressure as a function of the distance to the earth's surface. It also related a complicated concept such as entropy to probability. Many of you may have heard the second law of thermodynamics: "the entropy of an isolated system not at equilibrium will tend to increase over time, approaching a maximum value." Statistical thermodynamics relates entropy to probability. For example, when throwing coins on the floor, the probability of finding all heads is small (low entropy) compared to the probability of finding 50% heads and 50% tails (high entropy). Unfortunately, this particular law has been abused by evolutionists who claim that this fundamental physics law shows that life was created because physics shows that entropy increases. However, although this might convince the gullible, this is not exactly what the law says. Note the word isolated. Of course, I can get all heads, I just turn around the tails into heads. "But that's cheating!" you might say. No, I am just putting energy into the system, i.e. the system of coins is no longer an isolated system. This is exactly what happens with life. You have to keep putting energy in it. We all know what happens to life if you stop eating.

Anyway, we digress. Not all theories were as lucky as thermodynamics. Classical mechanics was shown to fail and a new theory was introduced at the beginning of the twentieth century: Quantum mechanics. In this theory even energy is quantized. This means, for example, that particles cannot take any velocity that they like. There are forbidden velocities! Note that the name itself reflects the discrete nature of the theory. This does not mean that classical mechanics is a useless theory. Classical mechanics fails when going to dimensions comparable to the size of atoms. For many purposes the use of classical mechanics is by far preferable over quantum mechanics. You do not want to build a bridge or send a rocket to the Moon using quantum mechanics. Another revolution occured in physics in the early twentieth century, when Albert Einstein developed the theory of relativity. This demonstrated another limitation of classical mechanics, namely when objects are travelling close to the speed of light. However, in some sense this still fits the "old-fashioned" type of theories in that it is continuous and infinitely divisible. Quantum mechanics and relativity were combined in the 1920's by the Englishman Paul Dirac.

Unfortunately, the theories involving the discrete nature of interactions are often complicated, not only from a mathematical point of view, but quite often (also for physicists) from an conceptual point of view. We will now only scratch the surface of some of these theories. The idea is to offer an alternative way to think about interactions, before we start treating them in a more conventional way. In quantum field theory (not really important what that is), the interaction between particles are described by exchange of particles. To communicate with each other particles continuously throw particles at each other. You can imagine this as, for example, two skaters interacting with each other by throwing balls between them. However, it is easy two imagine a repulsive force with such a picture, it is more difficult to imagine an attractive force like this. The skaters can attract each other by pulling the ball out of each others hand. This analogy is not entirely satisfactory, because forces can interact over large distances. You might say that you do not experience it that you are being constantly bombarded by particles such as photons from light and gravitons from gravity (although, it should be noted that the existence of gravitons has not been experimentally established). This is because so many particles are exchanged per second that we have no idea that this is going. More or less, in the same fasion as you do not notice that the television is changing only 50-60 times per second. We experience it as a continuous force or field. Later on we will forget about the particle exchange picture (in a large part because this theory, known as quantum field theory, can dazzle even the most seasoned theoretical physicist) in favor of the more standard models. However, let us continue for a little while and see where it brings us.

However, can we learn something from the picture of exchanging particles? Let us consider the forces between electric charges, known as the Coulomb force. How does it depend on the charge? Suppose we have to charges Q_1 and Q_2. Does the force depend on the total charge $Q_1 + Q_2$, or the difference $Q_1 - Q_2$ or Q_1Q_2, $(Q_1Q_2)^2$, etc.. Most of you will know the answer. Suppose the charges can be subdivided into discrete units (this is of course possible since the total charge is made up of unit charges e, the charge of a the proton and the electron). The charges interact with each other through the exchange of photons, as is shown schematically in Fig. 19. In that case, there are $N_1(= Q_1/e)$ charges in Q_1 and $N_2(= Q_2/e)$ in Q_2. A charge in Q_1 has not N_2 charges to interact with in Q_2. Since the are N_1 charges in Q_1, we can make $N_1 \times N_2$ pairs that interact with each other. Therefore, the force is proportional to N_1N_2 or, since we can always multiply it with a constant, $eN_1eN_2 = Q_1Q_2$.

How does the force depend on the distance between the particles r. If you increase the distance from a charge, the "particle" density becomes smaller, since the is a larger surface to be covered. Since the surface area is related to $4\pi r^2$, where r is the distance to the charge, the force decreases by $1/r^2$. The functional dependence is therefore given by

$$F \sim \frac{Q_1Q_2}{r^2}. \tag{351}$$

We now only need to add a constant, which is generally a fundamental constant and we end up with

$$F = f\frac{Q_1Q_2}{r^2}. \tag{352}$$

In the same way, we can write for the gravitational force

$$F = G\frac{M_1 M_2}{r^2}, \tag{353}$$

where M_1 and M_2 are the masses of the particles.

Of course, we cannot find this way the constants f and G. Their values are fundamental constants. Let us consider the relative strengths of the gravitational and Coulomb force

$$\text{EM} : \text{weak} : \text{gravity} = 1 : 10^{-36}. \tag{354}$$

We see that gravity is many order of magnitude weaker than the Coulomb force. This is not entirely a surprise because you might know already from experience that a magnet (EM force) can lift up a magnetic object even though the whole earth is pulling on it with its gravitational force. To understand why we are most aware of gravity, let us compare the gravitational and EM forces. We know that gravity works on mass and that EM forces work on charges. However, there is only one type of mass, but there are positive and negative charges. Therefore, an object always has a mass but its charge can be zero by having an equal amount of positive and negative charge, see Fig. 20. It is possible to charge an objective by having an imbalance between positive and negative charges. This is known as static electricity and nature does not really like this situation and solves this by a an electric current, a spark, lightning, or some other means to remove this charge imbalance.

B. Strong force

However, if you know something about atoms, then you directly realize that there must be even stronger forces. An atom has a nucleus made of protons and neutrons with a cloud of electrons around it. However, the protons in the nucleus repel each other. Since the radius of the nucleus is of the order of 10^{-15} m this repulsion must be very strong. However, the nucleus does not fall apart. The nucleus is kept together by the stong force.

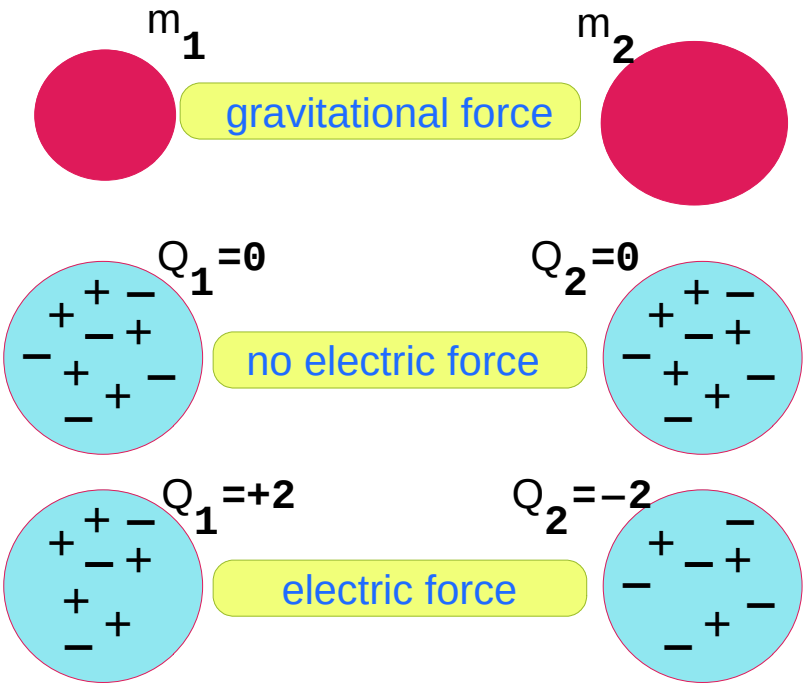


FIG. 20: The top figure shows two masses exerting a gravitational force on each other. No there is only one type of mass (always positive). And two objects with a finite mass will always exert a gravitational force on each. The situation is different for electric charge. Charge can be both positive and negative and the positive and negative charges can cancel each other making the object neutral. Neutral objects do no feel or exert electric forces, see the central figure. However, when both objects are charged there is a net electric force between the two.

The strong force works on what is known as color. We can distinguish three colors: red, green, and blue. Note that the colors have nothing to do with real colors. They are just a way to denote the different characters. It is just a choice to label them this way, they might have been called something entirely different such as Huey, Dewey, and Louie. The labels are abstract to us because we have absolutely no physical experience with the property color. We know what mass is, because we experience it every day. We have some idea about charge, which we can experience through say static electricity. But we will never experience color. For the strong force particles interact through gluons (indeed, from the word glue, helping to make everything stick together). Now something happens in the interaction. The colors are exchanged through the gluon. In an interaction, between a red and a blue particle, the red particle becomes blue and the blue particle becomes red, see Fig. 21. Particles of the same color repel each other, whereas particles of opposite color attract each other, much in the same way as electric charge, except we now have three kinds as opposed to two.

Again, nature prefers to make things neutral in the same way as a positive or negatively charge objects attract opposite charges to compensate the excess charge. However, now we need all three colors to make a neutral object (remember, the combined red, green, and blue spots on your television give a neutral white color), see Fig. 21.

Of course, you might say, what are these particles the have this red, green, and blue character. Not something that you know from daily life. Color is a property of quarks. Quarks are the tiny particles that make up the protons and the neutrons. Three quarks of different color therefore combine to make one color neutral object. Well-known particles that consist of three quarks are the proton and the neutron. Generally, we are mainly dealing with protons, neutrons, and electrons. Electrons are not made out of quarks, but are elementary particles by themselves and have no color. Therefore, in everyday life, we are mainly dealing with color neutral particles. Quarks are very strongly bound together.

Let us take a little detour in the world of quarks. There are six different flavors of quarks: down, up, strange, charm, top, bottom. Not your standard icecream flavors. If we want to have a color neutral particle we need three quarks. A color-neutral particle made up of three quarks is called a baryon. Out of 6 quarks, we should be able to make $6^3 = 216$ particles. The quarks most relevant to us in everyday life are the up, down, and strange quark. Let us restrict ourselves to those three as did Murray Gell-Man who postulated in 1964 (an independently of him George

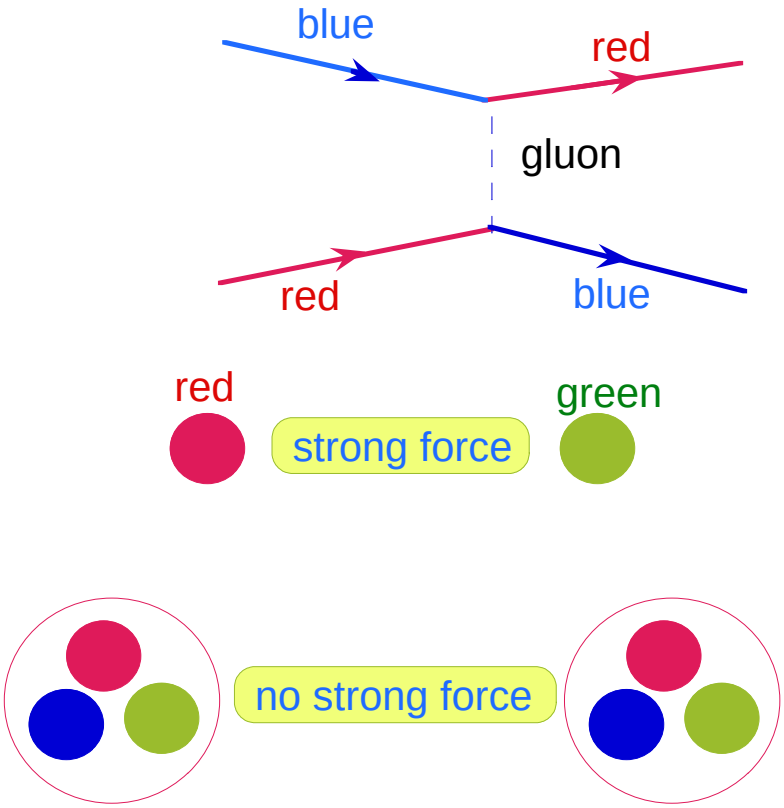


FIG. 21: Quarks interact though the strong force. The mediator particle is the gluon. Note that after the interaction the colors of the quarks are interchanged. Individual quarks interact through the strong force, such as in the red and green quark in the center figure. To make a neutral ("white") particle, three quarks of different color are combined.

Zweig) the quark model. He also coined the name quarks from James Joyce's book Finnegans Wake ("Three quarks for Muster Mark!"). This leaves us with $3^3 = 27$ possibilities. Other consideration, which are beyond the scope of these notes, allows you to divide these 27 into four groups with 1, 8, 8, and 10 particles. The ones lowest in energy (and hence the ones that you are most likely to encounter in real life) form a group of 8 particles (so if you ever hear high-energy physicsts talking of the "eight-fold" way, that is what they mean). The particles that we are most familiar with consist of the up and down quark. the protons. The up and down quark have fractional charges, $\frac{2}{3}$ for the up and and $-\frac{1}{3}$ for the down. The proton consists of two up quarks and and one down quark, giving a total charge of $\frac{2}{3}e + \frac{2}{3}e - \frac{1}{3}e = e$; the neutron consists of one up quark and two down quarks giving zero total charge, $\frac{2}{3}e - \frac{1}{3}e - \frac{1}{3}e = 0$, see Fig. 22.

High-energy physicists study the behavior of the many particle that can be made by combining the quarks. There are about 120 stable particles, known as baryons, made up of three quarks, like the proton and neutron. This can become quite confusing or, in the words of the famous physicist Enrico Fermi: "If I could remember the names of all these particles, I'd be a botanist." In order to detect quarks, we need to collide particles with extremely high velocities (very close to the speed of light) onto each other. This is done for example at Fermi National Laboratory in Batavia, Illinois, and at CERN in Geneva, Switzerland. Therefore, quarks and the strong force were not discovered until the twentieth century. The heaviest is the top quark, which is about 100,000 as heavy as the up and down quarks. The heavier the quark the harder the particles need to collide in the accelerator. That is why the top quark was the latest quark to be discovered experimentally.

Residual strong force. But hold on a second, you might say. We started this discussion by saying that there was a strong force holding the nucleus together. Now we find that the three quarks combine into color neutral particles, such as the proton and the neutron, which are color neutral and therefore do not interact with each other. Unfortunately, now it becomes rather complicated, and you can forget the following, if you wish. What happens is that the particles in the neutron exchange quarks with each other, see Fig. 23. A proton consists of two up quarks and one down quark, and a neutron consists of two down quarks and one up quark. So if the proton gives an up quark to the neutron and the neutron gives a down quark to the proton, the proton becomes a neutron and vice versa, see Fig. 23(a). However, this is not the way it goes. We see that there is an exchange of two quarks in both directions. What we would like is that there is an exchange from one particle to the other, i.e. the neutron is sending something to the proton or the proton is sending something to the neutron. Not both neutron and proton sending something. We can do this by letting one particle travel "back in time." This is called antimatter. Antimatter was predicted theoretically by Paul Dirac, who solving the relativistic Schrödinger equation (combining relativity and quantum mechanics), ended up with an equation like $x^2 = 1$, which has two solution $x = \pm 1$. Dirac concluded that besides have, for examples, electrons with a positive energy, there should also be "electron" with negative energy. This antielectron, also known as a positron, has the same mass as an electron, but a positive charge. If an electron and a positron meet, they annihilate

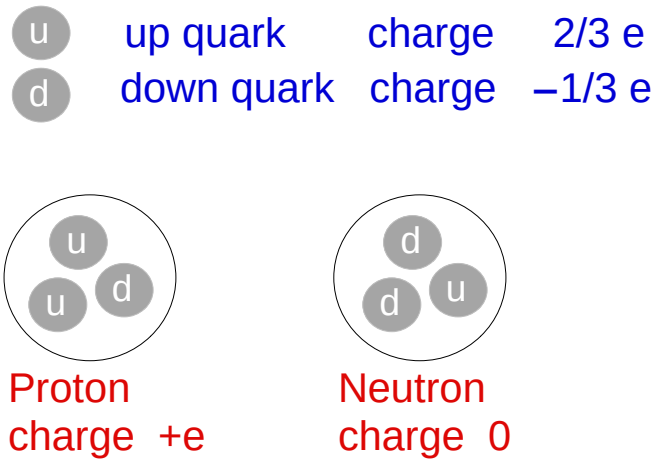


FIG. 22: The up and down quarks have fractional charges $\frac{2}{3}e$ and $-\frac{1}{3}e$, respectively. The proton consist of two up quarks and one down quark, giving a total charge of $+e$. The neutron consist of one up quark and two down quarks, giving a zero total charge. Although the colors of the three quarks in each particle are red, green, and blue, the are not indicated since the quarks interchange colors continuously. vv

each other under the emission high-energy radiation. It is great stuff for science fiction books and for example the Dan Brown novel "Angels and Demons," where a scientist siphons of significant amounts of matter and antimatter, which would create a huge explosion if the matter and antimatter recombine. However, here we have two different quarks, an up quark and an anti down quark, which will not annihilate each other. We see that we now have a combination of a quark and an antiquark. This is another way of making color neutral entities. Combining green with anti-green, (or blue with antiblue or red with anti-red), we obtain something color neutral. These type of particles are called mesons. The up quark-anti down quark meson is called a pion (π^+), which has a charge of $+e$ ($\frac{2}{3}e$ from the up quark and $\frac{1}{3}e$ from the anti down quark. Remember that the down quark has a charge of $-\frac{1}{3}e$). We could have done the same trick with a down quark-anti up quark, which gives also a pion (π^-), which has a charge of $-e$ and would be travelling from the neutron to the proton. So we can also redraw everything as a proton and a neutron exchanging a particle called a pion, see Fig. 23(c). This is similar to the diagrams that we saw for the Coulomb interaction and the gravitational force. However, now the particle that is exchanged has a mass. In addition, the proton turns into a neutron and the neutron turns into a proton. Note that the pion π^+ carries with it the charge $+e$ when it goes from the proton to the neutron. So why are human beings not interaction with each other through this residual strong force? Well, the pions are not stable so this force cannot interact over long distances and is only effective inside the nucleus whose size is of the order of 10^{-15}-10^{-14} m. Just for completeness. You might wonder what happened to the gluons. We had said earlier that the strong force was mediated by gluons. Well, this was temporarily shuffled under the carpet. The exchange of quarks is caused by the gluons. Just drawing some gluon exchanges will solve this problem, see Fig. 23(d).

