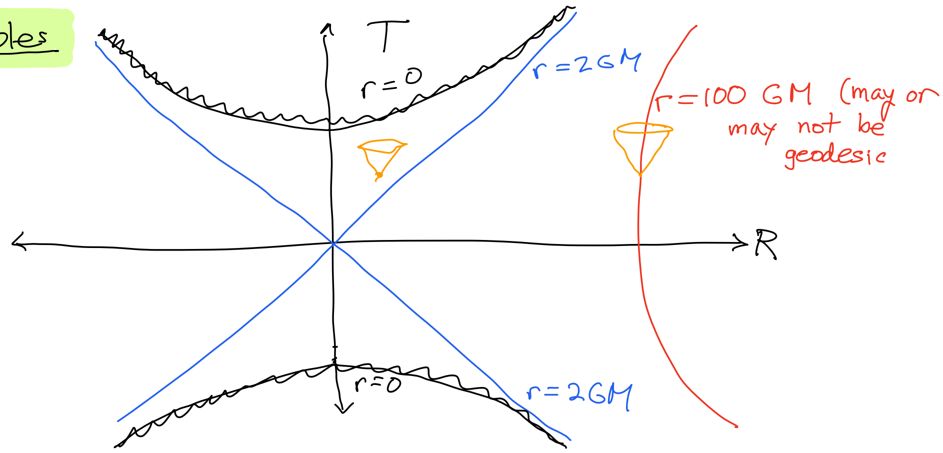
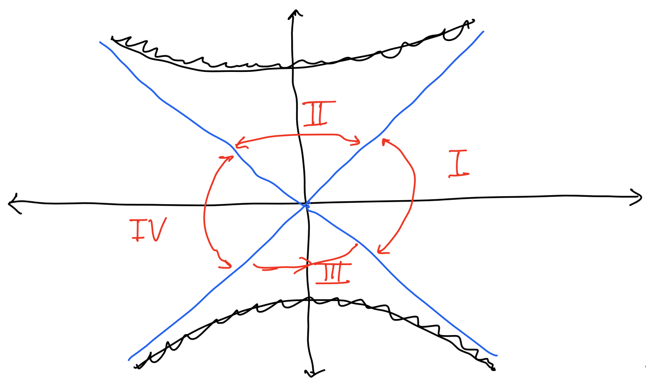
Eternal Black Holes



Comments:
- There are two $r=0$ singularities, past and future.
- Event horizons $r=2GM$ are null surfaces $\Leftrightarrow$ $T=\pm R$. (not "places")
- The singularities aren't places, they are times.
- For $r<2GM$, all paths within future light cones must hit the future singularity in finite time (both proper and coordinate).
- Kruskal chart completes the (t,r,θ,ϕ) chart.



I = asymptotically flat region covered by original chart, with $r>2GM$.

II = inside horizon = black hole

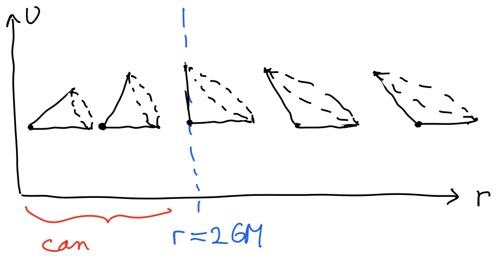
I & II covered by ingoing E-F chart (v,r,θ,ϕ)

III = time reverse of II = "white hole"
(doesn't exist for real-world black holes)

I & III covered by outgoing E-F chart (u,r,θ,ϕ).

$$ds^2 = -\left(1-\frac{2GM}{r}\right)du^2 - (du\,dr + dr\,du) + r^2\,d\Omega^2$$

Null lines: $$\frac{du}{dr} = \begin{cases} 0 & \text{(outgoing null)} \\ -2\left(1-\frac{2GM}{r}\right)^{-1} & \text{(ingoing null)} \end{cases}$$



But this is region III, not region II.

IV = another asymptotically flat spacetime region.

Just like us, but not us. We can never get there from I.

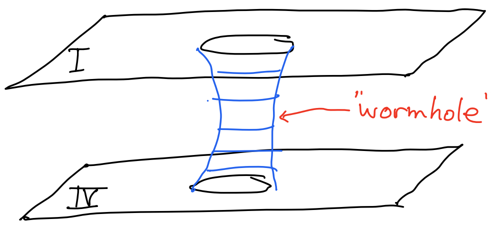
(Not in our future light cone = causally disconnected)

But, if we cross horizon, and someone from IV crosses their horizon, then we can meet in II. And then, die at the singularity.

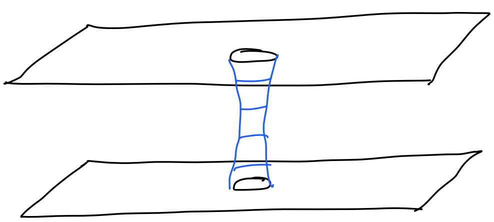
Someone in III can choose to join us in I, or enter IV, but not both.

