Physics 686 **Homework 2** **Due February 11, 2004**

Problem 1. On page 14 of the Phys 586 notes, the Lorentz transformation properties of Dirac spinors was discussed. Translate this into a discussion of the Lorentz transformation properties of Weyl two-component left-handed spinors and right-handed spinors, in terms of $\overline{\sigma}$ and σ matrices.

Problem 2. Suppose we have a Lagrangian consisting of Weyl two-component fermions ψ_i (where $i = 1, \ldots n$) and complex scalar fields ϕ_I (where $I = 1, \ldots N$). The Lagrangian is:

$$\mathcal{L} = i\psi^{\dagger i}\overline{\sigma}^\mu \partial_\mu \psi_i + \partial^\mu \phi^{*I} \partial_\mu \phi_I - \frac{1}{2}(M^{ij}\psi_i\psi_j + \text{c.c.}) - [m^2]_I^J \phi^{*I}\phi_J - \frac{1}{6}(y^{Ijk}\phi_I\psi_j\psi_k + \text{c.c.}).$$

Find the necessary and sufficient conditions on each of the masses M^{ij} and $[m^2]_I^J$ and the coupling y^{Ijk} so that the Lagrangian is invariant under the global symmetry transformation:

$$\begin{aligned} \phi_I &\rightarrow \phi_I + i\epsilon^a T_I^{aJ}\phi_J \\ \psi_i &\rightarrow \psi_i + i\epsilon^a t_i^{aj}\psi_j. \end{aligned}$$

(Here, T_a and t_a are the symmetry generators for the representations of the scalars and fermions, respectively.)

Problem 3. Consider the see-saw model for a single neutrino species, given in terms of two-component left-handed Weyl spinors ν and N by the Lagrangian:

$$\mathcal{L} = i\nu^\dagger \overline{\sigma}^\mu \partial_\mu \nu + iN^\dagger \overline{\sigma}^\mu \partial_\mu N - \left(m\nu N + \frac{M}{2}NN + \text{c.c.}\right)$$

where "c.c." means complex conjugate, and m and M are mass parameters. We want to find the mass eigenstates, so we will do a field redefinition of the form:

$$\begin{pmatrix} \nu' \\ N' \end{pmatrix} = U \begin{pmatrix} \nu \\ N \end{pmatrix} \tag{1}$$

where U is a unitary 2×2 matrix.

(a) Assume that m and M are positive and real, but do not assume anything about their

relative sizes (yet). Find an appropriate U to put the Lagrangian in the form:

$$\mathcal{L} = i\nu'^{\dagger}\overline{\sigma}^{\mu}\partial_{\mu}\nu' + iN'^{\dagger}\overline{\sigma}^{\mu}\partial_{\mu}N' - \frac{1}{2}\left(m_{\nu'}\nu'\nu' + M_{N'}N'N' + \text{c.c.}\right) \tag{2}$$

where $m_{\nu'}$ and $M_{N'}$ are *positive* quantities that you will find.

(b) Now suppose that $m \ll M$, and show that the masses are given by:

$$\begin{aligned} m_{\nu'} &= m^2/M + \ldots \\ m_{N'} &= M + \ldots, \end{aligned}$$

and find the next-order correction in each case.

(c) The interaction of the gauge-eigenstate neutrino ν_e with the W boson and the electron is given by a term in the Lagrangian:

$$\mathcal{L} = -\frac{g}{2}\left(e^{\dagger}\overline{\sigma}^{\mu}\nu\, W^{-}_{\mu} + \text{c.c.}\right).$$

(This is given in two-component Weyl notation; see equation (12.16) in the Phys 586 notes for the four-component spinor version.) By what proportion is the interaction strength for the mass eigenstate ν' smaller than that for ν ? (Give your answer to lowest non-trivial order in m/M.) What is your answer for $m = 100$ GeV and $M = 10^{16}$ GeV? This explains the statement that in the see-saw model, the neutrinos interact almost exactly the same way as in the Standard Model.

(d) The assumption that the original parameter m and M are real is actually completely unnecessary. Explain why, by describing how to modify your unitary matrix U so that $m_{\nu'}$ and $m_{N'}$ will be real and positive, no matter what the complex phases of the original parameters m and M were.