C. Weak force

We will not go into much detail about the weak force, just mention some things for completeness. The weak force is an absolutely bizarre force. The exchange particles are about 100 times heavier than the proton, in fact they are heavier than iron atoms. The lifetime of these exchange particles is very short (unlike photons and gravitinos), making the effective range of this interaction about 10^{-18} m, or about 0.1% of the diameter of a proton. It is the only interaction that can change the flavor of a quark. For example, it can change an up quark into a down quark, making it possible to change a proton into a neutron. However, even though the force is bizarre it is crucial for the burning of the Sun, since it is essential for deuterium formation allowing deuterium fusion to take place. The unification of the electromagnetic and weak forces into one theory (the "Standard Model") was one of the great triumphs of twentieth

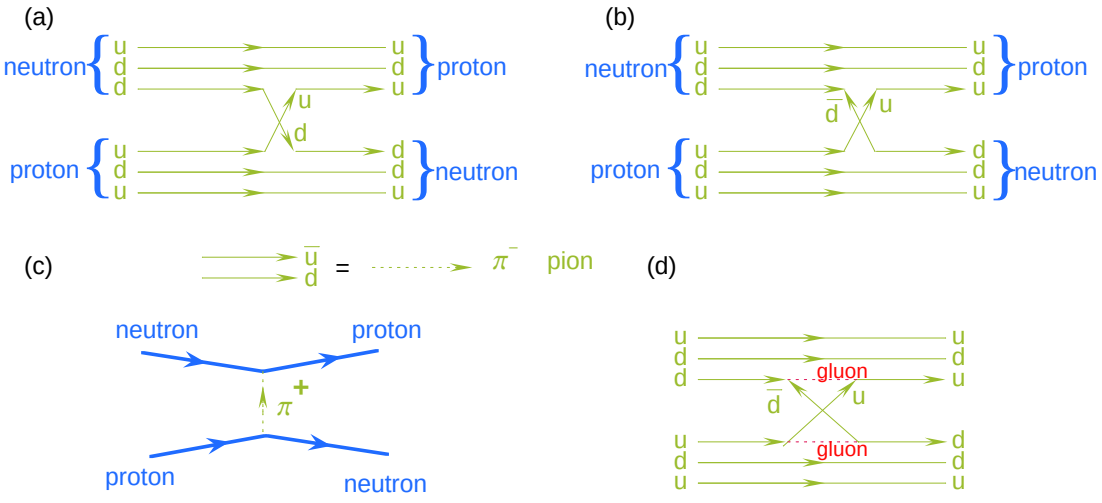


FIG. 23: Residual strong force (a) The proton and neutron interact with each other through the exchange of an up and a down quark. Note that after the interaction the proton has turned into a neutron and vice versa. (b) To make both particles move from the proton to the neutron, we replace the down quark by an anti down quark. (c) The combination of a up quark and an anti down quark is equivalent to a particle called a pion (denoted by π^+), which is going from the proton to the neutron. (d) Same as (b) but with the added gluon interactions that cause the interaction to take place. vv

century physics and led to several Nobel prizes.

D. What do we learn from all this?

You might wonder what do we learn from all of this? We are going to discuss Newton's laws, electricity and magnetism, etc. and not quantum field theory. All right, let us have a closer look at what is being exchanged. We learned that during an interaction particles are exchanging other particles. But what information is being exchanged. We saw that in some case they exchange color (the strong interaction) or charge (the residual strong interaction). But so far we have left out one of the most important quantities that is being exchanged in an interaction. Let us consider a collision between two macroscopic object, such as two cars. What is important for the collision? One thing is the velocity of the cars v and the other aspect is the mass m of the cars (arguments against SUV's is that they do not only guzzle gas, but also that they are a danger to other cars in head-on collisions). In physics, the two quantities are combined in a concept called momentum

$$p = mv, \qquad \text{or in vector form} \qquad \mathbf{p} = m\mathbf{v}. \tag{355}$$

Hold on again, you might say: "Photons do not have a mass, and I don't know about gravitons (because we have never observed one), but they probably don't have a large mass either." True, photons do not have a mass, but then again, they do move at the speed of light c. Remember, Einstein's famous equation for energy

$$E = mc^2, \tag{356}$$

one of the few physics equation with a certain cult status (yes, we are touching on all areas of physics). The m in there is the relativistic mass

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{357}$$

where m_0 is your mass (i.e., the number of kilograms) and v is your velocity. So even though the photon's mass m_0 is zero, so is the denomination $\sqrt{1 - \frac{v^2}{c^2}}$. So, we obtain zero divided by zero, which is undefined. What do we learn

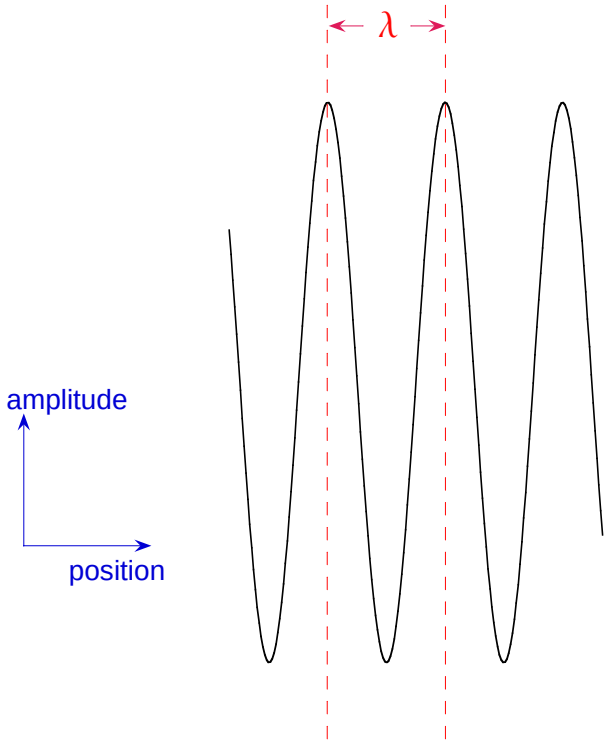


FIG. 24: Electromagnetic fields are waves. The wavelength λ is the distance between two maxima of the wave.

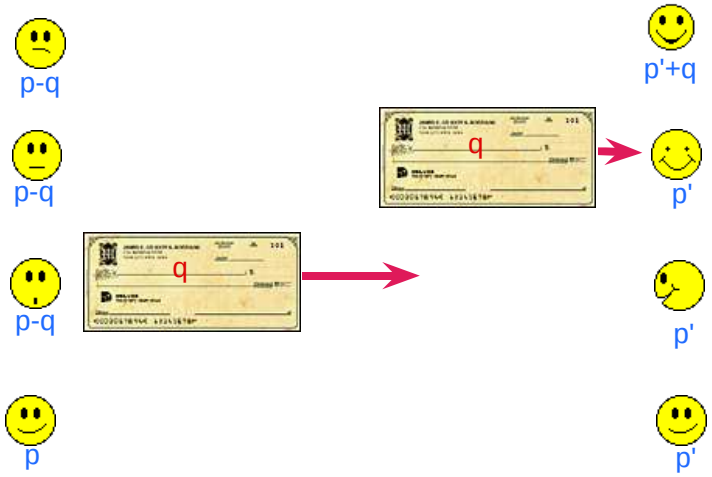


FIG. 25: Conservation of momentum. When to particles interact, they do that by exchanging other particles that carry momentum between the particles. Compare this to writing a check where momentum is the monetary unit of the particles. The particles start out with momentum p and p'. If the particle on the left, writes a check with q on it, he has $p-q$ left. If the particle on the right receives the check, he will have $p'+q$. However, note that the total momentum after the interaction $(p-q)+(p'+q)=p+p'$ is the same as the total momentum before the interaction.

from this. We learn that $p = mv$ is not the proper expression for the momentum of a photon. The right expression is

$$p = \frac{h}{\lambda}, \tag{358}$$

where h is Planck's constant, which is a fundamental constant in nature. Its value is not really important at the moment. The wavelength of the light is given by λ, see Fig. 24. An electromagnetic wave is very similar to vibrations of a string. A simple wave can be described in terms of sines and cosines. The amplitude can be given, for example, by

$$\textbf{amplitude} = A \sin \frac{2\pi x}{\lambda}, \tag{359}$$

where A is a constant determining the size of the wave and x is the position along the x axis. From Fig. 24, you can see that λ is equivalent to the distance between to maxima (or two minima) in the wave. From the equation $p = h/\lambda$, it follows that photons with shorter wavelengths have a higher momentum. Therefore, x-rays and γ-rays are the SUV's of electromagnetic radiation, whereas for example, visible and infrared light are the equivalent of Dodge Neon and Toyota Yaris.

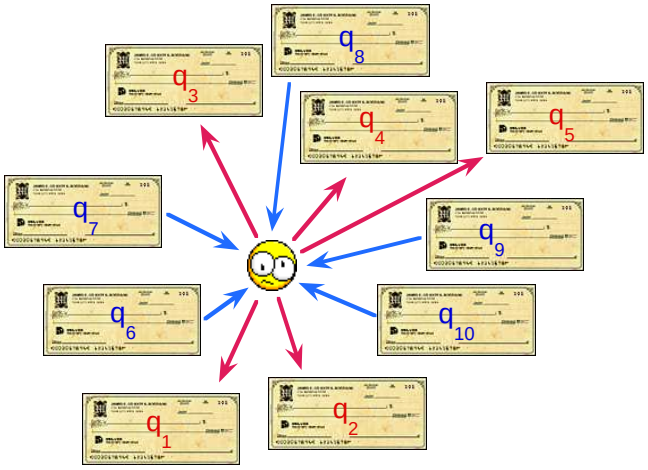


FIG. 26:

Momentum can be considered the currency of particles. Although, as humans we feel that position in space and speed are very important, for a lot of physical processes momentum is the most important quantity. Momentum is a bit like money. If you do not spend it or do not receive any, the amount of money remains the same. This is also the case for momentum, if nothing happens or, in physics terms, if there are no interactions with other objects, the momentum remain constant. So, in the absence of interactions of forces, an object moving at a certain velocity will keep moving at this velocity. This assumes that its mass does not change, which is the usual assumption in Newtonian mechanics. This is basically Newton's first law.

If we accept that momentum is the currency for particles, then let us consider what happens if two particles interact. Let us take two particles with momenta p and p', see Fig. 25. The particle with momentum p wants to interact with the particle with momentum p' and sends of an interaction particle (a photon, graviton, gluon) with momentum q. This is denoted by the check in Fig. 25. However, writing a check with momentum q, leaves that particle only $p - q$. When the other particle receives the check (= photon, graviton, gluon, etc.), then its momentum will be $p' + q$. Note that the total momentum after the interaction is

$$(p + q) + (p' - q) = p + p' \tag{360}$$

and equal to the momentum before the interaction. Therefore, the total momentum does not change. This is the *law of conservation of momentum*. If it schematically This can be compared with the law of conservation of money. Note, that momentum is a vector and not a scalar. Therefore, to treat it properly we have to consider size *and* direction. However, the basic concept remains the same.

Note, that this can be extended to many particles. Everytime time there is an interaction, there is an exchange of momentum between two particles. However, it is not important how many particles and how many interactions there are, the total momentum of all the particles remains constant. Again, think of the analogy between momentum and money, within a certain group of people the total amount of money is constant, even if there are a lot of exchanges of money between people.

When you are dealing with money, you are obviously interested in your own bank account. So what do you do? You keep track of all the checks that are coming in and out during the month and from that you can calculate how much you bank account has changed over that month, see Fig. 26. For particles, we want to know how the momentum is changing in time, or in equations, we like to calculate $\frac{dp}{dt}$. The equivalent in physics of the total amount of money on the checks coming in and out is the force. Therefore, if there are no checks coming in or out, your bank account remains constant. Or, in physics, if there is no force working on you, your momentum remains constant. However, also when the amount on the checks coming in equals the amount on the checks going out each month, your bank account does not change. The same with forces, if the sum of all the forces working on you is zero, the momentum will not change. For example, an object lying on the floor is feeling a constant gravitational force trying to pull it to the center of the Earth. However, there is an equal force working in the opposite direction cause by the surface on which the object is lying, preventing the object from moving towards the center of the Earth. Therefore, the object is going nowhere. If there is an imbalance between the total amount of money on the checks coming in compared to the total amount going out each month, your back account will change. You will become richer or poorer. The same with forces. For example, if there is only one force working on an object, the object will start to move in a certain direction. An object released above the ground only feels the gravitational force and will start to fall towards the center of the Earth since there is no surface (at least for the moment), holding it back.

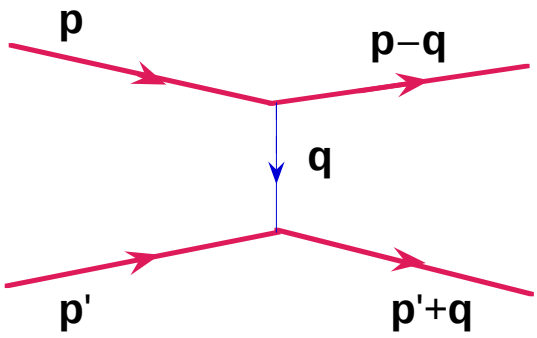


FIG. 27: Schematic picture of the exchange of momentum when two particles interact. Before the interaction (left side of the figure), the particles have momentum $\mathbf{p}$ and $\mathbf{p}'$. Then the particles interact by exchanging a particle (a photon, gluon, pion, graviton) which has a momentum $\mathbf{q}$. The net result is that one particle looses momentum $\mathbf{q}$, leaving it with $\mathbf{p} - \mathbf{q}$, whereas the other particle gain the momentum giving $\mathbf{p}' + \mathbf{q}$.

So what have we learnt from considering the interactions as an exchange of particles.

- We obtained some understanding about the functional dependence of the gravitational ($\frac{M_1 M_2}{r^2}$) and Coulomb ($\frac{Q_1 Q_2}{r^2}$) forces .
- We saw that neutral objects do not interact with each other. We can make neutral objects for charge (equal amount of positive and negative charge), color (equal amount of red, green, and blue quarks). However, we can never make particles with a neutral mass, since there is only one type of mass and it is always positive. This is the reason why the weakest force is so important to us: it is always there and why the strongest force (the "strong force") is not part of our daily experience since it is so strong that it is almost impossible to pull the neutral entities (3 quarks forming proton and neutrons) apart. We can still experience the Coulomb interaction, which also prefers neutral entities. However, the force is sufficiently weak that with some effort we can create charged particles. We will therefore be dealing only with gravitational and electromagnetic forces in the remainder of the course.
- We learnt that particles can send all kinds of information to each other: charge, color, and, most importantly for us, momentum. Momentum (mass × velocity) is the "money" of particles. Since interactions are the exchange of money from one particle to the other, the total momentum remains constant. This is still true for systems of many particles.