Spatial slices at $T=t=0$:

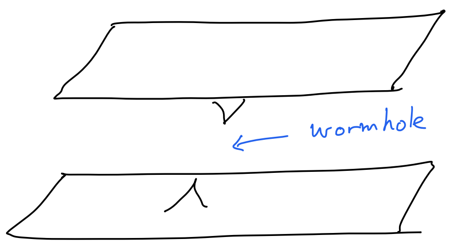
(falsely embedded in $\mathbb{R}^3$)

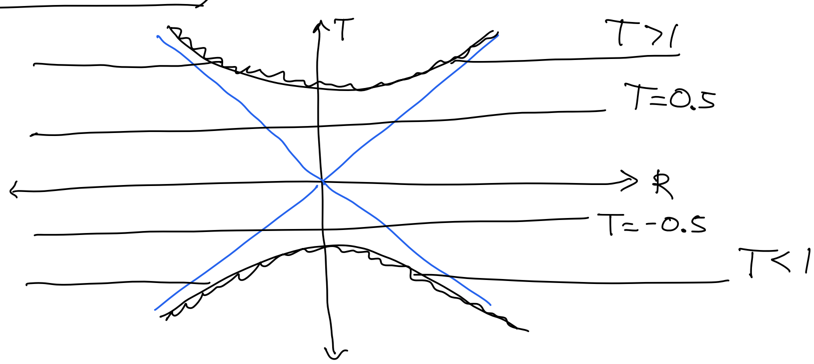


Spatial slices at $T=\pm T_0$ with $T_0<1$:



with $T_0>1$:

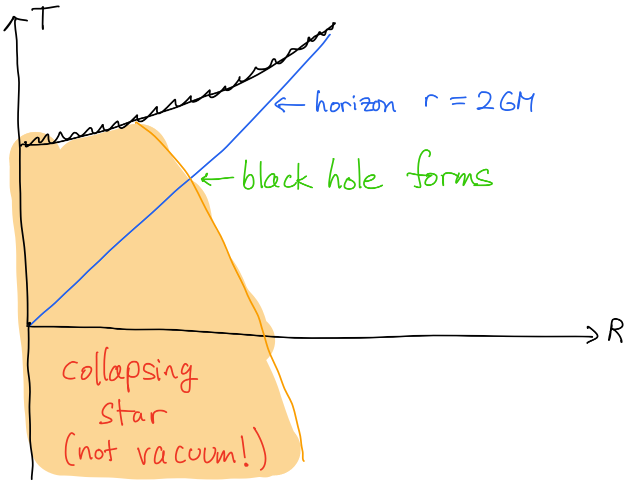




Spatial topology changes in time, never same as R^3.

All of that was for an idealized, primordial, vacuum black hole.

Real-world "astrophysical" black hole



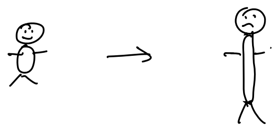
White hole, regions III, IV don't exist.

Spatial topology = boring R^3

Singularity horizon qualitatively same as Kruskal.

What happens as you fall in?

Tidal forces



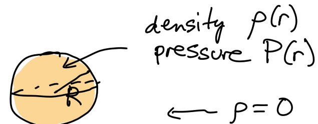
stretched toward singularity
squeezed ⊥ to "

At $r=2GM$ may be small if M is very large.

We may be crossing event horizon of some very distant, massive black hole, not knowing.

At $r=0$, tidal forces $\to \infty$ (curvature $\to \infty$).

Stable (non-BH) stars



$\leftarrow$ $\rho = 0$, $P = 0$ (vacuum) Schwarzschild metric (known)

Inside: $(r<R)$, $T_{\mu\nu} = (\rho + P) U_\mu U_\nu + P g_{\mu\nu}$

$U^\mu = (U^0, 0, 0, 0)$ for fluid inside.

Metric: $ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$ (static, spherically symmetric)

$U^\mu U_\mu = -1 \Rightarrow (U^0)^2 (-e^{2\alpha}) = -1 \Rightarrow U^0 = e^{-\alpha}$, $U_0 = -e^{\alpha}$

harmless typo, eq. (5.137)

So $T_{\mu\nu} = \mathrm{diag}(e^{2\alpha}\rho,\ e^{2\beta} P,\ r^2 P,\ r^2 \sin^2\theta\, P)$

Meanwhile: $\quad G_{\mu\nu} = \mathrm{diag}(G_{tt}, G_{rr}, G_{\theta\theta}, G_{\phi\phi}) = 8\pi G_N T_{\mu\nu}$

$$G_{tt} = \frac{1}{r^2} e^{2(\alpha-\beta)} (2r\beta' - 1 + e^{2\beta})$$

(From now on, $'$ means $\frac{d}{dr}$.)

$$G_{rr} = \frac{1}{r^2}(2r\alpha' + 1 - e^{2\beta})$$

$$G_{\theta\theta} = r^2 e^{-2\beta}\left[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{1}{r}(\alpha' - \beta')\right]$$

$$G_{\phi\phi} = \sin^2\theta \, G_{\theta\theta}$$

(nothing new learned)

So: tt eq: $\quad \frac{1}{r^2} e^{-2\beta}(2r\beta' - 1 + e^{2\beta}) = 8\pi G_N \rho$

rr eq: $\quad \frac{1}{r^2} e^{-2\beta}(2r\alpha' + 1 - e^{2\beta}) = 8\pi G_N P$

$\theta\theta$ eq: $\quad e^{-2\beta}\left(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{1}{r}(\alpha' - \beta')\right) = 8\pi G_N P$

Define $m(r)$ by $\quad e^{-2\beta} = 1 - \frac{2Gm(r)}{r}$. $\quad$ So

$$ds^2 = -e^{2\alpha}dt^2 + \left(1 - \frac{2Gm(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Interpret $m(r) =$ "mass enclosed inside radius r".

