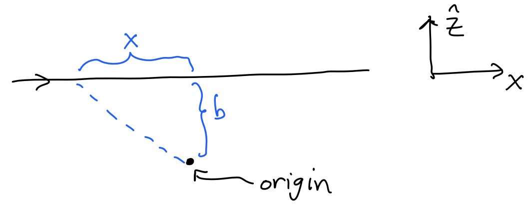
Example: point mass M at origin. Light approaches with impact parameter $\vec{b} = b\hat{z}$

Unperturbed path:



$$\Phi = -\frac{GM}{r} = \frac{-GM}{(z^2+x^2)^{1/2}} \qquad ds = dx$$

$$\vec{\nabla}_\perp \Phi = \hat{z} \text{ part of } \vec{\nabla}\Phi = -\left.\frac{GM\left(-\frac{1}{2}\right)(2z\hat{z})}{(z^2+x^2)^{3/2}}\right|_{z=b}$$

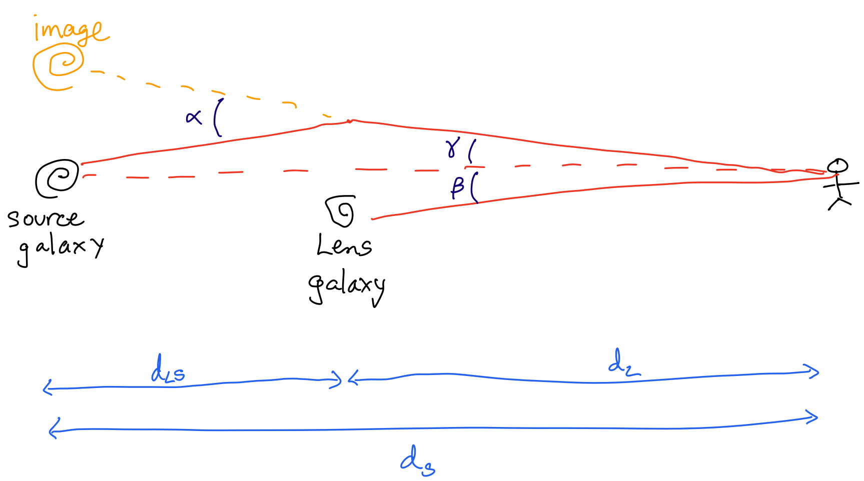
$$= \frac{GM}{(b^2+x^2)^{3/2}}\vec{b}$$

So, deflection angle $$\vec{\alpha} = 2\int_{-\infty}^{\infty} dx \frac{GM\vec{b}}{(b^2+x^2)^{3/2}} = 2GM\vec{b}\underbrace{\int_{-\infty}^{\infty}\frac{dx}{(b^2+x^2)^{3/2}}}_{\frac{2}{b^2}}$$

$$= \frac{4GM}{b}\hat{z}$$

✓ Did before, using Schwarzschild.

Gravitational Lensing



$$\gamma = \frac{d_{LS}}{d_S}\alpha$$

$$\beta = \theta - \frac{d_{LS}}{d_S}\alpha$$

where $\theta = \beta + \gamma$ = angle between lens, image (what is measured)

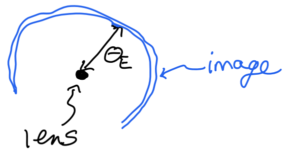
For a point lens galaxy w/ mass M, $\alpha = \frac{4GM}{b} = \frac{4GM}{d_L\theta}$

So $\beta = \theta - \frac{d_{LS}}{d_S d_L}\frac{4GM}{\theta}$.

Simple case: $\beta = 0$ (source exactly behind lens $\rightarrow$ max deflection)

Then $\boxed{\theta_E = \sqrt{\dfrac{4GM\, d_{LS}}{d_L\, d_S}}}$ = Einstein angle

See an Einstein ring:



Solar lensing: $M \sim M_{sun}$, $d_{LS}, d_S \sim$ tens of kpc (in our galaxy)

$\Rightarrow \theta_E \sim 10^{-3}$ arcsecs

Cosmological lensing: $M \sim 10^{11} M_{sun} \sim$ galaxy mass

$d_L, d_S, d_{LS} \sim$ Gpc

$\Rightarrow \theta_E \sim$ arcsecs

More generally, can have multiple images, severe distortion.
(Lens not point, source off center.)