XIII. CONSERVATION OF MOMENTUM

One can derive other useful conservation laws from Newton's laws. Let us introduce momentum

$$\mathbf{p} = m\mathbf{v}. \tag{361}$$

We can rewrite Newton's second law as

$$\sum \mathbf{F} = m\frac{d\mathbf{v}}{dt} = \frac{dm\mathbf{v}}{dt} = \frac{d\mathbf{p}}{dt}. \tag{362}$$

Let us now consider a set of object on which no external forces (such as gravity) work. For a certain object i we can write

$$\sum_j \mathbf{F}_{ij} = \frac{d\mathbf{p}_i}{dt}, \tag{363}$$

where $\mathbf{F}_{ij}$ is the force on i by another object j. However, since there are no external forces, the only forces in the system are those of the object on each other. However, for each force exerted by one object i on object j there is an equal but opposite force exterted by j on i, i.e. $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$. Therefore, one can sum over the particles i,

$$\sum_i \sum_j \mathbf{F}_{ij} = \sum_i \frac{d\mathbf{p}_i}{dt}. \tag{364}$$

The left-hand side is the equal to zero in the absence of external forces and we have

$$\sum_i \frac{d\mathbf{p}_i}{dt} = 0 \qquad \Rightarrow \qquad \frac{d}{dt}\left(\sum_i \mathbf{p}_i\right) = 0 \qquad \Rightarrow \qquad \sum_i \mathbf{p}_i = \text{constant}. \tag{365}$$

Note that this fails in the presence of an external force such as gravity

$$\mathbf{F}_i^{\text{ext}} + \sum_j \mathbf{F}_{ij} = \frac{d\mathbf{p}_i}{dt}, \tag{366}$$

Summing over all particles gives

$$\sum_i \mathbf{F}_i^{\text{ext}} = \sum_i \frac{d\mathbf{p}_i}{dt}, \tag{367}$$

where we have used the fact that the interaction between the particles cancel. This is clearly not zero and therefore the total momentum is not constant. Of course, you might say Earth is also an object and why don't we consider the Earth as part of our system removing the need for introducing external forces? This has some serious practical problems. First, the Earth is very heavy and although we would have conservation of momentum if you for example jump on the Earth, the momentum change with respect to the total momentum of the Earth is very small. Second, we usually not interested in the change in momentum of the Earth, but more in our relative change with respect to the Earth's surface. Thirdly, once we want to start considering our very small relative changes to the Earth's momentum, we are forced to start considering all the other small changes in the Earth's momentum caused by all the other 6 million human beings, plus animals. Plus the effect of the Moon and Sun on the Earth's momentum etc. Not very practical, indeed.

Gravity is continuously changing the momentum of the particles. For those who still remember the section on fundamental forces, can appreciate this in a different manner. Particles are constantly changing packages of momentum with each other. However, since they are only exchanging the total momentum is conserved. An external force means that something is constantly adding packages of momentum to the system, disturbing the total momentum.

In particular for two particles, we can write

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2' \tag{368}$$

When we are dealing with collisions the velocities before and after the collision are denoted without and with prime, respectively.

Example (see also Giancoli 9.3) Let us consider two railroad cars colliding into each other with a velocity of 24.0 m/s. The railcars lock and continue with a common speed after the collision. What is the velocity if the masses are equal?

Answer: The momenta before and after the collision are conserved so we find

$$m_1 v_1 = (m_1 + m_2)v' \quad \Rightarrow v' = \frac{m_1}{m_1 + m_2} v_1 \tag{369}$$

If the masses are equal we have

$$v' = \frac{1}{2} v_1 = 12 \text{ m/s}. \tag{370}$$

Example (see also Giancoli 9.4) What is the recoil of a $m_R = 5.0$ kg rifle that shoots a bullet with $m_B = 0.050$ kg at a speed of $v_B' = 120$ m/s.

Answer The momenta before and after the collision must be the same, therefore

$$m_r v_R + m_B v_B = m_r v_R' + m_B v_B' \quad \Rightarrow \quad 0 = 5.0 v_R + 0.050120 \;\; v_R' = -1.2 \;\; \text{m/s} \tag{371}$$

We have a billiard ball moving with a speed of 3.0 m/s in the positive x direction. It strikes a ball with equal mass which is initially at rest. After the collision, ball 1 is moving at an angle of $+45^\circ$ with the x-axis, whereas ball 2 is moving at an angle of -45° with the x-axis. What are the speeds of the two balls after the collision.

Answer: This is a problem in two dimension, so we have to work with vectors:

$$m\mathbf{v}_1 = m\mathbf{v}_1' + m\mathbf{v}_2'. \tag{372}$$

We can split that into components as

$$x: \qquad m v_1 = m v_1' \cos 45^\circ + m v_2' \cos(-45^\circ) \tag{373}$$

$$y: \qquad 0 = m v_1' \cos 45^\circ + m v_2' \cos(-45^\circ). \tag{374}$$

The second equation gives $v_1' = v_2'$. Inserting this in the first equation gives

$$m v_1 = m v_1' [\cos 45^\circ + \cos(-45^\circ)] = v_1' \sqrt{2} \quad \Rightarrow \quad v_1' = \frac{3.0}{\sqrt{2}} = 2.1 \;\; \text{m/s}. \tag{375}$$

A. Elastic and inelastic collisions

Let us reexamine the the examples that we did in the previous section to see if energy is conserved.
Let us calculate the kinetic energy after the collision,

$$E'_{\rm kin} = \frac{1}{2}(m_1+m_2)v'^2 = \frac{1}{2}(m_1+m_2)\frac{m_1^2v_1^2}{(m_1+m_2)^2} = \frac{1}{2}\frac{m_1^2}{m_1+m_2}v^2 < \frac{1}{2}m_1v_1^2 = E_{\rm kin} \tag{376}$$

Since these two are not equal, energy must have been lost in the collision process, most of it in the form of heat, although if the collision was rough energy could have been used to deform the locks. In the example of the rifle and the bullet, there is kinetic energy after the shooting whereas there was none before the shooting. However, this is clearly a result of the exploding gun powder that puts energy into the system (though no net momentum). Let us check the third example from the previous section. Let us square the momentum conservation

$$m^2\mathbf{v}_1^2 = m^2(\mathbf{v}'_1+\mathbf{v}'_2)^2 = m^2(\mathbf{v}'^2_1 + 2\mathbf{v}'_1\cdot\mathbf{v}'_2 + \mathbf{v}'^2_2) = m^2(\mathbf{v}'^2_1+\mathbf{v}'^2_2), \tag{377}$$

where we have made use of the fact that $\mathbf{v}_1$ and $\mathbf{v}_2$ are at a 90° angle. We can rewrite this as

$$\frac{1}{2}m\mathbf{v}_1^2 = \frac{1}{2}m(\mathbf{v}'^2_1+\mathbf{v}'^2_2), \tag{378}$$

showing that kinetic energy is conserved.

We can therefore distinguish two different types of collisions. In elastic collisions the kinetic energy is conserved, giving

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v'^2_1 + \frac{1}{2}m_2v'^2_2. \tag{379}$$

For inelastic collisions, we have

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v'^2_1 + \frac{1}{2}m_2v'^2_2 + \text{thermal and other forms of energy.} \tag{380}$$

A lot of collision are inelastic, certainly car collision. However, even collisions with billiard balls or puck on ice are inelastic. However, the amount of energy that goes into thermal energy is sufficiently small that we can treat the collisions as elastic.

B. Elastic collisions in one dimension

Let us consider an elastic collision in one dimension. We know that momentum is conserved,

$$m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2 \tag{381}$$

Note that if the masses and the two initial velocities are given, there remain two unknowns v'_1 and v'_2. In the previous examples, we were able to solve the final velocities, because more information was given. In the case of the colliding trains, we knew that the final velocities were equal, i.e. $v'_1 = v'_2$, eliminating one unknown. For the billiard balls, we were given the additional information that the paths of the billiard balls were at a 90° angle with each other after the collision. However, in general, we are stuck with one equation (conservation of momentum) and two unknowns. Therefore, we need to use some additional equation given by the conservation of energy

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v'^2_1 + \frac{1}{2}m_2v'^2_2. \tag{382}$$

Often if is convenient to rewrite this

$$\frac{1}{2}m_1(v_1^2 - v'^2_1) = \frac{1}{2}m_2(v'^2_2 - v_2^2). \tag{383}$$

Using the special product $a^2-b^2 = (a-b)(a+b)$, we can write

$$m_1(v_1-v'_1)(v_1+v'_1) = m_2(v'_2-v_2)(v'_2+v_2). \tag{384}$$

Rearranging the conservation of momentum gives

$$m_1(v_1 - v_1') = m_2(v_2' - v_2). \tag{385}$$

Dividing these to equations gives

$$v_1 + v_1' = v_2' + v_2 \quad \Rightarrow \quad v_1 - v_2 = v_2' - v_1' = -(v_1' - v_2'). \tag{386}$$

This results shows the magnitude of the relative velocity remains the same, but its sign changes.

Example: Equal masses. For two billiard balls with equal masses the conservation of momentum reduces to

$$mv_1 + mv_2 = mv_1' + mv_2' \quad \Rightarrow \quad v_1 + v_2 = v_1' + v_2' \tag{387}$$

In addition, we also have

$$v_1 - v_2 = v_2' - v_1'. \tag{388}$$

Adding the two equations gives $v_1 = v_2'$ and subtracting gives $v_2 = v_1'$. A particular case is when one of the balls is at rest $v_2 = 0$. This directly gives $v_1' = 0$ and $v_2' = v_1$. This is often observed when playing pools when the balls have equal masses and in the absence of spin on the balls.

General solution for a collision in one dimension. Two solve this problem, we have two equations: conservation of momentum

$$m_1(v_1 - v_1') = m_2(v_2' - v_2). \tag{389}$$

and the condition for the relative velocities derived above

$$v_1 - v_2 = v_2' - v_1' \quad \Rightarrow \quad v_1' = v_2' - v_1 + v_2. \tag{390}$$

Inserting this gives

$$m_1[v_1 - (v_2' - v_1 + v_2)] = m_2(v_2' - v_2) \quad \Rightarrow \quad 2m_1v_1 + (m_2 - m_1)v_2 = (m_1 + m_2)v_2', \tag{391}$$

giving for v_2',

$$v_2' = \frac{2m_1}{m_1 + m_2}v_1 + \frac{m_2 - m_1}{m_1 + m_2}v_2. \tag{392}$$

We can use this to derive v_1',

$$v_1' = v_2' - v_1 + v_2 = \frac{2m_1}{m_1 + m_2}v_1 + \frac{m_1 - m_2}{m_1 + m_2}v_2 + \frac{m_1 + m_2}{m_1 + m_2}(-v_1 + v_2) = \frac{m_1 - m_2}{m_1 + m_2}v_1 + \frac{2m_2}{m_1 + m_2}v_2 \tag{393}$$

Example: Target at rest. For the target at rest, $v_2 = 0$, we have

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2}v_1 \quad \text{and} \quad v_2' = \frac{2m_1}{m_1 + m_2}v_1. \tag{394}$$

Let us consider some limiting cases:
$m_1 \gg m_2$. In this limit, we obtain

$$v_1' \cong v_1 \quad \text{and} \quad v_2' \cong 2v_1. \tag{395}$$

Thus the velocity of m_1 is hardly unchanged, however, mass m_2 takes off with twice the velocity. This happens when a heavy bowling ball hits the pins.
$m_2 \gg m_1$. In this limit, we obtain

$$v_1' \cong -v_1 \quad \text{and} \quad v_2' \cong 0. \tag{396}$$

In this limit, the target remains at rest, and the incoming ball simply bounces back. This is comparable to the ball bouncing of a wall.

Example: Giancoli 9.9.

Pressure on a wall: A nice example of the use of momentum occurs in gas theory. In kinetic gas theory, the pressure on the wall is explained in terms of molecules colliding on the wall. Let us present here a heavily oversimplified version. Suppose we have n particles colliding against a wall per second. Let us assume that their velocities v are perpendicular to the wall. (Two simplifications here: obviously the velocities are not perpendicular. Furthermore, not all the molecules have the same velocity. In fact, there is a distribution of velocities (known as the Maxwell-Boltzmann distribution)). If the collision is elastic the velocity after the collision is $-v$. We saw this earlier in the collision between two balls where the target was much heavier than the moving ball. The momentum change is therefore $\Delta p = p' - p = -mv - mv = -2mv$. The momentum transferred to the wall is therefore $2mv$. Since we have n particles colliding on the wall per second the total momentum transferred per second is $2nmv$. The force exerted on the wall is then

$$F = \frac{\Delta p}{\Delta t} = \frac{2nmv}{1\text{ s}} = 2nmv. \tag{397}$$

You might bring up that this is not a continuous force, but a discrete one. However, since there are many particles (note one mole is of the order of 10^{23} particles), one can effectively treat is as a continuous force. Air pressure is also working on your body and I doubt whether you have ever noticed the discrete nature of the air pressure. The pressure p (due to a lack of letters also indicated with a p) is then the force divided by the surface area A

$$p = \frac{F}{A} = \frac{2nmv}{A}. \tag{398}$$

An additional thing to note is that with the use of gas theory one can show that the average velocity of the gas molecules is related to the temperature of the gas. This directly relates the pressure to the temperature.

Example: Rocket mechanics. So far, we have only considered examples where the mass is conserved. A very typical example where the mass is not conserved is the launching of a rocket. The idea of a rocket is that gases are expelled from the rocket. The principle is similar to trying to move yourself forward when you are skating on ice by throwing a ball. In that case, you will propel yourself in the direction opposite to where you are throwing the ball. Although, the gases ejected by the rocket are much lighter the principle is equivalent. Obviously, the rocket is losing mass very quickly. Let us consider the motion of the rocket in one dimension. Let us assume that the exhaust gases have the same relative velocity $-v_{\text{exh}}$ with respect to the rocket. The mass and velocity are both function of time. Before the dm of gas is exhausted, the momentum of the total system is

$$p(t) = m(t)v(t) \tag{399}$$

In a time interval, the rocket expels an amount of dm gas. Note that $dm < 0$ changing the velocity of the rocket by dv. After that, the momentum is

$$p(t) + dp = (m(t) + dm)(v(t) + dv) + (-dm)(v(t) - v_{\text{exh}}). \tag{400}$$

The change in momentum is therefore

$$dp = m(t)v(t) + m(t)dv + dmv(t) + dmdv - dmv(t) + dmv_{\text{exh}} - m(t)v(t) = m(t)dv + dmv_{\text{exh}}. \tag{401}$$

The term $dmdv$ is a product of two differentials and is small compared to all the other terms. We therefore neglect it. The change in momentum must equal the force times the time dt that the total force is working on the rocket

$$F_{\text{total}}dt = dp = m(t)dv + dmv_{\text{exh}} \quad \Rightarrow \quad F_{\text{total}} = m(t)\frac{dv}{dt} + \frac{dm}{dt}v_{\text{exh}}. \tag{402}$$

This is known as the rocket equation. Let us consider the situation far in space when there is no external force such as gravity working on the rocket. We then have

$$m(t)\frac{dv}{dt} + \frac{dm}{dt}v_{\text{exh}} = 0 \quad \Rightarrow \quad \frac{dv}{v_{\text{exh}}} = -\frac{dm}{m} \tag{403}$$

Integrating gives

$$\int_0^{v_f} \frac{dv}{v_{\text{exh}}} = -\int_{m_0}^{m} \frac{dm}{m} \quad \Rightarrow \quad \frac{v}{v_{\text{exh}}} = \ln m_0 - \ln m = \ln\frac{m_0}{m} \quad \Rightarrow \quad v = v_{\text{exh}} \ln\frac{m_0}{m} \tag{404}$$

where we have taken the rocket at rest initially.

C. Elastic collision in two and three dimensions

For an elastic collision in two and three dimension, we need to write conservation of momentum in terms of vectors:

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2'. \tag{405}$$

We can write this out in coordinates, for example in two dimensions

$$p_{1x} + p_{2x} = p_{1x}' + p_{2x}' \tag{406}$$
$$p_{1y} + p_{2y} = p_{1y}' + p_{2y}'. \tag{407}$$

When we are given the initial momenta, we are still left with two equations with four unknown. Since the collision is elastic, we can also use the law of conservation of energy

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2. \tag{408}$$

However, this is still insufficient to solve the problem. Therefore, we need an additional piece of information, for example determined by experiment. Let us rewrite the conservation of momentum in terms of the magnitude and the angle that the velocities (and momenta) make with the x-axis

$$mv_1 \cos\varphi + mv_2 \cos\theta = mv_1' \cos\varphi' + mv_2' \cos\theta' \tag{409}$$
$$mv_1 \sin\varphi + mv_2 \sin\theta = mv_1' \sin\varphi' + mv_2' \sin\theta'. \tag{410}$$

Since we had only three equation with four unknowns (if the initial conditions were given), an additional clue could be one of the final state angles. These is a typically determined in the collision of particles, see

Example: Giancoli 9.11

D. Inelastic collision in two and three dimensions

Example: Let us consider two objects colliding with each other at an angle of 30°. Object 1 with mass m is moving along the x axis. Object 2 has a mass $2m$. For their velocities, $v_1 = v_2 = v$ After the collision, the object move as one object at an angle φ with respect to the x-axis.
(a)Determine the final velocity $\mathbf{v}'$ and the angle φ in terms of v_1, v_2, and θ.
(b) Determine the change in kinetic energy.