Then tt eq becomes: $\quad m' = 4\pi r^2 \rho.$

Integrate w.r.t. r $\quad m(r) = 4\pi \int_0^r dr' \, r'^2 \rho(r')$

Schwarzschild mass: $\quad M = m(R) = 4\pi \int_0^R dr \, r^2 \rho(r)$

Total mass: $\quad \bar{M} = \int_{\text{star}} d^3\vec{r} \sqrt{\gamma}\, \rho(\vec{r}) = 4\pi \int_0^R dr\, r^2 e^{\beta(r)} \rho(r)$

determinant of spatial metric

$$\begin{pmatrix} e^{2\beta} & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix}$$

$$= 4\pi \int_0^R dr \frac{r^2 \rho(r)}{\left(1 - \frac{2Gm(r)}{r}\right)^{1/2}}$$

Binding energy $= \bar{M} - M > 0.$

rr equation: $\quad \alpha' = \dfrac{G[m(r) + 4\pi r^3 P]}{r^2\left(1 - \frac{2Gm(r)}{r}\right)}$

$\theta\theta$ equation: $P' = -(\rho+P)\,\alpha'$, so

$$\boxed{\frac{dP}{dr} = -(\rho+P)\,G\,\frac{[m(r)+4\pi r^3 P]}{r^2\left(1-\frac{2Gm(r)}{r}\right)}}$$

hydrostatic equilibrium (Tolman – Oppenheimer – Volkov)

To make progress, need either $\underbrace{P(\rho)\ \text{(eqn of state)}}_{\text{example: } P = K\rho^\gamma}$ or just $\rho(r)$.

Simpler: assume fluid incompressible, so $\rho = \begin{cases} \rho_0 & \text{for } r \le R \\ 0 & \text{for } r > R \end{cases}$

Then $m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & (r \le R) \\ \frac{4}{3}\pi R^3 \rho_0 & (r \ge R) \end{cases}$ So:

$$P' = -(\rho_0 + P)\left(\frac{\rho_0}{3} + P\right)\frac{4\pi G r}{\left(1 - \frac{8\pi G}{3}\rho_0 r^2\right)}.$$ This is solvable:

$$\frac{dP}{(\rho_0+P)\left(\frac{\rho_0}{3}+P\right)} = -4\pi G\,\frac{r\,dr}{\left(1-\frac{8\pi G\rho_0}{3}r^2\right)}.$$ Integrate both sides

$$-\frac{3}{2\rho_0}\ln\left(\frac{3P+3\rho_0}{3P+\rho_0}\right) = \frac{3}{4\rho_0}\ln\left(\frac{8\pi G\rho_0}{3}r^2 - 1\right) + C.$$ Take exp (both sides)

$$\frac{3P+3\rho_0}{3P+\rho_0} = \tilde{C}\left(1-\frac{8\pi G\rho_0}{3}r^2\right)^{-1/2}$$

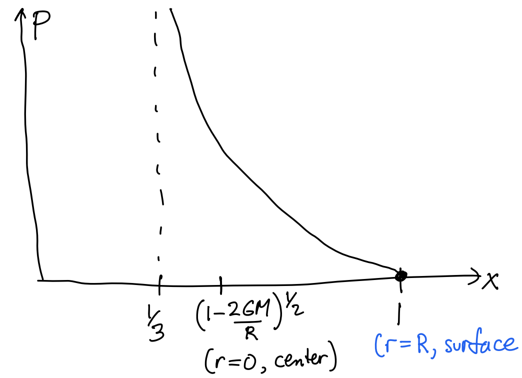
new constant. Choose so that $P=0$ at $r=R$.

$$3 = \tilde{C}\left(1-\frac{2GM}{r}\right)^{-1/2}.$$ Solve for $\tilde{C}$, plug in:

$$\frac{\rho_0+P}{\rho_0+3P} = \left(\frac{1-\frac{2GM}{R}}{1-2GMr^2/R^3}\right)^{1/2} \equiv x(r)$$

$\begin{cases} x = 1 & \text{at } r=R \\ x = \left(1-\frac{2GM}{R}\right)^{1/2} & \text{at } r=0 \\ x \text{ increases with } r \end{cases}$

Solve for P: $P(r) = \rho_0\left(\frac{x-1}{1-3x}\right)$ (Same as 5.158 Carroll)



For a stable solution, need $\left(1-\frac{2GM}{R}\right)^{1/2} > \frac{1}{3}$, so

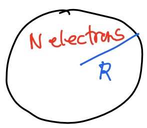
$1-\frac{2GM}{R} > \frac{1}{9}$, or $\boxed{M < \frac{4R}{9G}}$

If $M > M_{max} = \frac{4R}{9G}$, get inevitable collapse.