Grav. waves

$$ds^2 = -(1+2\Phi)dt^2 + w_i(dt\,dx^i + dx^i\,dt) + \left[(1-2\Psi)\delta_{ij} + 2s_{ij}\right]dx^i dx^j$$

Look for vacuum solns $(T_{\mu\nu} = 0)$, wavelike $(\Phi, \Psi, w^i = 0)$ $s_{ij} \neq 0$

Transverse gauge $\partial_i s^{ij} = 0$.

Recall: $G_{00} = 2\nabla^2 \cancel{\Psi} = 8\pi G \cancel{T_{00}} \rightarrow 0 = 0$ ✓

$G_{0j} = -\frac{1}{2}\nabla^2 \cancel{w_j} + 2\partial_0\partial_j \cancel{\Psi} = 8\pi G \cancel{T_{0j}}$ $\quad 0 = 0$ ✓

$G_{ij} = (\text{stuff} = 0) - \partial_\mu \partial^\mu s_{ij} = 8\pi G \cancel{T_{ij}}$

So, $\boxed{\partial_\mu \partial^\mu s_{ij} = 0}$ or $\left(-\dfrac{\partial^2}{\partial t^2} + \nabla^2\right)s_{ij} = 0$ 4-d wave eqn.

$$h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & 2S_{ij} & \\ 0 & & & \end{pmatrix}$$

← transverse, traceless gauge choice

with
$$\left.\begin{aligned} h^{TT}_{0\mu} = h^{TT}_{\mu 0} &= 0 \\ \partial^\mu h^{TT}_{\mu\nu} &= 0 \end{aligned}\right\} \text{transverse}$$
$$\eta^{\mu\nu} h^{TT}_{\mu\nu} = 0 \quad \} \text{ traceless}$$

To find sol'n, write $h^{TT}_{\mu\nu} = \mathrm{Re}[\tilde{h}_{\mu\nu}]$ ~ means complex; drop TT as notation

Then $\partial_\rho \partial^\rho \tilde{h}_{\mu\nu} = 0$ implies solutions are linear combinations of

$$\tilde{h}_{\mu\nu} = \underbrace{C_{\mu\nu}}_{\text{constant, symmetric, traceless, complex (0,2) tensor}} e^{ik\cdot x}$$ with k^μ satisfying $k^2 \equiv k^\mu k_\mu = 0$.

$$k\cdot x \equiv k_\mu x^\mu$$

Check: $$\partial_\rho \partial^\rho \tilde{h}_{\mu\nu} = C_{\mu\nu}\, \partial_\rho \underbrace{\partial^\rho e^{ik\cdot x}}_{ik^\rho e^{ikx}}$$
$$ik_\rho (ik^\rho e^{ik\cdot x}) = -k_\rho k^\rho e^{ik\cdot x} = 0 \ \checkmark$$

Also, must have
$$\left.\begin{aligned} C_{0\mu} = C_{\mu 0} &= 0 \\ k^\mu C_{\mu\nu} &= 0 \end{aligned}\right\} \text{transverse}$$
$$\eta^{\mu\nu} C_{\mu\nu} = 0 \quad \text{traceless}$$

W/o loss of generality, take wave moving in $+\hat{z}$ direction:
$k^\mu = (\omega, 0, 0, k^z)$. Then $k^\mu k_\mu = 0 \Rightarrow k^z = \omega$.

Also, $C_{3\mu} = C_{\mu 3} = 0$. What's left?

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Two linear polarizations: $\underbrace{C_{11}}_{=h_+}$ and $\underbrace{C_{12}}_{=h_\times}$

So, ...

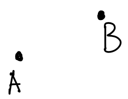
$$\tilde{h}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(z-t)}$$

What does this wave do to test masses?

(Carroll uses geodesic deviation eq, Riemann tensor.)

<u>Fact:</u> As wave passes, <u>coordinate</u> distance between test masses <u>unchanged</u> !!

Wave moving out of page.