Answer: From conservation of momentum, we have

$$m\mathbf{v}_1 + 2m\mathbf{v} = 3m\mathbf{v}'. \tag{411}$$

We can split this into x and y coordinates:

$$x: \quad mv_1 + mv_2 \cos\theta = 3mv' \cos\varphi \tag{412}$$
$$y: \quad mv_2 \sin\theta = 3mv' \sin\varphi. \tag{413}$$

The final state velocity is

$$\mathbf{v}' = \frac{1}{3}(1 + 2\cos\theta)v\hat{\mathbf{i}} + \frac{2}{3}\sin\theta v\hat{\mathbf{j}}. \tag{414}$$

The angle is determined by

$$\tan\varphi = \frac{2\sin\theta}{1 + 2\cos\theta} \tag{415}$$

(b) The kinetic energy before the collision is

$$E_{\text{kin}} = \frac{1}{2}mv^2 + \frac{1}{2}2mv^2 = \frac{3}{2}mv^2. \tag{416}$$

After the collision, the kinetic energy is

$$E'_{\rm kin} = \frac{1}{2}3m[\frac{1}{3}(1+2\cos\theta)^2+\frac{1}{3}(2\sin\theta)^2]v^2 \quad (417)$$

$$= \frac{1}{6}(1+4\cos\theta+2\cos^2\theta+4\sin^2\theta)v^2 = \frac{1}{6}m(5+4\cos\theta)v^2. \quad (418)$$

The difference in kinetic energy is therefore

$$\Delta E_{\rm kin} = E'_{\rm kin} - E_{\rm kin} = \frac{1}{6}m(5+4\cos\theta)v^2 - \frac{3}{2}mv^2 = -\frac{2}{3}mv^2 + \frac{2}{3}\cos\theta mv^2 = \frac{2}{3}m(1-\cos\theta)v^2 < 0. \quad (419)$$

Kinetic energy is therefore lost in the collision.

XIV. ELECTRICITY

The history of electricity can be traced back (as many things in science) to the ancient Greeks. They noticed the existence of static electricity. As you probably know, there are positive and negative charges. The positive charges are a result of the protons, which (together with the neutrons, which have no charge) form the nucleus of the atoms. Atoms are often strongly bound in solids. The negative charge is related to the electrons. Some electrons are very strongly bound to the nucleus, whereas other electrons are free to move around in solids. It is the electrons that cause the electric currents that you know from all your household applications. The choice of positive and negative is arbitrary and somewhat unfortunately chosen because if the current flow in one direction, the charge carriers (the electrons) are actually flowing in the other directions. We are all familiar with static electricity, by sparks from your comb or from the doorknob. Static electricity occurs because materials can loose or accept electrons relatively easily. Some materials easily let go of electrons when rubbed. One such materials is amber, fossiled resin, often with small bugs trapped in there. Amber was used as decoration (for example, buttons) by the ancient Greeks and was known to attract other things such as hair and, when rubbed more, to even spark. The name electricity is related to Latinized version of the Greek word $\eta\lambda\varepsilon\kappa\tau\rho o\nu$ (elektron), which is the Greek work form amber. This work was used for the first time by the English scientist William Gilbert. One of the problems with electrity is that normal it is not there. Gravity is always there, since there is only one type of mass and it is always positive. However, there are two types of electic charges, positive and negative. Charges with the same sign repel, and charges with the opposite sign attract. Therefore, If there is some finite charge somewhere, it will tend to attract opposite charges, until it becomes neutral, i.e. the total charge is zero. Therefore, we need a way to create or store electical charge. The first of such devices was the *Leyden* jar, invented at Leiden University by Pieter van Musschenbroek in 1745 (note that this is more than half a century later than Newton's theories). This is essentially a capacitor, which we will discuss in later sections. Another type of capacitor are storm clouds, where the discharge of positively and negatively charged clouds causese the well-known phenomenon of lightning. This was investigated by Benjamin Franklin in 1752. However, although he is often depicted as flying a kite in a thunderstorm (kids, do not do this at home), there are serious doubt whether he actually performed that experiment. In addition, to storing charge and creating currents through discharge, what we actually need is to create a current. This was done with the invention of the battery by Count Alessandro Giuseppe Antonio Anastasio Volta (1745-1827). Batteries essentially use chemical reactions to create an electrical current. The theory of electricity and its cousin magnetism really took of in the nineteenth century with people such as Michael Faraday, Andre-Marie Ampère and Georg Simon Ohm, who all got fundamental units named after them. The final unification of all the laws was done by James Clerk Maxwell. The equation are so beautiful that we just state them here (for vacuum)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{420}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{421}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{422}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \tag{423}$$

There are several ways of writing Maxwell's equations, but this is one of the more esthetically pleasing. We see that everything is defined by the inner and outer product of the operator ∇ with the electric and magnetic field. These are all the equations that you need in electricity and magnetism. All the other equations can be derived from it. Note that is does not contain Coulomb's law that we will be discussing in the next section. This is a special form of the first of Maxwell's equations (Gauss's law) for point charges. Note that the last two equations connect the elecric and the magnetic field with each other. In the third equation, the electric field can be related to a change in the magnetic field. This is important in electric generators where a moving magnetic causes an electric field leading to a currents that we all use in our houses, to light the streets, and so on.

With the theory on firm footing the twentieth century focused for a large part on applications. The turn of the twentieth century saw the great battle between Thomas Edison and Nikola Tesla to decide whether direct (DC) or alternating (AC) currents should be used. This was a mean battle. For example, the electric which uses alternating currents (promoted by Tesla), was invented in Edison's lab and used to demonstrate that alternating currents are more lethal than direct currents. The electrocution was a publicity disaster and damaged Edison's reputation more that that of alternating currents. However, eventually alternating currents won and Edison's company General Electric switched sides.

XV. COULOMB FORCE

One of the fundamental equations in electricity is the interaction between two point charges with charge Q_1 and Q_2, known as the Coulomb force. In scalar form,

$$F = k\frac{Q_1 Q_2}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r_{12}^2}, \tag{424}$$

where r_{12} is the distance between object 1 and 2. A charge is built up of fundamental charges. The form is very similar to Newton's gravitation law

$$F = -G\frac{M_1 M_2}{r_{12}^2}, \tag{425}$$

which describes the interaction between two masses. There are, however, a number of striking differences. Mass only comes in one type and is always positive. Charge, on the other hand, can be positive and negative. Charge is built up of fundamental units. Electrons have a negative charge of -1.6×10^{-19} C and protons have a positive charge of 1.6×10^{-19} C, where C stands of Coulomb, the unit of charge. The force between two (positive) masses is attractive. The force between two charges with equal signs, positive-positive and negative-negative, is repulsive. The force between two charges of opposite sign is attractive. The constant is given by

$$k = \frac{1}{4\pi\epsilon_0} = 8.988 \times 10^9 \ \ \frac{\mathrm{Nm}^2}{\mathrm{C}^2}. \tag{426}$$

The reason for the somewhat clumsy looking constant $\frac{1}{4\pi\epsilon_0}$ will become apparent later. In vector form, we can write Coulomb's law as

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r_{12}^2}\hat{\mathbf{r}}_{21}, \tag{427}$$

where $\hat{\mathbf{r}}_{21}$ is the unit vector pointing from object 1 to object 2.

A. Electric Field.

It is convenient to define an electric field

$$\mathbf{E} = \frac{\mathbf{F}}{q} = \frac{1}{q}\frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{\mathbf{r}}_{21} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}_{21}. \tag{428}$$

The introduction of the electric field allows us to study the effect of charge Q on the surroundings without introducing an additional charge q. We could have introduce a similar field when discussing gravity by dividing out one of the masses

$$\frac{F}{m} = \frac{1}{m}G\frac{mM}{r^2} = G\frac{M}{r^2}. \tag{429}$$

We know this quantity: it is nothing but the acceleration due to the mass M. In particular, on the Earth's surface the acceleration caused by the Earth is $g = GM_{\mathrm{Earth}}/r_{\mathrm{Earth}}^2$. Therefore, for gravity, it is not necessary to introduce an additional quantity. However, for electricity, we are dividing by the charge and not by the mass. There is another big difference between gravitational forces and electric forces. For gravity, we are often only dealing with the forces of large spherical objects. Gravity is very small and the only gravitational forces that are generally important are those of celestial objects, such as the Earth, the Moon, and the Sun, which often happen to be close to spherical. However, electric forces are a lot stronger and generally not spherical.

B. The electric field of different charged objects

Giancoli Example 21.9
Giancoli Example 21.10

Giancoli Example 21.11

Example: Electric field from a uniformly charged sphere.
We want to calculate the electric field at a point P at a distance r from the center of the sphere of radius R. Let us choose the origin O of our axes at the center of the sphere. The z-axis goes through OP. Let us define the charge density as

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}. \tag{430}$$

The electric field given by a portion of the sphere is then (in spherical coordinates)

$$dE = \frac{1}{4\pi\epsilon_0}\frac{\rho dv}{s^2}, \tag{431}$$

where dv is a small volume with coordinates r', θ', and φ'. The distance from volume dv to point P is s. Due to symmetry we can see that the only component that matters is the z component. We can therefore write

$$dE = \frac{1}{4\pi\epsilon_0}\frac{\rho}{s^2} r'^2 \sin\theta' dr' d\theta' d\varphi' \cos\alpha. \tag{432}$$

We would like to eleminate the angles from the problem. We can do this by using the cosine rule

$$s^2 = r^2 + r'^2 - 2rr'\cos\theta' \quad \Rightarrow \quad \cos\theta' = \frac{r^2 + r'^2 - s^2}{2rr'} \tag{433}$$

$$r'^2 = r^2 + s^2 - 2rs\cos\alpha \quad \Rightarrow \quad \cos\alpha = \frac{r^2 + s^2 - r'^2}{2rs}. \tag{434}$$

We would like to eliminate $\sin\theta' d\theta'$, taking the derivative of the expression for $\cos\theta'$, gives

$$-\sin\theta' d\theta' = -\frac{sds}{rr'}. \tag{435}$$

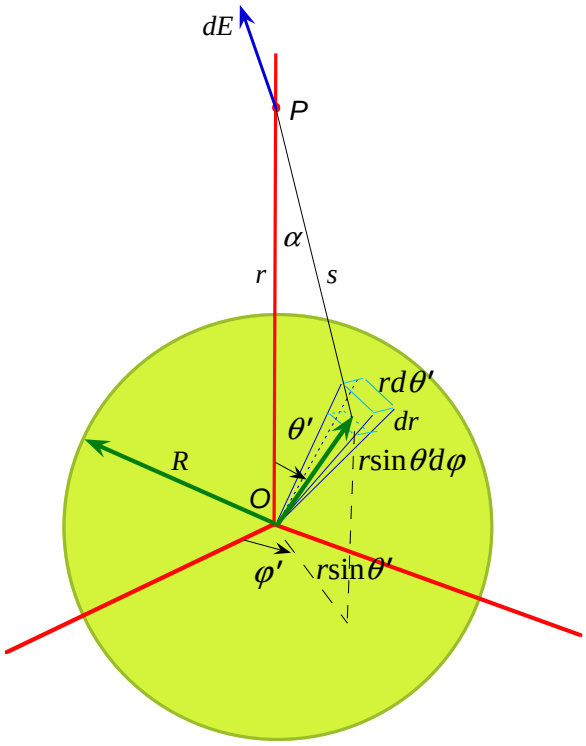


FIG. 28: An element dv produces an electric field at a point P at a distance r from the center of the sphere.

It is important to note that r is constant and that, when changing the angle θ', the distance to the center of the sphere r' is also constant. Collecting the results and integrating gives

$$E = \frac{\rho}{4\pi\epsilon_0}\int_0^R dr' \int_0^{2\pi} d\varphi' r'^2 \int_{r-r'}^{r+r'} ds \frac{1}{s^2}\left(\frac{s^2+r^2-r'^2}{2sr}\right)\frac{s}{rr'}. \tag{436}$$

The integral of φ' is trivial and gives $\int_0^{2\pi} d\varphi' = 2\pi$. Note that rotating the volume along the z axis, does not change r' and s. We then have

$$E = \frac{\pi\rho}{4\pi\epsilon_0 r^2}\int_0^R dr' r' \int_{r-r'}^{r+r'} ds \left(1+\frac{r^2-r'^2}{s^2}\right) \tag{437}$$

The integral over s gives

$$\int_{r-r'}^{r+r'} ds \left(1+\frac{r^2-r'^2}{s^2}\right)\left[s-\frac{r^2-r'^2}{s}\right]_{r-r'}^{r+r'} = r+r'-(r-r')-[r-r'-(r+r')] = 4r' \tag{438}$$

Leaving us with the integral

$$E = \frac{\pi}{4\pi\epsilon_0 r^2}\int_0^R dr' 4r' = \frac{1}{4\pi\epsilon_0 r^2}\frac{4}{3}\pi R^3 = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \tag{439}$$

C. Motion of a charged particle in an electric field

. Let us consider the acceleration of a particle with charge q under the influence of an electron field in the z direction (neglect gravity). The force and acceleration are given by

$$F = qE \quad \Rightarrow \quad a = \frac{F}{m} = \frac{qE}{m}. \tag{440}$$

We know from the equations of motion that

$$z = \frac{1}{2}at^2 \quad \Rightarrow \quad t = \sqrt{\frac{2z}{a}}. \tag{441}$$

After traveling a distance z, the velocity is then

$$v = at = \sqrt{2az} \quad \Rightarrow \quad v^2 = 2az = 2\frac{qE}{m}z. \tag{442}$$

Let us draw a parallel with acceleration as a result of a constant gravitational force. We know that we could write conservation of energy as

$$\frac{1}{2}mv^2 = mgz. \tag{443}$$

For a constant electric force, we obtain

$$\frac{1}{2}mv^2 = qEz. \tag{444}$$

In the case of gravitation, we know that the potential energy is given by mgz. Since the gravitational and Coulomb forces are very similar, we can expect that the electric forces can also be written in terms of a potential (we will prove that later on). The potential energy for the electric force is therefore qEz. However, for electric forces, we saw that it is convenient to introduce the electric field though $E = F/q$. This allows us to study the effects of a particular charge density. We can do the same thing with the potential energy $V = U/q = Ez$. This quantity is known as the electric potential whose unit is volts 1 V=1 J/C. This is something, you are all familiar with on electrical appliances, 110 V comes out of the socket, a battery produces 12 V. (In the same fashion, we can introduce a gravitational potential $V = U/m = gz$. Although this allows us to study of the effects of gravitation by itself, this quantity is rarely introduced).

D. An electric force is conservative

Let us consider a charge q moving in the electric field of another point charge Q. The work done in moving the charge q from a point a to a point b is

$$W = \int_a^b \mathbf{F} \cdot d\boldsymbol{l} = \int_a^b q\mathbf{E} \cdot d\boldsymbol{l}. \tag{445}$$

To see whether a force is conservative, we need to calculate the work done in a closed loop, for example from a back to a,

$$W = q\int_a^a \mathbf{E} \cdot d\boldsymbol{l} = q\oint \mathbf{E} \cdot d\boldsymbol{l} = \frac{qQ}{4\pi\epsilon_0}\oint \frac{1}{r^2}\mathbf{r} \cdot d\boldsymbol{l}. \tag{446}$$

The inner product $\mathbf{r} \cdot d\boldsymbol{l}$ is equal to the change in the path in the radial direction, i.e. $dr = \mathbf{r} \cdot d\boldsymbol{l}$. The integral then reduces to

$$W = \frac{qQ}{4\pi\epsilon_0}\oint \frac{1}{r^2}dr = \frac{qQ}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_a^a = 0. \tag{447}$$

Therefore, the force of an electric point charge is conservative. In the same way as we saw in the section on classical mechanics, the fact that $\oint \mathbf{F} \cdot d\boldsymbol{l} = 0$, implies that the work done in going from a point a to b is independent of the path taken. Therefore, it can only depend on the initial and final positions. This means that we can write

$$W_{ab} = \int_a^b \mathbf{F} \cdot d\boldsymbol{l} = -[U(\mathbf{r}_b) - U(\mathbf{r}_a)], \tag{448}$$

where $U(\mathbf{r})$ is the potential energy. Again, we have a minus sign by convention. However, for electric forces one often studies the integral over the electric field

$$\frac{W_{ab}}{q} = \int_a^b \mathbf{E} \cdot d\boldsymbol{l} = -[V(\mathbf{r}_b) - V(\mathbf{r}_a)]. \tag{449}$$

The scalar function $V(\mathbf{r})$ is called the potential as opposed to the potential energy $U(\mathbf{r}) = qV(\mathbf{r})$. We can express the potential as an integral

$$\int \mathbf{E} \cdot d\boldsymbol{l} = -\int dV. \tag{450}$$

This shows that

$$\mathbf{E} \cdot d\boldsymbol{l} = -dV = -\nabla V \cdot d\boldsymbol{l}, \tag{451}$$

where the operator nabla ∇ is defined as

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}. \tag{452}$$

The change in potential is therefore

$$\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz. \tag{453}$$

This result is obvious in one dimension where $\frac{dV}{dx}dx = dV$. In three dimensions, the partial derivative with respect to x, y, and z indicates how fast the potential is changing in the x, y, and z direction, respectively. This we then multiply times the amount that we are moving in that direction. For example, if the potential is $-E_z z$, the electric field is

$$\mathbf{E} = -\nabla V = \frac{\partial(E_z z)}{\partial x}\hat{\mathbf{i}} + \frac{\partial(E_z z)}{\partial y}\hat{\mathbf{j}} + \frac{\partial(E_z z)}{\partial z}\hat{\mathbf{k}} = 0 + 0 + E_z\hat{\mathbf{k}} = E_z\hat{\mathbf{k}}. \tag{454}$$

The work done is therefore

$$W = \int_{\mathbf{r}}^{\mathbf{r}+\Delta\mathbf{r}} E_z \hat{\mathbf{k}} \cdot d\boldsymbol{l} = \int_{z}^{z+\Delta z} E_z dz = E_z \Delta z. \tag{455}$$

We see that there is only work done in the z direction, since this is the only direction where the partial derivative of the potential is not equal to zero.