True even if $\rho \neq \rho_0 =$ constant, this was a best case scenario to avoid collapse.

In real stars, nuclear fusion $H \to He$ provides hot gas pressure.

Star collapses to white dwarf, supported by electron degeneracy pressure.



$\frac{\text{electrons}}{\text{volume}} \sim \frac{N}{R^3} \implies \lambda_e \sim \frac{R}{N^{1/3}}$

Fermi momentum: $p_F = \frac{\hbar}{\lambda} \sim \frac{\hbar N^{1/3}}{R}$

Energy/electron $= c\,p_F \sim \frac{\hbar c N^{1/3}}{R}$ (relativistic)

So total electron kinetic energy: $\frac{\hbar c N^{4/3}}{R}$.

Meanwhile:

Gravitational potential energy $\sim \frac{G(aN\, m_p)^2}{R}$

nucleon mass

$a = \frac{\text{nucleons}}{\text{electron}} = \begin{cases} 2 & \text{for He} \\ 1 & \text{for H} \end{cases}$

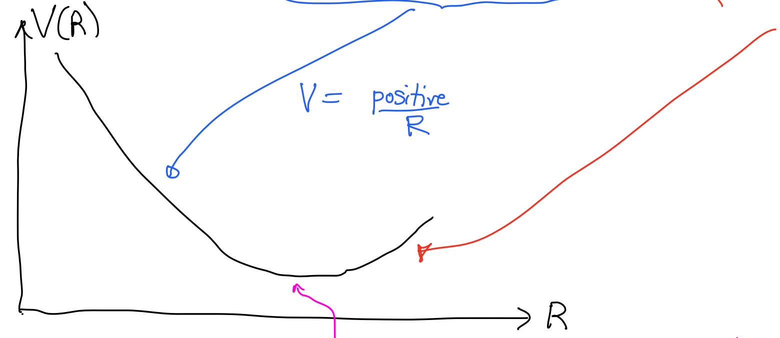
≈ 2 for stars that have used up most of H.

So total energy: $V(R) = \frac{\hbar c N^{4/3}}{R} - \frac{G a^2 N^2 m_p^2}{R} = \frac{1}{R}\left[\hbar c N^{4/3} - G a^2 N^2 m_p^2\right]$

Fermi pressure, resists collapse

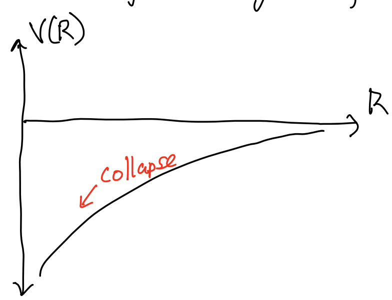
gravitational potential, favors collapse.

For small N, Fermi pressure wins, but (non-relativistic effects take over at large R)



Stable minimum = white dwarf.

For large enough N, $V = \dfrac{\text{negative}}{R}$



The boundary between these two scenarios: $\hbar c N_{crit}^{4/3} = G(a N_{crit} m_p)^2$

or $N_{crit} \sim \left(\dfrac{\hbar c}{G a^2 m_p}\right)^{3/2} \sim 10^{57}$

Chandrasekhar limit on white dwarf mass:

$$M_{crit} = a m_p N_{crit} \sim \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{a^2 m_p^2} = \frac{5.87}{a^2} M_{sun}$$

put back factors of $2, \pi$

$$= 1.4 \ M_{sun}$$

When white dwarf reaches Chandrasekhar limit, collapses to neutron star. (Typical radius ~ 10 km) due to $e^- + p \to n + \nu_e$ ← escapes

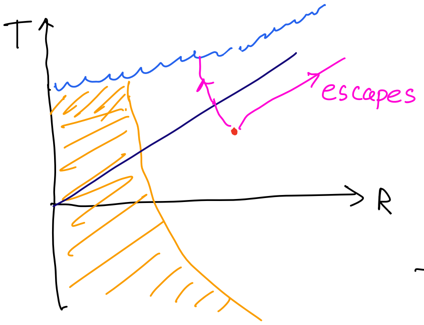
Neutron degeneracy pressure prevents further collapse, until/unless $M \gtrsim 2.5 \, M_{sun}$. Then collapse is inevitable $\Rightarrow$ black hole.

Some have $M \sim$ few $\times M_{sun}$.

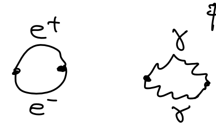
Galactic centers have black holes with masses $M \sim 10^6$ to $10^9 \, M_{sun}$.

Hot material falling in shields blackness.

Hawking effect Black holes (even in vacuum) emit radiation with a characteristic temperature, depends on mass.