Before wave hits: $x_A^\mu = (\tau, 0, 0, 0)$ $\quad U_A^\mu = \frac{dx_A^\mu}{d\tau} = (1,0,0,0)$

$x_B^\mu = (\tau, x_B, y_B, z_B)$ $\quad U_B^\mu = \frac{dx_B^\mu}{d\tau} = (1,0,0,0)$

Geodesic eqn: $\frac{d^2 x^i}{d\tau^2} = -\Gamma^i_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$ for A, B.

Recall $x^\mu = x^\mu_{(0)} + x^\mu_{(1)}$.

So $\frac{d^2}{d\tau^2}\left(x^i_{(1)}\right) = -\Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\Gamma^i_{00} = \mathcal{O}(h^2)$

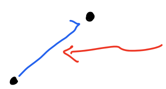
at least 1st order in h

1st order in h, unless $\mu=\nu=0$.

So $\delta x^i_{(1)} = 0$ for each of A, B (up to $\mathcal{O}(h^2)$)

<u>Physical</u> distance <u>does</u> change.

Take $\hat{z}$ out of board,



$L_0 \vec{n}$ — unit vector $n_x\hat{x} + n_y\hat{y}$

average separation

Physical distance $= L(t) = \int_0^{L_0} dx \left[1 + n^i n^j h_{ij}(t,0)\right]^{1/2}$

at $z=0$ = constant

$\approx L_0\left(1 + \frac{1}{2} n^i n^j h_{ij}(t,0)\right)$

So $\boxed{\frac{\delta L}{L_0} = \frac{1}{2} n^i n^j h_{ij}(t,0)}$ for test masses in $z=0$ plane

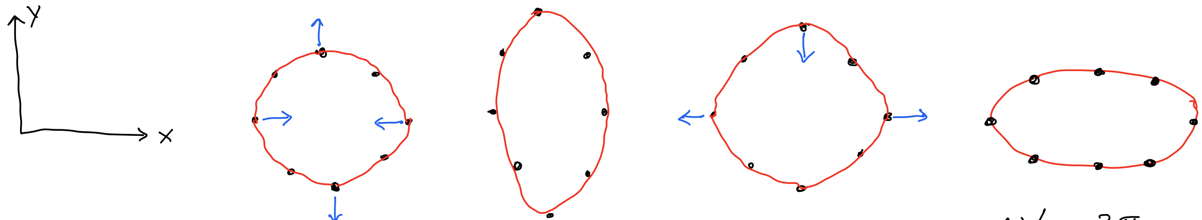
Consider a + polarization wave: in $i,j = xy$ subspace,

$$h_{ij} = \begin{pmatrix} h_+ & 0 \\ 0 & -h_+ \end{pmatrix} \sin(\omega(z-t)). \qquad \text{Take } z=0$$

For $\vec{n} = \pm\hat{x}$, $\quad \dfrac{\delta L}{L_0} = -\dfrac{1}{2} h_+ \sin(\omega t)$

$\hat{n} = \pm\hat{y} \quad \dfrac{\delta L}{L_0} = +\dfrac{1}{2} h_+ \sin(\omega t)$

$\hat{n} = \dfrac{\pm\hat{x} \pm \hat{y}}{\sqrt{2}} \quad \dfrac{\delta L}{L_0} = 0$



$\omega t = 0, 2\pi, 4\pi \qquad \omega t = \frac{\pi}{2} \qquad \omega t = \pi \qquad \omega t = \frac{3\pi}{2}$

Exaggerated! in reality, $h_+ \sim 10^{-20}$ or less.

+ pattern of strain on test masses

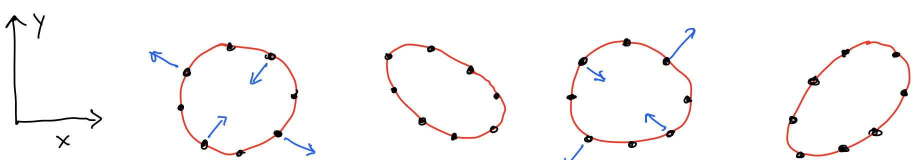
For × polarized wave: $\quad h_{ij} = \begin{pmatrix} 0 & h_\times \\ h_\times & 0 \end{pmatrix} \sin(\omega(t-z))$ in $i,j = x,y$ subspace