The potential of a point charge. For a point charge charge, we can write the potential as

$$V = -\int \mathbf{E} \cdot d\boldsymbol{l} = -\frac{1}{4\pi\epsilon_0} \int \frac{Q}{r^2} \hat{\mathbf{r}} \cdot d\boldsymbol{l} = -\frac{1}{4\pi\epsilon_0} \int \frac{Q}{r^2} dr = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} + \text{constant}. \tag{456}$$

Since the work only depends on the difference in potential, we can always add an arbitrary constant. Often, we can take this constant zero.

So far we have only proven that the electric force is conservative for a point charge. However, this can be generalized to any charge distribution by simply adding the electric field produced from the different charges. The same applies to the potential. The total charge is then

$$Q = \int \rho(\mathbf{r}) dv = \int \rho(x, y, z) dx dy dz, \tag{457}$$

where the volume element is $dv = dxdydz$ (indicated by a lower case v to distinguish it from the potential V). The potential is now

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r})}{r} dv = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} dx dy dz. \tag{458}$$

Electric field lines: See Giancoli.

XVI. GAUSS'S LAW

Before we can derive Gauss's law, we have to introduce the concept of flux. The flux is defined as the product of the electric field and the surface. For example, for a constant electric field E and a surface A perpendicular to it, the flux is

$$\Phi = EA. \tag{459}$$

The situation is somewhat more complex if the surface is not perpendicular to the electric field. In this case, we have

$$\Phi = \mathbf{E} \cdot \mathbf{A}, \tag{460}$$

where $\mathbf{A}$ is a vector perpendicular to the surface with a magnitude equal to the size of the surface. When the electric field is not constant or when the surface is not flat, we have to divide the surface into infinitesimally small pieces $d\mathbf{a}$ with a flux

$$d\Phi = \mathbf{E} \cdot d\mathbf{a}, \tag{461}$$

and integrate over the surface

$$\Phi = \int d\Phi = \int \mathbf{E} \cdot d\mathbf{a}. \tag{462}$$

For a point charge, this amounts to

$$d\Phi = \mathbf{E} \cdot d\mathbf{a} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}} \cdot d\mathbf{a}. \tag{463}$$

The term $\frac{\hat{\mathbf{r}}\cdot d\mathbf{a}}{r^2}$ is the solid angle $d\Omega$. This can be seen as follows. Consider a surface in the z direction. This surface is not necessarily perpendicular to the vector $\mathbf{r} = z\hat{\mathbf{z}}$. However, after projecting the surface vector in $\mathbf{r}$ we are only looking at the surface area perpendicular to $\mathbf{r}$, i.e.

$$\hat{\mathbf{r}} \cdot d\mathbf{a} = \hat{\mathbf{z}} \cdot d\mathbf{a} = da_z = dxdy, \tag{464}$$

where where $dxdy$ is the surface perpendicular to $\mathbf{r}$. Now we have to divide by r^2, giving

$$\frac{dxdy}{r^2} = \frac{dx}{r}\frac{dy}{r} = d\theta_x d\theta_y = d\Omega, \tag{465}$$

where $d\theta_x$ and $d\theta_y$ are the changes in angle in the x and y direction, respectively. We can therefore write

$$d\Phi = \frac{1}{4\pi\epsilon_0} Q d\Omega \quad \Rightarrow \quad \Phi = \int d\Phi = \frac{1}{4\pi\epsilon_0} Q \int d\Omega = \frac{1}{4\pi\epsilon_0} Q 4\pi = \frac{Q}{\epsilon_0}, \tag{466}$$

where 4π is the solid angle over a sphere (note that the surface of a sphere is $4\pi r^2$. Note that the 4π cancels that in the constant $\frac{1}{4\pi\epsilon_0}$, which looked so clumsy when introducing Coulomb's law. The total result can now be written as

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\epsilon_0}, \tag{467}$$

this is known as Gauss's law. Let us first consider the case, where there is no charge:

$$\int \mathbf{E} \cdot d\mathbf{a} = 0. \tag{468}$$

This says that all the electrical field coming inside the surface should also leave again. Compare this with waterflow. If you study the waterflow in, say, the sea and you consider a certain closed surface enclosing a certain vloume then the amount of water flowing in that volume, should be equal to the amount of water flowing out of the volume. This does not have to be the case if there is a drain or a source of water inside that volume (leaving aside the problem how the pipes to the drain or source get there. Let us just assume that the water just appears or disappears). If there is a drain, then water can just keep flowing into the volume. A negative charge can be compared to a drain of electric field. Oppositely, a positive charge can be seen as a source of electric field, like a watertap. Let us jump ahead a little. We have shown already Maxwell's equations. In there, it also states that

$$\int \mathbf{B} \cdot d\mathbf{a} = 0. \tag{469}$$

That is, there is no source or drain for magnetic fields. We saw for electric charges that they can start or end at a certain point, just look at the electric field lines for a positive and negative point charge. Aparently, this does not happen for magnetic field lines. Since they cannot start or end somewhere, the only option left is that they go in circles. However, we all experience that. Everybody who has been playing with a magnet knows that one side attracts and the other side repels another magnet. This is a direct result of this theorem. If magnetic field is pointing outwards of the magnet at one side, it has to be pointing inwards at the other side.

Examples: Giancoli 22.3-7

We can express this in a more complicated fashion. Let us take an infinitesimally small cube with a volume $v = dxdydz$. Let us consider the flux in the x-direction. The flux enters the cube at the side at $x - \frac{dx}{2}$. Since the cube is infinitesimally small, we can calculate the electric field there by linearization

$$E_x(x - \frac{dx}{2}, y, z) = E_x(x, y, z) - \frac{\partial E_x}{\partial x}\frac{dx}{2}. \tag{470}$$

The flux is then

$$d\Phi_L = \left(E_x(x, y, z) - \frac{\partial E_x}{\partial x}\frac{dx}{2}\right) dydz. \tag{471}$$

In the same way, we can derive the the flux going through the side of the cube at $x + \frac{dx}{2}$:

$$d\Phi_R = \left(E_x(x, y, z) + \frac{\partial E_x}{\partial x}\frac{dx}{2}\right) dydz. \tag{472}$$

The difference between the flux going through the different sides is

$$d\Phi = d\Phi_R - d\Phi_L = \frac{\partial E_x}{\partial x} dxdydz. \tag{473}$$

We can do the same thing for all the sides of the cube, giving

$$d\Phi = \frac{\partial E_x}{\partial x} dxdydz + \frac{\partial E_y}{\partial y} dxdydz + \frac{\partial E_z}{\partial z} dxdydz. \tag{474}$$

We can also express this with the nabla operators ∇,

$$\nabla \cdot \mathbf{E} = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}} \right) \cdot \left(E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}} \right) \cdot = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}. \tag{475}$$

Note that since both ∇ and $\mathbf{E}$ are vectors, we have to use the inner product and we end up with a scalar. This is different from ∇V that we used before, where V is a scalar. The inner product $\nabla \cdot \mathbf{E}$ is usually called the divergence of $\mathbf{E}$. After multiplying, we end up with a vector, $\mathbf{E} = -\nabla V$. We can therefore express the flux through our infinitesimally small volume $dxdydz$ as

$$\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) dv = \nabla \cdot \mathbf{E} dv. \tag{476}$$

On the other hand we can write the charge as an integral over the charge density

$$Q = \int \rho(x, y, z) dxdydz. \tag{477}$$

For an infinitesimally small volume, there is no need to integrate and we have

$$dQ = \rho dxdydz = \rho dv. \tag{478}$$

According to Gauss's law, this should be related to the flux going through the cube

$$d\Phi = \nabla \cdot \mathbf{E} dv = \frac{dQ}{\epsilon_0} = \frac{\rho}{\epsilon_0} dv. \tag{479}$$

We then end up with the differential form of Gauss's law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}. \tag{480}$$

Relation to gravity. Although very few books treat it, Gauss's law is also applicable to gravity in a somewhat modified form. Note that the forces are given by

$$F_{\text{Coulomb}} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \quad \text{and} \quad F_{\text{grav.}} = -G\frac{mM}{r^2}. \tag{481}$$

Note that the acceleration $\mathbf{g} = \mathbf{F}_{\text{grav.}}/m$ is the equivalent of the electric field $\mathbf{E} = \mathbf{F}_{\text{Coulomb}}/q$. Note that the constants are related through $-G \leftrightarrow \frac{1}{4\pi\epsilon_0}$ or $-4\pi G \leftrightarrow \frac{1}{\epsilon_0}$. We can therefore modify Gauss's law for gravity as

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\epsilon_0} \quad \leftrightarrow \quad \int \mathbf{g} \cdot d\mathbf{a} = -4\pi GM. \tag{482}$$

We can use this to calculate the gravitational acceleration inside the Earth. Let us assume that the density of the Earth is constant (which it is not, but we don't want to make our lifes to complicated). Let us consider g at a sphere with radius r inside the Earth. The enclosed mass is then

$$M_{\text{enclosed}} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3_{\text{Earth}}} M. \tag{483}$$

We can now use Gauss's law for gravity

$$g4\pi r^2 = -4\pi G\frac{r^3}{R^3_{\rm Earth}}M \quad \Rightarrow \quad g = -G\frac{r}{R^3_{\rm Earth}}M. \tag{484}$$

So that g is increasing linearly with r the closer you get to the Earth's surface. Note that on the Earth's surface $r = R_{\rm Earth}$, we have

$$g = -G\frac{M}{R^2_{\rm Earth}}. \tag{485}$$

Outside the Earth's surface ($r > R_{\rm Earth}$), we have We can now use Gauss's law for gravity

$$g4\pi r^2 = -4\pi GM \quad \Rightarrow \quad g = -G\frac{M}{r^2}, \tag{486}$$

which is the gravitation law as if the whole Earth was a point mass at the center of the Earth.
Relation to continuity equation. So far we have been discussing Gauss's law

$$\int \mathbf{E}\cdot d\mathbf{a} = \frac{Q}{\epsilon_0}. \tag{487}$$

We saw that we have electric field flowing into and out of surfaces. This looks very much like, for example, fluid flow. In fact, flux is just the Latin word for flow. We also saw that positive charges are sources of electric field and negative charges are sinks. Again, sounds very much like fluid flow. Let us investigate that analogy a bit further. Suppose we have a current $\mathbf{J}$. Again the flux for a constant flow in one direction and a flat surface A would be defined as $\Phi = \mathbf{J}\cdot\mathbf{A} = JA\cos\theta$, where θ is the angle between the direction of the flow and the surface normal. Thus, if the flow is perpendicular to the surface ($\theta = 0°$), then the flux is maximum, and if the current is parallel to the surface ($\theta = 90°$), the flux is zero. Obviously, we would like to consider more complicated current pattterns and surfaces. We can follow the same approach as for the electric field by subdividing the surface in infinitessimally small surfaces $d\mathbf{a}$ and calculate the flux through all these small surfaces. The total flux is then

$$\Phi = \int \mathbf{J}\cdot d\mathbf{a}. \tag{488}$$

Now we need to relate this flux to something. Let us take a closed surface. The current in and out of this surface is related to the change in mass dm/dt inside that surface. If the current is out of the surface ($\Phi > 0$) then the mass inside the surface is decreasing, i.e. $dm/dt < 0$. Vice versa, if the current is into the surface ($\Phi < 0$) then the mass inside the surface is increasing, i.e. $dm/dt > 0$. Therefore, we can relate the change in mass to the flux through the closed surface:

$$\int \mathbf{J}\cdot d\mathbf{a} = -\frac{dm}{dt}. \tag{489}$$

A. Electric field of a dipole

The complexity of calculating an electric field can increase rapidly. Already the field of two point charges is far from trivial. Let us consider the field produced by a positive and negative charge at close proximity, known as a dipole. Let us place the charges on the z axis with the charge $+q$ at $(\frac{a}{2}, 0, 0)$ and the charge the charge $-q$ at $(-\frac{a}{2}, 0, 0)$, see Fig. 29. The magnitude of the electric field $E_\pm = |\mathbf{E}_\pm|$ of the charges separately at a point $(x, 0, z)$ (note that the problem is symmetric around the z-axis, so we can always choose a plane such that $y = 0$) is then

$$E_\pm = \frac{1}{4\pi\epsilon_0}\frac{q}{(z \mp \frac{a}{2})^2 + x^2}. \tag{490}$$

Since we assume that the two charges are close together, we can make the approximation for the limit $a \ll r$,

$$E_\pm = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2 \mp az + \left(\frac{a}{2}\right)^2 + x^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2 \mp az + \left(\frac{a}{2}\right)^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2[1 \mp \frac{az}{r^2} + \left(\frac{a}{2r}\right)^2]} \cong \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}[1 \pm \frac{az}{r^2} - \left(\frac{a}{2r}\right)^2]$$

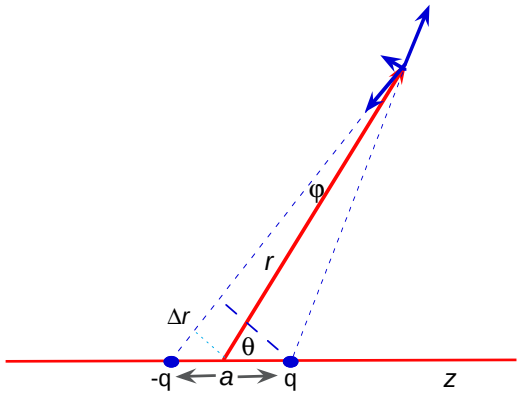


FIG. 29: A dipole is the field produced by a positive and a negative charge.

with $r = x^2 + z^2$. We can neglect the last term, which is of the order $\frac{a^2}{r^2}$ and therefore much smaller than the term $\frac{az}{r^2}$. We have to separate two components: one along the direction from the dipole to the point where we are determining the electric field, indicated by $\hat{\mathbf{r}}$. The other component is perpendicular to that. Let us denote this by $\hat{\boldsymbol{\theta}}$. We can know write the electric field of the positive and negative charges

$$\mathbf{E}_\pm = \pm\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}(1 \pm \frac{az}{r^2})[\cos\varphi\hat{\mathbf{r}} \pm \sin\varphi\hat{\boldsymbol{\theta}}], \tag{491}$$

where θ is the angle of the position vector with the z axis. Note that we are considering the limit that $r \gg a$. In that limit, the length of the thin dotted lightblue line is given by $\frac{a}{2}\sin\theta$. The angle φ is then $\varphi = \frac{a}{2r}\sin\theta$. Since $\varphi \ll 1$, we have $\cos\varphi \cong 1$ and $\sin\varphi \cong \varphi$.

$$\mathbf{E}_\pm = \pm\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}(1 \pm \frac{az}{r^2})[\hat{\mathbf{r}} \pm \frac{a\sin\theta}{2r}\hat{\boldsymbol{\theta}}] \cong \pm\frac{1}{4\pi\epsilon_0}\frac{q}{r^2} + \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\frac{az}{r^2}\hat{\mathbf{r}} + \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\frac{a\sin\theta}{2r}\hat{\boldsymbol{\theta}}, \tag{492}$$

where again we have only retained terms of the order a, neglecting terms of the order a^2. Notice that we have three contribution. The first term looks like the electric field of a positive or negative charge at the origin an have different signs. The second term is a result of the fact that the charges are not at the same distance but that there is a difference of $\Delta r = a\cos\theta = \frac{az}{r}$ between them. Note that this gives the same sign for both charges. For one charge, electric field is less, for the other charge the electric field is more, but the charge is opposite. The third term results from the fact that there is a difference in angles of $\varphi = \frac{a}{2r}\sin\theta$. The field of the dipole is given by the sum of the two charges:

$$\mathbf{E}_+ + \mathbf{E}_- = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left[2a\cos\theta\hat{\mathbf{r}} + a\sin\theta\hat{\boldsymbol{\theta}}\right]. \tag{493}$$

Note that the $1/r^3$ dependence. The field of a dipole therefore decays more rapidly than that of a single point charge.