Cartoon version: virtual particle/antiparticle pairs



Thermal (nearly black-body) radiation

Temperature $k_B T \sim \dfrac{c^3 \hbar}{8\pi G M}$

Black hole evaporates in time $T_{BH} \sim \left(\dfrac{M}{m_{Planck}}\right)^3 t_{Planck}$

$t_{Planck} = 5.4 \times 10^{-44}$ sec

$m_{Planck} = 1.2 \times 10^{19}$ GeV $= 2.2 \times 10^{-8}$ kg

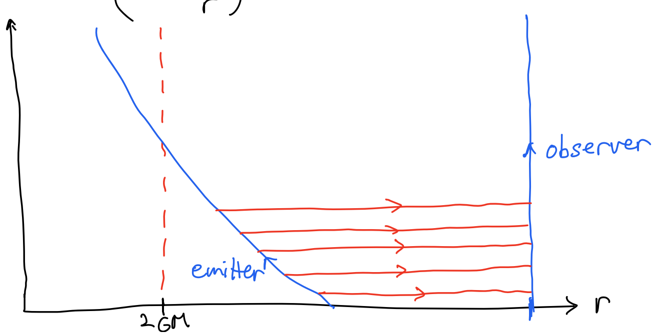
So $T_{BH} \approx \left(\dfrac{M}{M_{sun}}\right)^3 \times \underbrace{10^{71} \text{ sec}}_{10^{53} \times \text{ current age of universe}}$

$\approx \left(\dfrac{M}{1 \text{ kg}}\right)^3 \times 10^{-21}$ sec

Another look at gravitational redshift

Consider observer fixed at $r = \infty$, emitter in free fall.
Use outgoing E-F chart, so

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du^2 - (du\, dr + dr\, du) + r^2 d\Omega^2$$



Outgoing light moves on $u =$ constant lines

$$\frac{\omega_{em}}{\omega_\infty} = \frac{1/(\Delta\tau)_{em}}{1/(\Delta t)_\infty} = \frac{(\Delta\tau)_\infty}{(\Delta\tau)_{em}} \underset{t=\tau \text{ at } \infty}{=} \frac{(\Delta t)_\infty}{(\Delta\tau)_{em}} \underset{\substack{t=u+\text{const}\\ \text{for fixed } r}}{=} \frac{(\Delta u)_\infty}{(\Delta\tau)_{em}} = \frac{(\Delta u)_{em}}{(\Delta\tau)_{em}} = \left(\frac{du}{d\tau}\right)_{em}$$

(recall $u = t - r - 2GM\ln\left(\frac{r}{2GM}-1\right)$)

So, want to find $\frac{du}{d\tau}$ for the emitter geodesic.

Recall: $E = \left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau}$ = conserved energy of emitter.

$$\frac{dr}{d\tau} = -\sqrt{E^2 - 1 + \frac{2GM}{r}}$$

infalling emitter

So, $du = dt - \frac{dr}{1-\frac{2GM}{r}} \implies \frac{du}{d\tau} = \frac{dt}{d\tau} - \frac{1}{\left(1-\frac{2GM}{r}\right)}\frac{dr}{d\tau}$

$$= \left(1-\frac{2GM}{r}\right)^{-1}\left[E + \sqrt{E^2 - \left(1-\frac{2GM}{r}\right)}\right]$$

What redshift does the observer at $r=\infty$ see as $t\to\infty$?

$\left(\frac{\omega_\infty}{\omega_{em}}\right)$ as emitter nears horizon, as a function of t?

$\frac{du}{d\tau} \approx \frac{2E}{1-\frac{2GM}{r}}$ (r — emitter). How does r_{em} depend on t?

$$\frac{du}{dr} = \frac{du/d\tau}{dr/d\tau} \approx \frac{\frac{2E}{1-\frac{2GM}{r}}}{-E} \approx \frac{-2}{1-\frac{2GM}{r}} \implies u \approx -4GM\ln\left(1-\frac{2GM}{r}\right) + \text{const.}$$

So $1-\frac{2GM}{r} \approx e^{-u/4GM} \approx e^{-t_{obs}/4GM}$.

So $\frac{\omega_\infty}{\omega_{em}} \propto \left(\frac{du}{d\tau}\right)^{-1} \propto e^{-t/4GM}$

Light observed at large fixed distance redshifts exponentially with t_{obs}.

Example: Particle dropped into BH from rest at ∞. Find relations between τ, t, r on path. Already found

$$\frac{dt}{d\tau} = \frac{E}{1-\frac{2GM}{r}} \qquad \text{and} \qquad \frac{dr}{d\tau} = -\left(E^2 - 1 + \frac{2GM}{r}\right)^{1/2}$$

falling in

At $r \to \infty$, $\frac{dr}{d\tau} = 0$, so $E = 1$.

So $\frac{dr}{d\tau} = -\left(\frac{2GM}{r}\right)^{1/2} \Rightarrow r^{1/2}\, dr = -(2GM)^{1/2}\, d\tau \Rightarrow$

$$\frac{2}{3} r^{3/2} = -(2GM)^{1/2}\,\tau + \text{constant} \Rightarrow \boxed{\tau = \text{constant} - \frac{2}{3}\left(\frac{r^3}{2GM}\right)^{1/2}}$$

$$\frac{dt}{dr} = \frac{dt/d\tau}{dr/d\tau} = \frac{1/(1-\frac{2GM}{r})}{-\left(\frac{2GM}{r}\right)^{1/2}} \Rightarrow dt = \frac{-\left(\frac{r}{2GM}\right)^{1/2}}{\left(1-\frac{2GM}{r}\right)}\, dr \Rightarrow$$

$$t = -\frac{2}{3}\left(\frac{r^3}{2GM}\right)^{1/2} - 4GM\left(\frac{r}{2GM}\right)^{1/2} + 2GM \ln\left[\frac{\left(\frac{r}{2GM}\right)^{1/2} + 1}{\left(\frac{r}{2GM}\right)^{1/2} - 1}\right]$$

Homework: similar case, observer falls from rest from $r = R$. What is τ as a function of r?