For $\vec{n} = \pm\hat{x} \quad \dfrac{\delta L}{L_0} = 0$

$\vec{n} = \pm\hat{y} \quad \dfrac{\delta L}{L_0} = 0$

$n = \pm\dfrac{(\hat{x}+\hat{y})}{\sqrt{2}} \quad \dfrac{\delta L}{L_0} = -\dfrac{1}{2} h_\times \sin(\omega t)$

$n = \pm\dfrac{(\hat{x}-\hat{y})}{\sqrt{2}} \quad \dfrac{\delta L}{L_0} = +\dfrac{1}{2} h_\times \sin(\omega t)$



× pattern of strain

$\omega t = 0, 2\pi, 4\pi, \ldots \qquad \omega t = \frac{\pi}{2} \qquad \omega t = \pi \qquad \omega t = \frac{3\pi}{2}$

Linearized gravity $\Longleftrightarrow$ can take superpositions of solutions $\to$ solutions

Not true of black holes!

So, can also have circular polarizations:

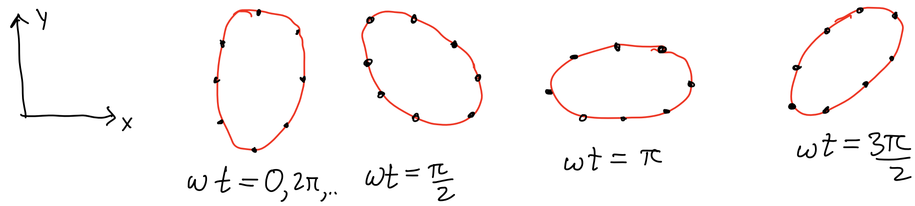
$$\tilde{h}_{\mu\nu} = \frac{h_R}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & -i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik\cdot x} \quad \text{or} \quad \frac{h_L}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik\cdot x}$$

R-circular L-circular

So $$h_{\mu\nu} = \mathrm{Re}[\tilde{h}_{\mu\nu}] = h_R \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos(\omega(z-t)) & \sin(\omega(z-t)) & 0 \\ 0 & \sin(\omega(z-t)) & -\cos(\omega(z-t)) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Consider test particles in a circle in coordinates. $x^2+y^2=L^2$

Not equidistant in physical distance from center $x=y=0$.



rotates in RH direction as an ellipse.

For L-circular polarization, rotates in opposite direction $\sin \to -\sin$.

(Note: physical distances describe ellipses, but these are not elliptical polarization!)

Production of gravity waves

Matter source: $T_{\mu\nu}(t,\vec{x}) \neq 0$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

Define $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h\, \eta_{\mu\nu}$ (recall $h \equiv \eta^{\mu\nu} h_{\mu\nu}$).

(To recover $h_{\mu\nu}$ from $\bar{h}_{\mu\nu}$: $\bar{h} \equiv \eta^{\mu\nu}\bar{h}_{\mu\nu} = h - \frac{1}{2}h(4) = -h$

So $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\,\eta_{\mu\nu}$.)

Einstein tensor $G_{\mu\nu} = -\frac{1}{2}\partial_\rho \partial^\rho \bar{h}_{\mu\nu} = 8\pi G\, T_{\mu\nu}$.

So $\boxed{\partial_\rho \partial^\rho \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}}$

Analogy:

	EM	Linearized GR
source	J^μ = current	$T_{\mu\nu}$ = stress-energy
potential	$A_\mu = (V, \vec{A})$	$\bar{h}_{\mu\nu}$
gauge choice	$\partial^\mu A_\mu = 0$	$\partial^\mu \bar{h}_{\mu\nu} = 0$
eqn	$\partial_\rho \partial^\rho A_\mu = -4\pi J_\mu$	$\partial_\rho \partial^\rho \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}$
solution	retarded potential	" "

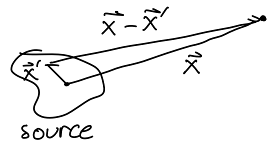
For EM: ret. pot. $A_\mu(t,\vec{x}) = \int d^3\vec{x}' \, \dfrac{J_\mu(t - |\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x}-\vec{x}'|}$

For gravity: $\bar{h}_{\mu\nu}(t,\vec{x}) = \dfrac{4G}{c^2} \int d^3\vec{x}' \, \dfrac{T_{\mu\nu}(t - |\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x}-\vec{x}'|}$

$t_r = t - |\vec{x}-\vec{x}'|$ = retarded time

Expand in large distances $|\vec{x}| \gg |\vec{x}'|$

throughout source



In EM, get dipole radiation at leading order in $\dfrac{|\vec{x}'|}{|\vec{x}|}$.