We could also approach this problem by using the potential. This is in fact simpler since the potential is a scalar function and not a vector. Note that there is a difference in distance of $\Delta r = a\cos\theta$ between the two charges and the point where we want to calculate the potential. This gives a potential

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r - \frac{\Delta r}{2}} + \frac{1}{4\pi\epsilon_0}\frac{(-q)}{r + \frac{\Delta r}{2}} = \frac{1}{4\pi\epsilon_0}\frac{q\Delta r}{(r - \frac{\Delta r}{2})(r + \frac{\Delta r}{2})} = \frac{1}{4\pi\epsilon_0}\frac{q\Delta r}{r^2 - \left(\frac{\Delta r}{2}\right)^2}. \tag{494}$$

The term $\left(\frac{\Delta}{2}\right)^2$ is small compared to the r^2 term and we can neglect it, giving for the potential

$$V = \frac{1}{4\pi\epsilon_0}\frac{q\Delta r}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{qa\cos\theta}{r^2}. \tag{495}$$

The product $p = qa$ is called the dipole moment. Often this is also denoted as a vector. Note that the moment is pointing along the positive z direction, so we can write $\mathbf{p} = p\hat{\mathbf{k}}$. Note that θ is the angle between the position vector $\mathbf{r}$

and the z axis, and therefore also the moment, i.e. the inner product between the dipole moment and the unit vector $\hat{\mathbf{r}}$ is $\mathbf{p} \cdot \hat{\mathbf{r}} = p\cos\theta = qa\cos\theta$. We can therefore also write the potential as

$$V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}. \tag{496}$$

That is very nice you might say, but suppose we wanted to know the electric field and not the potential. However, we know that there is an exact relationship between the potential and the electric field

$$\mathbf{E} = -\nabla V. \tag{497}$$

However, so far we have only used $\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}$. However, the expression $\mathbf{E} = -\nabla V$ is valid in any coordinate system (this is the great advantage of using this funny triangle). We just have to find the ∇ for the coordinate system we are working in. In fact, we are working now in a spherical coordinate system (for the experts, substituting ρ and φ for r and θ, cylindrical coordinates would also work, since we are working in a plane). Looking up ∇ (and everybody looks those up after maybe deriving it once in their lives) we find

$$\nabla = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\boldsymbol{\varphi}}. \tag{498}$$

Notice that the last term contains a partial derivative with respect to φ. Since our expression does not contain φ (due to the fact that we considered the problem only in a plane), this derivative will be zero. We are therefore left with

$$\mathbf{E} = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}} = \frac{1}{4\pi\epsilon_0}\frac{2aq\cos\theta}{r^3}\hat{\mathbf{r}} + \frac{1}{4\pi\epsilon_0}\frac{qa\sin\theta}{r^3}\hat{\boldsymbol{\theta}}, \tag{499}$$

reproducing the result found above.

Homework 9/8/2005, Return 9/22/2005

1. Use the limit definition to compute the derivative of $\frac{1}{1+x}$.

2. Use the limit definition to compute the derivative of $\sqrt{x}$. Do not use a series expansion of $\sqrt{x}$.

3. Differentiate

$$\ln \frac{1}{1+x^2}, \tag{500}$$

make use of the fact that $(\ln x)' = 1/x$.

4. Differentiate

$$\frac{4+\frac{3}{x^3}}{x^2} \tag{501}$$

5. Differentiate

$$\cos^5 x\sqrt{\sin x} \tag{502}$$

make use of the fact that $(\cos x)' = -\sin x$ and $(\sin x)' = \cos x$.

Answers Homework 9/8/2005

1.

$$\lim_{h\to 0}\frac{\frac{1}{1+x+h}-\frac{1}{1+x}}{h}=\lim_{h\to 0}\frac{1}{h}\frac{1+x-(1+x+h)}{(1+x)(1+x+h)}=\lim_{h\to 0}\frac{1}{h}\frac{-h}{(1+x)(1+x+h)}=\frac{-1}{(1+x)^2} \tag{503}$$

2.

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\lim_{h\to 0}\frac{x+h-x}{(\sqrt{x+h}+\sqrt{x})h}=\lim_{h\to 0}\frac{h}{(\sqrt{x+h}+\sqrt{x})h}=\frac{1}{2\sqrt{x}} \tag{504}$$

3. The "hard" way

$$\frac{1}{\frac{1}{1+x^2}}\frac{-1}{(1+x^2)^2}2x=-\frac{2x}{1+x^2}. \tag{505}$$

The "easier" way

$$\ln\frac{1}{1+x^2}=-\ln(1+x^2)\qquad\Rightarrow\qquad(-\ln(1+x^2))'=-\frac{2x}{1+x^2} \tag{506}$$

4.

$$(\frac{4+\frac{3}{x^3}}{x^2})'=(\frac{4}{x^2}+\frac{3}{x^5})'=-\frac{8}{x^3}-\frac{15}{x^6}=-\frac{8x^3+15}{x^6} \tag{507}$$

5.

$$5\cos^4 x(-\sin x)\sqrt{\sin x}+\cos^5 x\frac{1}{2}(\sin x)^{-1/2}\cos x=\frac{-10\cos^4 x\sin^2 x+\cos^6 x}{2\sqrt{\sin x}}, \tag{508}$$

Homework 9/22/2005, Return 10/6/2005

1. A ball is seen to pass upward by a window 25 m above the street with a vertical speed of 14 m/s. If the ball was thrown from the street,
(a) What was the initial velocity?
(b) What altitude does it reach?
(c) When was it thrown?
(d) When does it reach the street again?

2. A skier is accelerating down a 30° hill at 3.8 m/s^2.
(a) What is the vertical component of her acceleration?
(b) How long will it take her to reach the bottom of the hill, assuming she starts from rest and accelerates uniformly, if the elevation change is 250 m?

3. An object is thrown horizontally with a velocity v from a height y_0. A second object is dropped at a distance Δx from a height $y_0 + \Delta y$.
(a) What is the condition for Δy in order for the two objects to hit each other.
(b) Find the expression for v, such that the two objects hit each other at $y_0/2$.

Answers Homework 9/22/2005

1. (a) The height as a function of time is given by

$$y(t) = -\frac{1}{2}t^2 + v_0 t + y_0 \tag{509}$$

From this it follows that $y = 0$ for

$$y = \frac{1}{g}\left\{v_0 \pm \sqrt{v_0^2 + 2gy_0}\right\} = \frac{1}{9.8}(14 \pm \sqrt{14^2 + 29.825}) = \left\{\begin{array}{c} -1.24 \\ 4.10 \end{array}\right. \tag{510}$$

The initial velocity was therefore

$$v(t) = -gt + v_0 = -9.8(-1.24) + 14 = 26.2 \text{ m/s} \tag{511}$$

(b) At the maximum altitude, the velocity is zero

$$v(t) = -gt + v_0 = 0 \qquad \Rightarrow \qquad t = \frac{v_0}{g} = \frac{14}{9.8} = 1.43 \tag{512}$$

(or the average of -1.24 and 4.10). The height is then

$$y(t) = -\frac{1}{2}t^2 + v_0 t + y_0 = -\frac{1}{2}9.8(1.43)^2 + 14 \times 1.43 + 25 = 35 \text{ m} \tag{513}$$

(c),(d) at $t = -1.24$ and 4.1 s, respectively, see (a).

2 (a) The vertical component is

$$a_y = a \sin 30^\circ = 3.8 \times \frac{1}{2} = 1.9 \text{ m/s}^2 \tag{514}$$

(b) The elevation is 250 m. Therefore the distance on the slope is 250/sin 30°=500 m. The acceleration is along the slope. Therefore

$$\frac{1}{2}at^2 = 500 \qquad \Rightarrow \qquad t = \sqrt{\frac{2 \times 500}{3.8}} = 16.2 \ s \tag{515}$$

3. For object 1, the trajectories are given by

$$x_1 = vt \qquad \text{and} \qquad y_1 = -\frac{1}{2}gt^2 + y_0 \tag{516}$$

For object 2, the trajectories are given by

$$x_2 = \Delta x \qquad \text{and} \qquad y_2 = -\frac{1}{2}gt^2 + y_0 + \Delta y. \tag{517}$$

(a) For the two objects to hit each other, we need $x_1(t_{\rm hit}) = x_2(t_{\rm hit})$ and $y_1(t_{\rm hit}) = y_2(t_{\rm hit})$. From the second equality, we directly see the $y_0 = y_0 + \Delta y$. Therefore, $\Delta y = 0$.
(b) The two objects hit each other at the time $vt_{\rm hit} = \Delta x$, therefore $t_{\rm hit} = \Delta x/v$. The objects are then at a height

$$y = -\frac{1}{2}gt_{\rm hit}^2 + y_0, \tag{518}$$

which should equal $y_0/2$. Therefore,

$$\frac{1}{2}gt_{\rm hit}^2 = \frac{1}{2}y_0 \qquad \Rightarrow \qquad \frac{1}{2}g\left(\frac{\Delta x}{v}\right)^2 = \frac{1}{2}y_0 \qquad \Rightarrow \qquad v = \sqrt{\frac{g}{y_0}}\Delta x \tag{519}$$

Homework 10/6/2005

1. Giancoli 4.68.
2. Giancoli 5.21.
3. Giancoli 5.83.
4. Giancoli 5.92.

Answers Homework 10/20/2005

1. Both blocks are at rest, therefore

$$\mathbf{F}_{g1} + \mathbf{F}_{N1} + \mathbf{F}_{T1} = 0 \tag{520}$$
$$\mathbf{F}_{g2} + \mathbf{F}_{N2} + \mathbf{F}_{T2} = 0. \tag{521}$$

Choosing the x-axis along the slope, we can write down in this direction

$$m_1 \sin\theta_1 - F_T = 0 \tag{522}$$
$$-m_2 \sin\theta_2 + F_T = 0. \tag{523}$$

Adding gives

$$m_1 \sin - m_2 \sin\theta_2 = 0 \qquad \Rightarrow \qquad \frac{m_1}{m_2} = \frac{\sin\theta_1}{\sin\theta_2}. \tag{524}$$

For the tension force, we find

$$F + T = m_1 \sin\theta_1 = m_2 g \sin\theta_2. \tag{525}$$

Substituting the numbers is left up to the reader.

2. For block 1,

$$\mathbf{F}_{r1} + \mathbf{F}_{g1} + \mathbf{F}_{N1} + \mathbf{F}_{T1} = m\mathbf{a} \tag{526}$$

For block 2,

$$\mathbf{F}_{r2} + \mathbf{F}_{g2} + \mathbf{F}_{N2} + \mathbf{F}_{T2} = m\mathbf{a}. \tag{527}$$

Let us take the x-axis along the slope. We then have for block 1,

$$mg\sin\theta - F_T - F_{r1} = ma \tag{528}$$
$$mg\cos\theta - F_{N1} = 0 \tag{529}$$

and for block 2,

$$mg\sin\theta + F_T - F_{r1} = ma \tag{530}$$
$$mg\cos\theta - F_{N2} = 0. \tag{531}$$

Since the blocks have equal mass, they both have a normal force $F_{N1} = F_{N2} = mg\cos\theta$. We can rewrite the equations in the x-direction as

$$mg\sin\theta - F_T - \mu_1 mg\cos\theta = ma \tag{532}$$
$$mg\sin\theta + F_T - \mu_2 mg\cos\theta = ma. \tag{533}$$

Adding gives,

$$2mg\sin\theta - (\mu_1 + \mu_2)mg\cos\theta = 2ma \qquad \Rightarrow \qquad a = mg\sin\theta - \frac{1}{2}(\mu_1 + \mu_2)mg\cos\theta. \tag{534}$$

The tension force is then

$$F_t = ma - mg\sin\theta + \mu_2 mg\cos\theta = mg\sin\theta - mg\sin\theta - \frac{1}{2}(\mu_1 + \mu_2)mg\cos\theta + \mu_2 mg\cos\theta \tag{535}$$
$$= \frac{1}{2}(\mu_2 - \mu_1)mg\cos\theta. \tag{536}$$

Substituting the numbers is left up to the reader.

3. The system is at rest, therefore

$$\mathbf{F}_g + \mathbf{F}_N + \mathbf{F}_r = 0. \tag{537}$$

For our axis system, we choose x along the sphere and y perpendicular to the sphere. We then have

$$\begin{aligned} -mg\cos\theta + F_N &= 0, & (538)\\ mg\sin\theta - F_r &= 0. & (539) \end{aligned}$$

From this we find $F_N = mg\cos\theta$ and

$$mg\sin\theta - \mu_s mg\cos\theta = 0 \qquad \Rightarrow \qquad \tan\theta = \mu_s. \tag{540}$$

4. For the equation of motion, we have

$$\mathbf{F}_g + \mathbf{F}_N = m\mathbf{a}. \tag{541}$$

or

$$\begin{aligned} F_N\sin\theta &= m\frac{v^2}{l} & (542)\\ -mg + F_N\cos\theta &= 0. & (543) \end{aligned}$$

From this we find

$$F_N = \frac{mg}{\cos\theta} \qquad \Rightarrow \qquad mg\tan\theta = \frac{mv^2}{r\sin\theta} = \frac{m(2\pi f r\sin\theta)^2}{r\sin\theta} = m4\pi^2 f^2 r\sin\theta. \tag{544}$$

We can write this as

$$\cos\theta = \frac{g}{(2\pi f)^2 r} \tag{545}$$

Substituting gives

$$\cos\theta = \frac{9.8}{(2\pi 4)^2 0.2} \cong 0.078 \tag{546}$$

Which gives $\theta = 85.5°$.

Homework 3/11/2005, Return 17/11/2005

1. Two objects, each with speed v make a completely inelastic collision (i.e. after the collision they move together as one objects). Just before they hit, their velocities are given by

$$\mathbf{v}_1 = v\hat{\mathbf{i}} \qquad \text{and} \qquad \mathbf{v}_2 = v\cos\theta\hat{\mathbf{i}} + v\sin\theta\hat{\mathbf{j}}. \tag{547}$$

The masses are $m_1 = m$ and $m_2 = 2m$.
(a) Find the velocity of the composite object after the collision and the angle φ that this velocity makes with the x-axis.
(b) What is the total kinetic energy of the objects?

2. Giancoli 8.85.

3. Let us take a uniformally charged rod of length l with a positive charge Q. The rod is lying in the x direction with its center at $x = 0$. Calculate the electric field in the x direction at a distance a from the end of the rod.

Answers Homework 3/11/2005

1(a). In a collision the total momentum is conserved.

$$p_{1x} + p_{2x} = P'_x \qquad \Rightarrow \qquad mv + 2mv\cos\theta = 3mv'_x\cos\varphi \tag{548}$$
$$p_{1y} + p_{2y} = P'_y \qquad \Rightarrow \qquad 0 + 2mv\sin\theta = 3mv'\sin\varphi = 3mv'_y \tag{549}$$
$$\tag{550}$$

From this we find

$$\mathbf{v}' = \frac{v}{3}(1+2\cos\theta)\hat{\mathbf{i}} + \frac{2v}{3}\sin\theta\hat{\mathbf{j}} \tag{551}$$

and

$$\tan\varphi = \frac{2\sin\theta}{1+2\cos\theta} \tag{552}$$

(b)The kinetic energy of the composite particle before the collision is

$$E_{\rm kin} = \frac{1}{2}mv^2 + \frac{1}{2}2mv^2 = \frac{3}{2}mv^2 \tag{553}$$

The kinetic energy of the composite particle after the collision is

$$E'_{\rm kin} = \frac{1}{2}3mv'^2 = \frac{1}{6}m\left\{(1+2\cos\theta)^2 + (2\sin\theta)^2\right\}v^2 = \frac{1}{6}m(1+4\cos\theta+4\cos^2\theta+4\sin^2\theta)v^2 = \frac{1}{6}m(5+4\cos\theta)v^2 \tag{554}$$

The difference is therefore

$$\Delta E_{\rm kin} = E'_{\rm kin} - E_{\rm kin} = \frac{1}{6}m(5+4\cos\theta) - \frac{3}{2}mv^2 = -\frac{2}{3}m(1-\cos\theta). \tag{555}$$

Note that this is less than zero. Therefore, kinetic energy has been lost in the collision.