Conformal diagrams = Penrose = Carter – Penrose   Appendix H

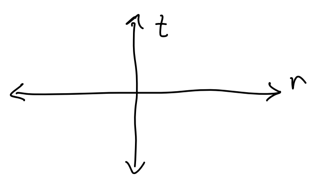
Tool for expressing causal structure.
what events can affect other events (inside light cone)

Choose coordinates so that:

1) T, R = time, radial coordinates
2) Point on diagram $\leftrightarrow$ 2-sphere (or point)
3) Causal structure: light cones = $45°$ lines in R, T plane.
4) Infinity (spatial, timelike, null) is finite coordinate distance away.
proper distance $= \infty$ proper time $= \infty$ from $R = T = 0$.

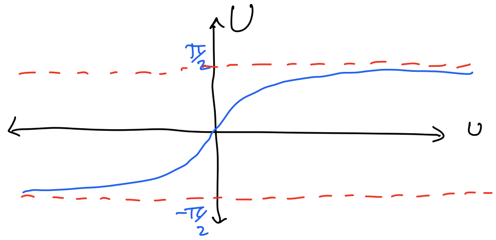
Example: Minkowski $ds^2 = -dt^2 + dr^2 + r^2\, d\Omega^2$

$-\infty < t < \infty$
$0 \leq r < \infty$



Let $\left.\begin{array}{l} u = t - r \\ v = t + r \end{array}\right\}$ $\begin{array}{l} -\infty < u < \infty \\ -\infty < v < \infty \end{array}$ $u \leq v.$ $ds^2 = -\frac{1}{2}(du\,dv + dv\,du) + \frac{1}{4}(v-u)^2 d\Omega^2$

Now let $U = \arctan(u)$
$V = \arctan(v)$



So $-\frac{\pi}{2} < U < \frac{\pi}{2}$, $-\frac{\pi}{2} < V < \frac{\pi}{2}$, and $U \leq V.$

Now take $\left.\begin{array}{l} T = V + U \\ R = V - U \end{array}\right\}$ $\begin{array}{l} 0 \leq R < \pi \\ |T| < \pi - R \end{array}$

$ds^2 = \frac{1}{\omega^2}\left(-dT^2 + dR^2 + \sin^2 R\, d\Omega^2\right)$ where $\omega = \cos T + \cos R$ = conformal factor

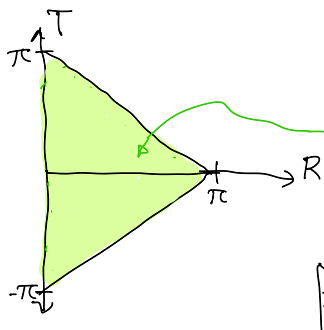
Conformal transformation on metric $ds^2 \to d\tilde{s}^2 = C^2 ds^2$

fn of coordinates, but same for all metric components

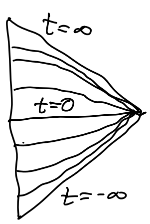
Choose $C^2 = \omega^2$, then we have an unphysical metric:

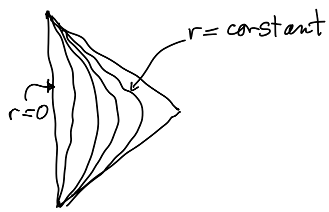
$$d\tilde{s}^2 = \underbrace{-dT^2 + \underbrace{dR^2 + \sin^2 R\, d\Omega^2}_{\text{3-sphere}}}_{R \times S^3 \text{ with Lorentzian metric}}$$

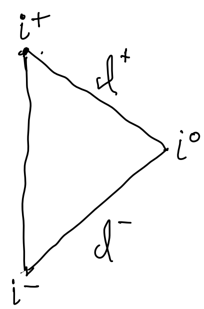
$0 \leq R < \pi$
$-\infty < T < \infty$



Whole Minkowski space is squeezed into this triangle







Five types of ∞:

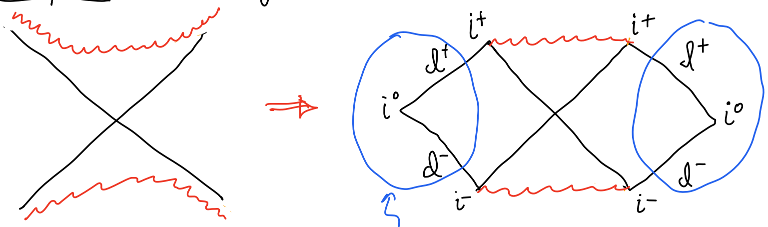
i^0 = spatial ∞. $T = 0$, $R = \pi$
i^+ = future timelike ∞ $T = \pi$, $R = 0$
i^- = past timelike ∞ $T = -\pi$, $R = 0$
} Points in unphysical metric

$\mathscr{I}^+$ = future null ∞ $T = \pi - R$, $0 < R < \pi$ "scri plus"

$\mathscr{I}^-$ = past null ∞ $T = R - \pi$, $0 < R < \pi$ "scri minus"

For causal structure, ignore conformal factor $\frac{1}{\omega^2}$. Doesn't affect light cones.