In gravity, leading is quadrupole.

Define quadrupole moment tensor (3-d spatial tensor)

$I_{ij}(t) = \int d^3\vec{y} \; y^i y^j \; T_{00}(t,\vec{y})$. Then $\boxed{\bar{h}_{ij}(t,\vec{x}) = \dfrac{2G}{c^4 r} \ddot{I}_{ij}(t_r)}$

relabel of $\vec{x}'$ (large r limit)

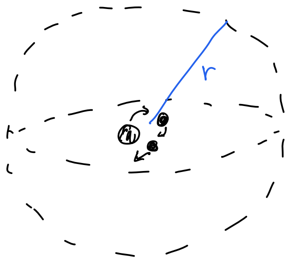
Get $\bar{h}_{0i} = \bar{h}_{i0}$, and $\bar{h}_{00}$, from $\partial^\mu \bar{h}_{\mu\nu} = 0$.

[Compare to EM: $\vec{A}(t,\vec{x}) = \dfrac{\dot{\vec{p}}(t_r)}{r}$ $\vec{p}(t)$ = dipole moment]

Now define reduced quadrupole moment $J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl} I_{kl}$

(traceless part)

Then, radiated power $\boxed{P = \frac{G}{5c^5}\sum_{i,j}\left(\dddot{I}_{ij}(t_r)\right)^2}$ (detected on large sphere of radius r)



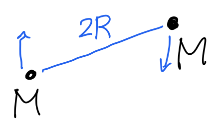
[Compare to EM: radiated power $= \frac{2}{3}\left(\ddot{p}(t_r)\right)^2$]

Derivatives w.r.t. $\Rightarrow$ ω = characteristic frequency of source

$$I \approx \underset{\text{size of source}}{R^2}\, \underset{\text{mass of source}}{M}$$

So, naively, power $\sim \frac{G}{c^5} M^2 R^4 \omega^6$. ← depends on M, R, often.

<u>Example</u> Binary star system, circular orbit

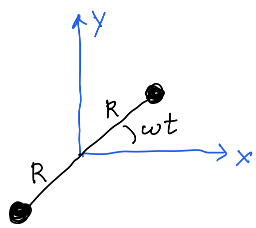


Let's find ω, using Newtonian approximation.

$$F = Ma = M\left(\frac{v^2}{R}\right) = M\left(\frac{(\omega R)^2}{R}\right) = \frac{GMM}{(2R)^2}$$

$$\text{So } \omega^2 = \frac{GM}{4R^3} \Rightarrow \omega = \sqrt{\frac{GM}{4R^3}}$$

Compute Quad. Moment Tensor. Start with positions of masses:



mass 1: $x = R\cos(\omega t)$
$y = R\sin(\omega t)$
$z = 0$

mass 2: $x = -R\cos(\omega t)$
$y = R\sin(\omega t)$
$z = 0$.

Energy density: $T^{00}(t,\vec{x}) = M\left[\delta^{(3)}(x_1 - R\cos(\omega t),\ x_2 - R\sin(\omega t),\ x_3) + \delta^{(3)}(x_1 + R\cos(\omega t),\ x_2 + R\sin(\omega t),\ x_3)\right]$

$$I_{ij} = \int d^3\vec{y}\; y_i y_j\, T^{00}(t,\vec{y})$$ ← easy, δ-fns.