2. At the top, we have

$$F_N + mg = \frac{mv^2}{R}. \tag{556}$$

At the bottom, we have

$$F'_N - mg = \frac{mv'^2}{R}. \tag{557}$$

The velocity follows from conservation of energy

$$\frac{1}{2}mv^2 + mg(2R) = \frac{1}{2}mv'^2 \qquad \Rightarrow \qquad mv'^2 = mv^2 + 4mgR \tag{558}$$

This gives

$$F'_N = \frac{mv'^2}{R} + mg = \frac{mv^2}{R} + 5mg = F_N + 6mg \tag{559}$$

3. The electric field from a piece of the rod at a distance x from where we want to determine the electric field.

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{dQ}{x^2}\hat{\mathbf{i}} \tag{560}$$

The charge is given by $dQ = \frac{Q}{l}dx$. We can therefore write for the electric field

$$\mathbf{E} = \int_a^{l+a}\frac{1}{4\pi\epsilon_0}\frac{Q}{lx^2}dx\hat{\mathbf{i}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{l}\left[-\frac{1}{x}\right]_a^{l+a}\hat{\mathbf{i}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{l}\left(\frac{1}{a} - \frac{1}{a+l}\right)\hat{\mathbf{i}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{a(a+l)}\hat{\mathbf{i}} \tag{561}$$

Final 252, 12/6/2005 noon-1.50pm

1. Two balls of mass m are shot at each other. One ball is shot with a velocity v under an angle θ with the ground in the positive x direction from the position $x = -d$. The other ball is shot in the negative x direction with the same velocity v and angle θ from a position $x = d$.
(a) For what angle θ do the two balls hit each other at the maximum height?
(b) Suppose the balls form one ball (say, they are made out of clay). What is the kinetic energy lost in the collision?
(c) What is the velocity when the balls hit the ground?

2. Let us consider charges arranged in the following way

(a) What is the electric field at a point at a distance r from the charge $-2Q$ in the positive z direction. Find the approximate the expression for $r \gg d$.
(b) What is the electric field at a point at a distance r from the charge $-2Q$ in the positive y direction. Find the approximate the expression for $r \gg d$.

3. An insulating washer (see Figure) with an inner radius a and an outer radius b has a charge Q uniformly distributed on its surface

(a) Calculate the electric potential at a point P located at a distance z from the washer along the symmetry axis perpendicular to the washer.
(b) What is the electric field component E_z as a function of z.
(c) What is the electric field at $z = 0$ m.

Final 252, 12/6/2005 noon-1.50pm 1(a) For the first ball, we have

$$x = v\cos\theta t - d \tag{562}$$

$$y = -\frac{1}{2}gt^2 + v\sin\theta \tag{563}$$

The balls hit at $x = 0$ at a time $t = d/v\cos\theta$. To hit at a maximum, we need

$$v_y = \frac{dy}{dt} = -gt + v\sin\theta = 0 \qquad \Rightarrow \qquad -g\frac{d}{v\cos\theta} + v\sin\theta = 0 \qquad \Rightarrow \qquad \tan\theta = \frac{gd}{v^2} \tag{564}$$

(b) Before the collision, the kinetic energy was $2 \times \frac{1}{2}mv^2\cos^2\theta$. This is also the energy lost in the collision.
(c) The maximum height reached is

$$y = v\sin\theta t = v\sin\theta\frac{d}{v\cos\theta} = d\tan\theta \tag{565}$$

From conservation of energy, it follows

$$2mgd\tan\theta = \frac{1}{2}2mv^2 \qquad \Rightarrow \qquad v = \sqrt{2gd\tan\theta} \tag{566}$$

2(a) In the z direction,

$$E = \frac{1}{4\pi\epsilon_0}Q\left(\frac{1}{(r-d)^2} - \frac{2}{r^2} + \frac{1}{(r+d)^2}\right) \tag{567}$$

$$= \frac{1}{4\pi\epsilon_0}Q\left(\frac{1}{r^2(1-d/r)^2} - \frac{2}{r^2} + \frac{1}{r^2(1+d/r)^2}\right) \tag{568}$$

$$\cong \frac{1}{4\pi\epsilon_0}Q\left(\frac{1}{r^2(1-d/r)^2} - \frac{2}{r^2} + \frac{1}{r^2(1+d/r)^2}\right) \tag{569}$$

Homework 9-25-2006

Problem 1

(a) Give the derivative of

$$f(x) = (3x + x^2)e^{x^2} \tag{570}$$

(b) Find the derivative of $f(x) = \frac{1}{x}$ using

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{571}$$

Problem 2

A swimmer is standing on the edge of a jump board 5 m above a swimming pool. The diver jumps up with a velocity of 3 m/s. Consider only the movements in the vertical direction and neglect the movement in the horizontal direction. You can assume that the jump board is rigid and neglect air resistance.

(a) After jumping up, the swimmer will pass the jump board again. From that moment, how long does it take for the swimmer to reach the water?

(b) What is the velocity of the swimmer when she passes the jump board again?

(c) What is the velocity when the swimmer reaches the water? Give the velocity in km/h.

Problem 3

An object has an acceleration

$$a(t) = 2t - 5. \tag{572}$$

Its velocity and position at $t = 0$ s are $v_0 = 6$ m/s and $x_0 = 4$ m, respectively.

(a) When is the velocity zero?

(b) Determine $x(t)$.

Answers Homework 9-25-2006

Problem 1

(a)

$$f(x) = (3x + x^2)e^{x^2} \quad \Rightarrow \quad f'(x) = (3 + 2x)e^{x^2} + (3x + x^2)e^{x^2}2x) = (3 + 2x + 6x^2 + 2x^3)e^{x^2} \tag{573}$$

(b)

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h} \frac{x - (x + h)}{(x + h)x} = \lim_{h\to 0} \frac{1}{h} \frac{-h}{(x + h)x} = \lim_{h\to 0} \frac{-1}{(x + h)x} = -\frac{1}{x^2} \tag{574}$$

Problem 2

(a) The equation of motion is given by

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0 = -4.9t^2 + 3t + 5. \tag{575}$$

We are asked to calculate to times

$$y(t) = y_0 \quad \text{and} \quad y(t) = 0. \tag{576}$$

The first can be calculated by solving

$$-\frac{1}{2}gt^2 + v_0t + y_0 \;=\; y_0 \quad \Rightarrow \quad -\frac{1}{2}gt^2 + v_0t = 0 \tag{577}$$

$$\Rightarrow \quad t = 0 \;\text{ and }\; t = \frac{2v_0}{g} = 0.61224 \text{ s} \tag{578}$$

The second time can be obtained from

$$-\frac{1}{2}gt^2 + v_0t + y_0 = 0 \quad \Rightarrow \quad t_{1,2} \;=\; \frac{1}{g}(v_0 \pm \sqrt{v_0^2 + 2gy_0}) \tag{579}$$

$$=\; \frac{1}{9.8}(3 \pm \sqrt{9 + 2 \times 9.8 \times 5}) = -0.7493 \;\text{ and }\; 1.361 \;\text{ s} \tag{580}$$

The second time is the one we need, so the time difference is $\Delta t = 1.361 - 0.6122 = 0.75$ s.

(b) The velocity when the swimmer passes the jumpboard again is

$$v(0.6122) = -9.8 \times 0.6122 + 3 = -3 \;\text{ m/s} \tag{581}$$

(c) The velocity when the swimmer passes the jumpboard again is

$$v(1.36164) = -9.8 \times 1.36164 + 3 = -10.34 \;\text{ m/s} = -10.34 \times \frac{10^{-3} \text{ km}}{\frac{1}{3600} \text{ h}} = -10.34 \times 3.6 = -37.2 \;\text{ km/h}. \tag{582}$$

Problem 3

(a) The velocity is given by

$$a(t) = 2t - 5 \quad \Rightarrow \quad v(t) = t^2 - 5t + 6 \;\text{ m/s}. \tag{583}$$

This is zero for

$$v(t) = t^2 - 5t + 6 = (t-2)(t-3) = 0 \quad \Rightarrow \quad t = 2 \ \text{ and } \ 3 \ \text{ s.} \tag{584}$$

(b) Integrating again gives

$$x(t) = \frac{1}{3}t^3 - \frac{5}{2}t + 6t + 4 \ \text{ m.} \tag{585}$$

Homework 10-9-2006

Problem 1
Two people are shooting objects into the air and they want their objects to hit. Person 1 is at the ground and shoots an object with a velocity $v = 2$ m/s at an angle of 30° with the surface. Person 2 is shooting back at a distance $x_0 = 10$ m at an angle of 45° with the same velocity. Person 2 is standing at a height y_0.

(a) What is the x position when the objects hit?

(b) At what height y_0 does person 2 have to stand if they want the two objects to hit each other?

(c) At what height do the objects hit?

Problem 2
Two blocks are on a slope that makes an angle θ with the horizontal. We will consider the motion of the two blocks down the slope and ignore any motion sideways. Block 1 with mass m_1 is higher on the slope. Block 2 with mass m_2 is lower on the slope and attached to block 1 via a massless cord. Block 1 and 2 have kinetic friction coefficients μ_k^1 and μ_k^2, respectively, with the surface. (Suppose we could change the friction coefficients by some lubricant).

(a) Give the expressions for the accelerations for object 1 and 2 if $\mu_k^1 > \mu_k^2$?

(b) Give the expressions for the accelerations for object 1 and 2 if $\mu_k^1 < \mu_k^2$ (think carefully!)?

Answers Homework 10-9-2006

Problem 1

(a)

$$\text{object 1}: \; x_1(t) = v\cos 30^\circ t \;\text{ and }\; y_1(t) = -\frac{1}{2}gt^2 + v\sin 30^\circ \tag{586}$$

$$\text{object 2}: \; x_2(t) = -v\cos 45^\circ t + x_0 \;\text{ and }\; y_2(t) = -\frac{1}{2}gt^2 + v\sin 45^\circ + y_0 \tag{587}$$

Let us assume that hit at a time t_h. We then have

$$x_1(t_h) \;=\; x_2(t_h) \;\Rightarrow\; v\cos 30^\circ t_h = -v\cos 45^\circ t_h + x_0 \tag{588}$$

$$\Rightarrow\; t_h = \frac{x_0}{v\cos 30^\circ + v\cos 45^\circ} = \frac{x_0}{\frac{1}{2}\sqrt{3}v + \frac{1}{2}\sqrt{2}v} = \frac{10}{2(\frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{2})} = 3.18 \text{ s} \tag{589}$$

Note that this looks like the time needed for one object to travel at distance x_0 at a velocity $v\cos 30^\circ + v\cos 45^\circ$. The objects combine their velocities in the x-direction to travel the distance x_0. The position is then

$$x_1(3.14) = v\cos 30^\circ t_h = 2\frac{1}{2}\sqrt{3}3.18 = 5.51 \text{ m}. \tag{590}$$

(b) When they hit each other, we have the condition

$$y_1(t_h) \;=\; y_2(t_h) \;\Rightarrow\; -\frac{1}{2}gt_h^2 + v\sin 30^\circ t_h - \frac{1}{2}gt_h^2 + v\sin 45^\circ t_h + y_0 \tag{591}$$

$$\Rightarrow\; y_0 = v\sin 30^\circ t_h - v\sin 45^\circ t_h = \frac{v\sin 30^\circ - v\sin 45^\circ}{v\cos 30^\circ + v\cos 45^\circ}x_0 = \frac{\sin 30^\circ - \sin 45^\circ}{\cos 30^\circ + \cos 45^\circ}x_0 \tag{592}$$

$$= \frac{1-\sqrt{2}}{\sqrt{3}+\sqrt{2}}x_0 = -1.32 \text{ m}. \tag{593}$$

This means that person 2 has to dig a hole in the ground. Does this make sense? Yes, because object 2 is shot at a steeper angle an therefore will get higher.

(c) The height is then

$$y_1(t_h) = -\frac{1}{2}gt_h^2 + v\sin 30^\circ t_h = -\frac{1}{2}9.8(3.18)^2 + 2\frac{1}{2}3.18 = -46.37 \text{ m} \;\text{ or (as check)} \tag{594}$$

$$y_2(t_h) = -\frac{1}{2}gt_h^2 + v\sin 45^\circ t_h + y_0 = -\frac{1}{2}9.8(3.18)^2 + 2\frac{1}{2}\sqrt{2}3.18 - 1.32 = -46.37 \text{ m} \tag{595}$$

Therefore, the people should stand at the edge of a cliff.

Problem 2
(a) If $\mu_k^1 > \mu_k^2$, block 2 would like to slide down the slope faster than block 1. The cord will be tight and there will be a tension force and the block more with the same acceleration. For block 1, we have

$$\mathbf{F}_{g1} + \mathbf{F}_{N1} + \mathbf{F}_{fr,1} + \mathbf{F}_{T1} = m_1\mathbf{a} \tag{596}$$

Taking the positive x-axis down the slope gives

$$x: \quad m_1 g\sin\theta + F_T - F_{fr,1} = m_1 a \tag{597}$$

$$y: \quad -m_1 g\cos\theta + F_{N1} = 0 \tag{598}$$

The second equation gives $F_{N1} = m_1 g\cos\theta$. Inserting in the equation for x gives

$$m_1 g\sin\theta + F_T - \mu_k^1 m_1 g\cos\theta = m_1 a \tag{599}$$

For block 2, we can do (almost) the same thing

$$\mathbf{F}_{g2} + \mathbf{F}_{N2} + \mathbf{F}_{fr,2} + \mathbf{F}_{T2} = m_2\mathbf{a} \tag{600}$$

Taking the positive x-axis down the slope gives

$$x: \quad m_2 g\sin\theta - F_T - F_{fr,2} = m_2 a \tag{601}$$
$$y: \quad -m_2 g\cos\theta + F_{N2} = 0 \tag{602}$$

The second equation gives $F_{N2} = m_2 g\cos\theta$. Inserting in the equation for x gives

$$m_2 g\sin\theta - F_T - \mu_k^2 m_2 g\cos\theta = m_2 a \tag{603}$$

Adding the expressions for a for the different objects gives

$$(m_1+m_2)a = (m_1+m_2)g\sin\theta - (\mu_1^2 m_1 g + \mu_k^2 m_2 g)\cos\theta \tag{604}$$

or

$$a = g\sin\theta - \frac{\mu_1^2 m_1 + \mu_k^2 m_2}{m_1+m_2} g\cos\theta. \tag{605}$$

(b) If $\mu_k^1 < \mu_k^2$, block 1 would like to slide down the slope faster than block 2. The cord will be loose and there will be no tension force. (at some point the objects will bump into each other but let us ignore that. The blocks therefore move independently:

$$m_1 g\sin\theta - \mu_k^1 m_1 g\cos\theta = m_1 a \quad \Rightarrow \quad a = g\sin\theta - \mu_k^1 g\cos\theta \tag{606}$$

and for block 2

$$m_2 g\sin\theta - \mu_k^2 m_2 g\cos\theta = m_2 a \quad \Rightarrow \quad a = g\sin\theta - \mu_k^2 g\cos\theta \tag{607}$$

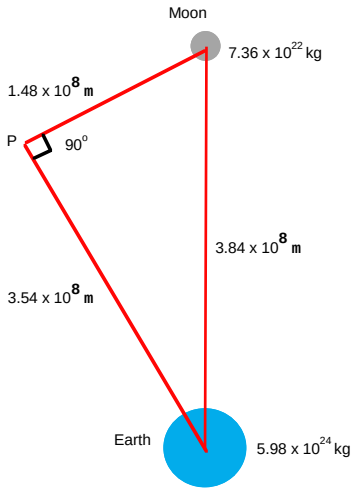


FIG. 30:

Homework 10-23-2006

Problem 1

(a) What is the total acceleration caused by both the Earth and the Moon at point P in the Figure above?

(b) What is the magnitude of the total acceleration at this point?

(c) What is the total gravitational force on a spacecraft at location P? The mass of the spacecraft is 1200 kg. What is the magnitude of the total gravitational force on the spacecraft?

Problem 2

An object is sitting is on a cone that is rotating with a certain velocity v, see Figure. There is a static friction coefficient $\mu_s = 0.8$. Calculate the maximum velocity that the cone can turn before the object starts to slide. Assume that the object is small with respect to the cone.
(originally posted as 0.4, however, object will always slide at 0.4).