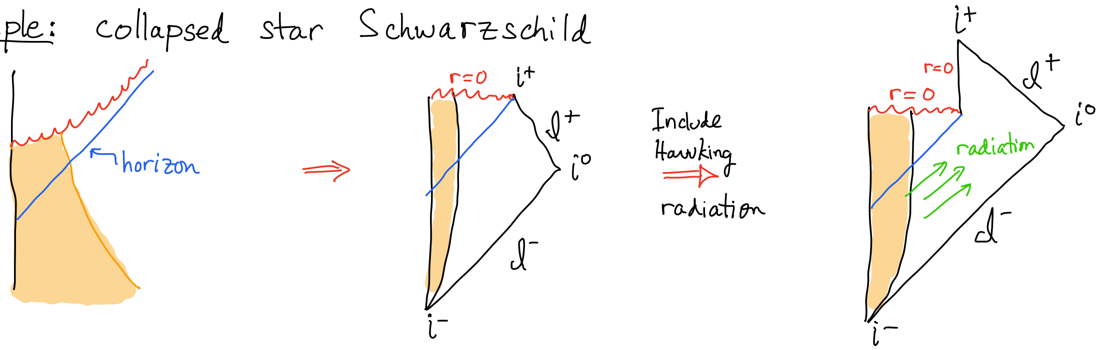
<u>Example:</u> maximally extended Schwarzschild.



asymptotically Minkowski region
same i^0, $\mathscr{I}^+$, $\mathscr{I}^-$, not necessarily i^+ and i^-

another asymptotically Minkowski region

<u>Example:</u> collapsed star Schwarzschild



For more complicated spacetimes, conformal diagram can be shockingly complicated.

<u>General black holes</u> Chapter 6

Can have mass M, angular momentum J, electric charge Q.

<u>No-hair Theorem</u> Vacuum solutions to GR that are stationary and asymptotically flat and have no singularities outside horizon are fully characterized by M, J, Q.

has a Killing vector timelike at i_0, $\mathscr{I}^+$, $\mathscr{I}^-$.

(If other long-range forces besides EM, add corresponding charges.)

Any black holes resulting from collapse will be determined by M, J, Q.

Baryon number, lepton number not conserved!

Throw protons into black hole

Thermal radiation
$B=0$.

This is consistent with results in quantum field theory; B, L both violated by non-perturbative electroweak effects.

Information loss paradox Problem with unitarity in QM. What happens to information encoded in matter states falling into black hole?

In QM, given state $|\Psi(t)\rangle$, can reconstruct state at earlier times via unitary time evolution: $|\Psi(t_0)\rangle = e^{iH(t-t_0)}|\Psi(t)\rangle$.

But, all black holes with same M, J, Q will evolve to <u>same</u> thermal state. What happened to info in state before collapse?

Possible resolutions:

(1) the radiation is not quite thermal, encodes "lost" information

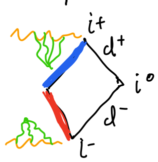
(2) the state after black hole evaporation "fuzzball" encodes lost information.

Singularity Theorems (Hawking & Penrose)

Generic collapse always leads to singularities.

technical definitions required.

"Future event horizon" = Surface inside of which timelike curves cannot escape to i_+ (future timelike ∞) = boundary of causal past of $\mathscr{I}^+$



"Past event horizon" = Surface inside of which timelike curves could not have come from i^-

Event horizons are <u>null surfaces</u>. Vectors tangent to them are null.

In charts/metrics we will use, horizons $\Leftrightarrow$ $g^{rr} = 0$ $\Leftrightarrow$ $g_{rr} = \infty$

($r = 2GM$ for Schwarzschild)

Cosmic Censorship Conjecture

Naked singularities can't occur in generic collapse of non-singular space.

not hidden by horizon, signal escapes to $\mathscr{I}^+$?

No precise proof, but circumstantial evidence, including numerical.

Counterexamples fine-tuned, not "generic".

Hawking's Area theorem Assuming cosmic censorship and reasonable matter,

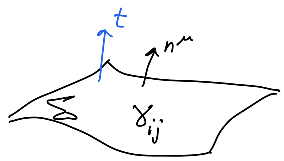
weak energy condition $T_{\mu\nu} t^\mu t^\nu \geq 0$ for all timelike t^μ. Violated by Hawking radiation

the area of a future horizon can't decrease.

(Recall $R_s = R_{horizon} = 2GM$. Throw more stuff in, horizon gets bigger.)

Define mass, charge, spin of a region:
$M \quad Q \quad J$

not a point!



Conserved quantity $\quad q = \int_\Sigma d^3\vec{x}\, \sqrt{\gamma}\, n_\mu j^\mu$

conserved current

normal vector to Σ

spacelike region (solid sphere?)