$I_{11} = 2M\left(R\cos(\omega t)\right)^2 \qquad I_{22} = 2M\left(R\sin(\omega t)\right)^2 \qquad I_{33} = 0$

$I_{12} = I_{21} = 2R\left(R\cos(\omega t)\right)\left(R\sin(\omega t)\right),$

$I_{13} = I_{31} = I_{23} = I_{32} = 0.$

So, summarizing, $I_{ij} = 2MR^2 \begin{pmatrix} \cos^2(\omega t) & \cos(\omega t)\sin(\omega t) & 0 \\ \cos(\omega t)\sin(\omega t) & \sin^2(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Now can get perturbed metric at large distances:

$$\bar{h}_{ij}(t,\vec{x}) = \frac{2G}{c^4 r}\ddot{I}_{ij} = -\frac{8GM}{c^4}\frac{1}{r}\omega^2 R^2 \begin{pmatrix} \cos(2\omega t_r) & \sin(2\omega t_r) & 0 \\ \sin(2\omega t_r) & -\cos(2\omega t_r) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Can get $h_{\mu\nu}$ from this.

Notes: 1) non-zero components are in plane of stars' motion.

2) Observed at large $\pm z$, see circular polarizations.

3) Observed at large r with $z=0$, see linear h_+ polarizations

To find radiated power:

First, $\delta^{kl} I_{kl} = 2MR^2$. So:

$$J_{ij} = I_{ij} - \tfrac{1}{3}\delta_{ij}(2MR^2) = 2MR^2 \begin{pmatrix} \cos^2(\omega t) - \frac{1}{3} & \cos(\omega t)\sin(\omega t) & 0 \\ \cos(\omega t)\sin(\omega t) & \sin^2(\omega t) - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

$$= MR^2 \begin{pmatrix} \cos(2\omega t) + \frac{1}{3} & \sin(2\omega t) & 0 \\ \sin(2\omega t) & -\cos(2\omega t) + \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}$$

So, $\dddot{J}_{ij} = MR^2(2\omega)^3 \begin{pmatrix} +\sin(2\omega t) & -\cos(2\omega t) & 0 \\ -\cos(2\omega t) & -\sin(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\sum_{i,j} \left(\dddot{J}_{ij}\right)^2 = \left(8MR^2\omega^3\right)^2 \left(\underbrace{\sin^2 + \sin^2 + \cos^2 + \cos^2}_{2}\right)$$

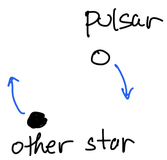
$$= 128\, M^2 R^4 \omega^6$$

So $\text{Power} = \frac{G}{5c^5} \sum_{ij} \left(\dddot{J}_{ij}\right)^2 = \frac{128}{5} \underbrace{\frac{G}{c^5} M^2 R^4 \omega^6}_{\text{naive estimate from above}}$

$$= \frac{2G^4M^5}{5c^5R^5} \quad \text{for this system.}$$

As the system loses energy, $\left\{ \begin{array}{l} R \text{ decreases} \\ \omega \text{ increases} \end{array} \right\}$ with time.

Hulse-Taylor binary pulsar (discovered 1974, Nobel 1993)



$M_{pulsar} = 1.442\ M_{sun}$

$M_{other} = 1.386\ M_{sun}$

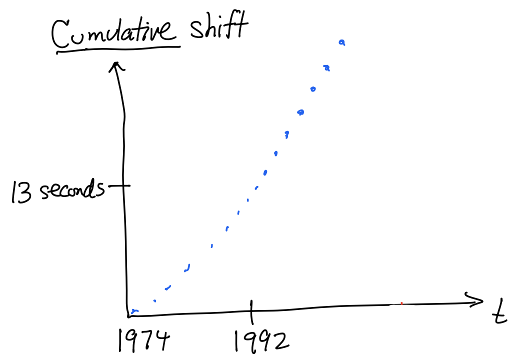
$T = 2.79 \times 10^4 \text{ sec} = 7.75 \text{ hours}$

$\omega = 2.25 \times 10^{-4} \text{ sec}^{-1}$

$R_{min} \approx 1.1\ R_{earth\text{-}sun}$

The pulsar is an accurate clock. $T_{pulsar} = 0.059$ seconds
Known to 1 part in 10^{12}

$$\frac{\Delta T}{\Delta t} = 7.65 \times 10^{-5} \ \frac{\text{seconds}}{\text{year}}.$$



Agrees with GR prediction.

(Reality: orbits not circular.)

Indirect evidence for gravity waves.