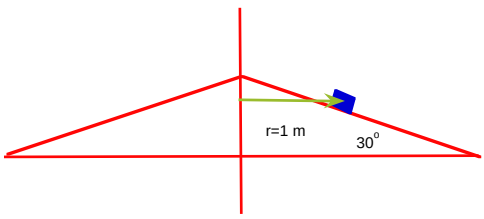


FIG. 31:

Answers Homework 10-23-2006

Problem 1

(a) The accelaration is given by

$$g_{\rm M} = \frac{GM}{r^2}. \tag{608}$$

Since the position vectors of the Moon and Earth make an angle of 90° with respect to each other, it is covenient to choose P-Earth as the x-axis and P-Moon as the y-axis. This gives as acceleration

$$\begin{aligned} \mathbf{g} &= \frac{GM_{\rm Earth}}{r_{\rm P-Earth}}\mathbf{i} + \frac{GM_{\rm Moon}}{r_{\rm P-Moon}}\mathbf{j} = \frac{6.67\times10^{-11}5.98\times10^{24}}{(3.54\times10^8)^2}\mathbf{i} + \frac{6.67\times10^{-11}\ 7.36\times10^{22}}{(1.48\times10^8)^2}\mathbf{j} & (609)\\ &= 3.18\times10^{-3}\mathbf{i} + 2.24\times10^{-4}\mathbf{j}. & (610)\end{aligned}$$

(b) The magnitude is then

$$g = \sqrt{(3.18\times10^{-3})^2 + (0.22\times10^{-3})} = 3.19\times10^{-3}\ \ {\rm m/s^2} \tag{611}$$

(c) The force is then

$$F = mg = 1200(3.18\times10^{-3}\mathbf{i} + 2.24\times10^{-4}\mathbf{j}) = 3.82\mathbf{i} + 0.26\mathbf{j}\ \ {\rm N} \tag{612}$$

with a magnitude

$$F = mg = 1200\times3.19\times10^{-3} = 3.83\ \ {\rm N} \tag{613}$$

Problem 2

The equation of motion is

$$\mathbf{F}_g + \mathbf{F}_{\rm fr} + \mathbf{F}_N = m\mathbf{a}. \tag{614}$$

Splitting in components gives

$$\begin{aligned} x: \quad F_{\rm fr}\cos30^\circ - F_N\sin30^\circ &= m\frac{v^2}{r} & (615)\\ y: \quad F_{\rm fr}\sin30^\circ + F_N\cos30^\circ - mg &= 0. & (616)\end{aligned}$$

Since $F_{\rm fr} = \mu_s F_N$, we find

$$\begin{aligned} (\mu_s\cos30^\circ - \sin30^\circ)F_N &= m\frac{v^2}{r} & (617)\\ (\mu_s\sin30^\circ + \cos30^\circ)F_N - mg &= 0. & (618)\end{aligned}$$

The second equation gives

$$F_N = \frac{mg}{\mu_s\sin30^\circ + \cos30^\circ}, \tag{619}$$

inserting in the equation for the x direction leads to

$$\frac{\mu_s\cos30^\circ - \sin30^\circ}{\mu_s\sin30^\circ + \cos30^\circ} = m\frac{v^2}{r} \tag{620}$$

which gives for v,

$$v = \sqrt{\frac{\mu_s\cos30^\circ - \sin30^\circ}{\mu_s\sin30^\circ + \cos30^\circ}rg} = \sqrt{\frac{0.8\frac{1}{2}\sqrt{3} - \frac{1}{2}}{0.8\frac{1}{2} + \frac{1}{2}\sqrt{3}}\times9.8} = 0.41\ \ {\rm m/s} \tag{621}$$

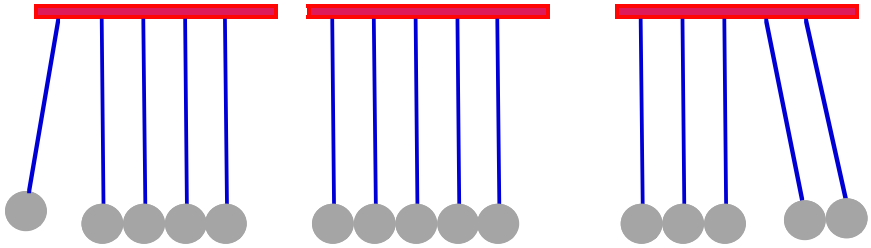


FIG. 32: Problem 1

Homework 11-6-2006

Problem 1
Let us have a look at the toy where steel balls of equal mass bump into each other, see Figure. The collisions of the steel balls can be considered elastic. Therefore, we have conservation of energy and momentum. If we let one ball collide with the other balls, see Figure, why don't we see two balls swinging out at the other side?

Problem 2
A model rocket is fired from the ground in a parabolic trajectory. At the top of the trajectory, at a horizontal distance of 260 m from the launch point, an explosion occurs within the rocket, breaking it into two fragments. One fragment, having one-third of the mass of the rocket falls straight down to Earth as if it had been dropped from rest at that point. At what horizontal distance from the launch point does the other fragment land?

Problem 3
Two pendulums of equal length $l = 50$ cm are suspended from the same point. The pendulum bobs are steel spheres of masses 140 and 390 g. The more massive bob is drawn back to make a 15° angle with the vertical, see Figure. When it is released the bobs collide elastically. What is the maximum angle made by the less massive pendulum?

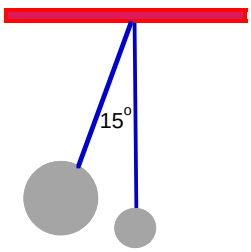


FIG. 33: Problem 3

Answers Homework 11-6-2006

Problem 1
First, we have conservation of momentum

$$mv = mv' + mv' \quad \Rightarrow \quad v' = \frac{1}{2}v. \tag{622}$$

However, we also need conservation of energy

$$\frac{1}{2}mv^2 \neq \frac{1}{2}mv'^2 + \frac{1}{2}mv'^2 = \frac{1}{4}mv^2. \tag{623}$$

Therefore, this would mean that kinetic energy is lost in the collision, which is in contradiction with the statement that the collision were elastic.

Problem 2
Since we are at the top of the trajectory, the momentum is in the horizontal direction

$$p_x = mv_x. \tag{624}$$

Object 1 with mass $\frac{1}{3}m$ falls down as if it were dropped, implying that $p'_{1x} = p'_{1y} = 0$. Since we have conservation of momentum, this means that

$$mv_x = \frac{1}{3}mv'_{1x} + \frac{2}{3}mv'_{2x} = \frac{2}{3}mv'_{2x} \tag{625}$$

$$mv_y = \frac{1}{3}mv'_{1y} + \frac{2}{3}mv'_{2y} = \frac{2}{3}mv'_{2y} \tag{626}$$

This means that $v'_{2x} = \frac{3}{2}v_x$ and $v'_{2y} = 0$. Since the velocity is increased, by a factor $\frac{3}{2}$, the distance travelled before hitting the ground will also increase by a factor $\frac{3}{2}$. The total distance is therefore

$$\Delta x = 260 + \frac{3}{2} \times 260 = 650 \ \text{ m}. \tag{627}$$

Problem 3
The velocity of the heavy ball M can be determined from the law of conservation of energy

$$Mgh = \frac{1}{2}Mv_1^2 \quad \Rightarrow \quad v_1 = \sqrt{2gh}. \tag{628}$$

The length g can be expressed in terms of the length of the pendulum l and the angle $\theta = 15°$:

$$h = l - l\cos\theta \quad \Rightarrow \quad v_1 = \sqrt{2gl(1-\cos\theta)} = \sqrt{2 \times 9.8 \times 0.5(1-\cos 15°)} = 0.577 \ \frac{\text{m}}{\text{s}}. \tag{629}$$

At the moment of the collision, we have conservation of momentum

$$Mv_1 = Mv'_1 + mv'_2. \tag{630}$$

We also have the relationship between the relative velocities

$$v_1 = v'_2 - v'_1 \quad \Rightarrow \quad v'_1 = v'_2 - v_1. \tag{631}$$

Substituting this in the equation above

$$Mv_1 = M(v'_2 - v_1) + mv'_2 \quad \Rightarrow \quad v'_2 = \frac{2M}{m+M}v_1 = \frac{2 \times 0.39}{0.39 + 0.14}0.577 = 0.85 \ \frac{\text{m}}{\text{s}} \tag{632}$$

The height than can be reached is then

$$mgh' = \frac{1}{2}mv'^2_2 = \frac{1}{2}m\left(\frac{2M}{m+M}v_1\right)^2 = \frac{1}{2}m\left(\frac{2M}{m+M}\right)^2 2gl(1-\cos\theta) \tag{633}$$

$$h' = \frac{4M^2}{(m+M)^2}l(1-\cos\theta) = \frac{4 \times 0.39^2}{(0.39+0.14)}0.5(1-\cos 15°) = 0.036 \ \text{ m} \tag{634}$$

$$\tag{635}$$

Giving for the angle

$$\varphi' = \arccos(1 - \frac{h'}{l}) = 22.2^\circ \tag{636}$$

Homework 11-20-2006

Problem 1
An insulating washer with an inner radius a and an outer radius b has a charge Q uniformly distributed on its surface.
(a) Calculate the electric potential at a point P located at a distance z from the washer along the symmetry axis perpendicular to the washer.
(b) What is the electric field component E_z as a function of z.
(c) What is the electric field in the following cases:

- $z = 0$.
- $b \to \infty$.
- $a \to 0$.

Problem 2
Let us take a uniformally charged rod of length l with a positive charge Q. The rod is lying in the x direction with its center at $x = 0$. Calculate the electric field in the x direction at a distance a from the end of the rod.

Homework 11-20-2006

Problem 1
The charge density is given by

$$\rho = \frac{Q}{\pi r^2} \quad \Rightarrow \quad Q = \sigma \pi r^2 \quad \Rightarrow \quad dQ = \sigma 2pirdr. \tag{637}$$

The potential of a ring is given by

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r} = \frac{1}{4\pi\epsilon_0}\frac{2\sigma\pi r dr}{r^2+z^2}. \tag{638}$$

The total potential of the washer is then

$$\int_a^b dV = \frac{1}{4\pi\epsilon_0}\int_a^b \frac{2\sigma\pi r}{r^2+z^2}dr = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+z^2}\right]_a^b = \frac{\sigma}{2\epsilon_0}(\sqrt{b^2+z^2}-\sqrt{a^2+z^2}) \tag{639}$$

(b) The electric field is given by

$$E = -\frac{dV}{dz} = \frac{\sigma}{2\epsilon_0}\left[\frac{z}{\sqrt{a^2+z^2}} - \frac{z}{\sqrt{b^2+z^2}}\right] \tag{640}$$

(c)

- z=0 gives $E = 0$.
- $b \to \infty$ gives

$$E = \frac{\sigma}{2\epsilon_0}\frac{z}{\sqrt{a^2+z^2}} \tag{641}$$

- $a \to 0$ gives

$$E = -\frac{dV}{dz} = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{b^2+z^2}}\right] \tag{642}$$

Problem 2
The electric field from a piece of the rod at a distance x from where we want to determine the electric field.

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{dQ}{x^2}\hat{\mathbf{i}} \tag{643}$$

The charge is given by $dQ = \frac{Q}{l}dx$. We can therefore write for the electric field

$$\mathbf{E} = \int_{a-l/2}^{a+l/2}\frac{1}{4\pi\epsilon_0}\frac{Q}{lx^2}dx\hat{\mathbf{i}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{l}\left[-\frac{1}{x}\right]_{a-l/2}^{a+l/2}\hat{\mathbf{i}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{l}\left(\frac{1}{a-\frac{l}{2}} - \frac{1}{a+\frac{l}{2}}\right)\hat{\mathbf{i}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{a^2-\frac{l^2}{4}}\hat{\mathbf{i}} \tag{644}$$

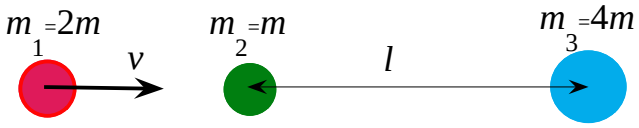


FIG. 34: Problem 1

Final 252, 12-11-2006, 12.00

Problem 1
Three balls with masses $m_1 = 2m$, $m_2 = m$, and $m_3 = 4m$ go into a series of head-on collisions. At $t = 0$, ball 1 hits ball 2 with a velocity v. Both ball 2 and 3 are initially at rest, and separated by a distance l, see Figure. At what time (expressed in l and v) do ball 1 and 2 hit again?

Problem 2
(a) The Figure below shows a quadrupole consisting of two positive charges q separated by $2s$ and a negative charge $-2q$ in the middle. Show that the potential at a distance x perpendicular to the quadrupole (point P in the figure) is

$$V = -\frac{1}{4\pi\epsilon_0}\frac{qs^2}{x^3}, \qquad (x \gg s) \tag{645}$$

in the limit $x \gg s$. Make use of the fact that $(1+y)^n \cong 1 + ny$ if $y \ll 1$ for any real value of n.
(b) Calculate the size and direction of the electric field at point P.

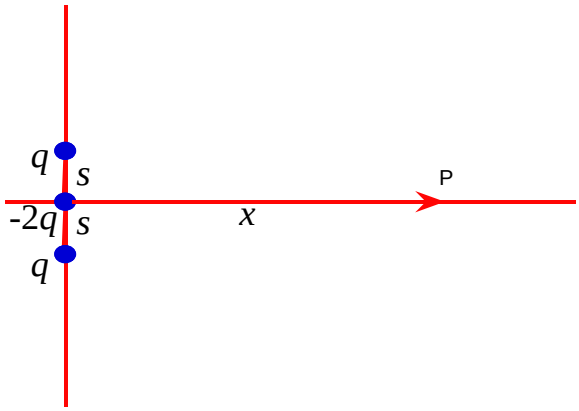


FIG. 35: Problem 2

Answers Final 252, 12-11-2006, 12.00

Problem 1
For head-on collisions with the target at rest, we have conservation of momentum

$$m_1 v_1 = m_1 v_1' + m_2 v_2'. \tag{646}$$

In addition, the relative velocities are conserved: $v_1 = v_2' - v_1'$. Substituting v_2' gives

$$m_1 v_1 = m_1 v_1' + m_2(v_1 + v_1') \quad \Rightarrow \quad (m_1 - m_2)v_1 = (m_1 + m_2)v_1' \quad \Rightarrow \quad v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1. \tag{647}$$

We also have

$$v_2' = v_1 + v_1' = \frac{2m_1}{m_1 + m_2} v_1. \tag{648}$$

For the collision between 1 and 2, we obtain

$$v_1' = \frac{2m - m}{2m + m} v = \frac{1}{3} v \quad \text{and} \quad v_2' = \frac{2 \times 2m}{2m + m} v = \frac{4}{3} v. \tag{649}$$

It takes ball 2 $t = l/v_2' = \frac{3}{4}\frac{l}{v}$, to travel the distance to ball 3. Ball 1 has travelled a distance $v_1' t = \frac{1}{3} v \frac{3}{4}\frac{l}{v} = \frac{1}{4} l$ (of course, since its velocity is four times as small). We then have the collision between balls 2 and 3:

$$v_2'' = \frac{m - 4m}{4m + m} v_2' = -\frac{3}{5} v_2' = -\frac{12}{15} v \quad \text{and} \quad v_3' = \frac{2 \times m}{4m + m} v_2' = \frac{2}{5} v_2' = \frac{8}{15} v. \tag{650}$$

Ball 2 is now going backwards again and will hit ball 1. The equations of motion are

$$x_1 = \frac{1}{3} v t + \frac{1}{4} l \quad \text{and} \quad x_2 = -\frac{4}{5} v t + l. \tag{651}$$

When they hit the positions have to be equal

$$\frac{1}{3} v t + \frac{1}{4} l = -\frac{4}{5} v t + l \quad \Rightarrow \quad \frac{17}{15} v t = \frac{3}{4} l \quad \Rightarrow \quad t = \frac{45}{68}\frac{l}{v}. \tag{652}$$

The total time is therefore $t = (\frac{3}{4} + \frac{45}{68})\frac{l}{v} = \frac{24}{17}\frac{l}{v}$.

Problem 2
(a) The potential is given by

$$V = \frac{q}{4\pi\epsilon_0}\left(2 \times \frac{1}{\sqrt{x^2 + s^2}} - \frac{2}{x}\right) = \frac{q}{4\pi\epsilon_0}\left(\frac{2}{x\sqrt{1 + \left(\frac{s}{x}\right)^2}} - \frac{2}{x}\right) \cong \frac{q}{4\pi\epsilon_0}\left(\frac{2}{x}[1 - \frac{1}{2}\left(\frac{s}{x}\right)^2] - \frac{2}{x}\right) = -\frac{q}{4\pi\epsilon_0}\frac{s^2}{x^3}.$$

(b) The electric field is given by

$$\mathbf{E} = -\frac{dV}{dx}\hat{\mathbf{i}} = -\frac{q}{4\pi\epsilon_0}\frac{3s^2}{x^4}\hat{\mathbf{i}} \tag{653}$$