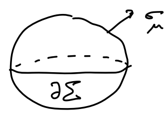
determinant of 3-metric on Σ

Electric charge Recall Maxwell's eqns $\quad \nabla_\nu F^{\mu\nu} = 0 \qquad \nabla_\mu J^\mu_{EM} = 0$

Let $Q = -\int_\Sigma d^3\vec{x}\, \sqrt{\gamma}\, n_\mu J^\mu_{EM} = -\int_\Sigma d^3\vec{x}\, \sqrt{\gamma}\, n_\mu \nabla_\nu F^{\mu\nu}$ $\quad$ Use Stokes' Thm

convention

$$\boxed{Q = -\int_{\partial\Sigma} d^2\vec{x}\, \sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu F^{\mu\nu}}$$



boundary of Σ

outward normal to ∂Σ (spacelike)

This is curved space version of Gauss' Law, integral form.

Test this for Minkowski, with charge Q at $r=0$. $\quad n^\mu = (1,0,0,0) \qquad n_\mu = (-1,0,0,0)$

(t,r,θ,ϕ) $\qquad \sigma^\mu = (0,1,0,0) \qquad \sigma_\mu = (0,1,0,0)$

$$\gamma^{(2)}_{ij} = r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \Rightarrow \sqrt{\gamma^{(2)}} = \sqrt{r^2\left(r^2 \sin^2\theta\right)} = r^2 \sin\theta$$

$$d^2\vec{x} = d\theta\, d\phi$$

$$d^2\vec{x}\, \sqrt{\gamma^{(2)}} = r^2\, d(\cos\theta)\, d\phi$$

$$\text{So} \quad Q = -\underbrace{\int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi}_{4\pi}\; r^2 \underbrace{\left(-F^{01}\right)}_{-E_r = -\frac{Q}{4\pi r^2}} = Q\ \checkmark$$

For BH, $\quad Q = -\int_{\partial\Sigma} d^3x\, \sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu F^{\mu\nu}$

2-sphere enclosing horizon. For example, i^0 = spacelike ∞.

Mass (= total energy) Need a Killing vector timelike at ∞.

$$E = \int_\Sigma d^3\vec{x}\, \sqrt{\gamma}\, n_\mu J^\mu$$

what is this?

Try: $J^\mu \stackrel{?}{=} J^\mu_T = K_\nu T^{\mu\nu}$ ✗ No good! In Schwarzschild, vanishes everywhere except singularity

Try: $J^\mu = J^\mu_R = K_\nu R^{\mu\nu} = \nabla_\nu \nabla^\mu K^\nu$ ← eq. 3.177, follows from Killing's equation $\nabla_{(\mu} K_{\nu)} = 0$

Conserved: $\nabla_\mu J^\mu_R = \nabla_\mu (K_\nu R^{\mu\nu}) = \underbrace{\nabla_\mu K_\nu}_{=0} R^{\mu\nu} + \underbrace{K_\nu \underbrace{\nabla_\mu R^{\mu\nu}}_{=\frac{1}{2}\nabla^\nu R}}_{=0}$ Bianchi identity (3.140)

$= 0$ $\qquad K_\nu \nabla^\nu R = 0$ (3.178)

So $\boxed{E = \frac{1}{4\pi G} \int_\Sigma d^3\vec{x}\, \sqrt{\gamma}\, n_\mu \nabla_\nu (\nabla^\mu K^\nu).}$

arbitrary constant, to agree with usual definition of mass.

Use Stokes' thm again: $E = \frac{1}{4\pi G} \int_{\partial\Sigma = i^0 = \text{spacelike } \infty} d^2\vec{x}\, \sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu \nabla^\mu K^\nu =$ Komar mass (Conserved, independent of t)

Example: Komar mass for Schwarzschild

Need n_μ (timelike), σ_μ (radial), K^ν (timelike at ∞).

$n_\mu n^\mu = -1$ and $\sigma_\mu \sigma^\mu = 1$

$n_\mu = (n_t, 0, 0, 0)$ and $n_\mu n^\mu = n_\mu n_\nu g^{\mu\nu} = -\left(1 - \frac{2GM}{r}\right)^{-1} (n_t)^2$

So $n_t = \sqrt{1 - \frac{2GM}{r}}$.

Also $\sigma_\mu \sigma^\mu = g^{rr} (\sigma_r)^2 = \left(1 - \frac{2GM}{r}\right)(\sigma_r)^2$, so $\sigma_r = \left(1 - \frac{2GM}{r}\right)^{-1/2}$

So $n_\mu \sigma_\nu \nabla^\mu K^\nu = -\nabla^t K^r$.

Killing vector $K^\mu = (1, 0, 0, 0)$. $\quad \left(K^r = 0, \text{ but...}\right)$

$\nabla^t K^r = g^{tt} \nabla_t K^r = g^{tt} (\cancelto{0}{\partial_t K^r} + \Gamma^r_{t\mu} K^\mu)$

$= g^{tt} \Gamma^r_{tt} \underbrace{K^t}_{1} = -\left(1 - \frac{2GM}{r}\right)^{-1} \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right) = -\frac{GM}{r^2}$

Already found $d^2\vec{x}\, \sqrt{\gamma^{(2)}} = r^2\, d(\cos\theta)\, d\phi$

So $E = \frac{1}{4\pi G} \int \underbrace{d(\cos\theta)\, d\phi}_{4\pi}\, r^2 \frac{GM}{r^2} = M$ ✓

Other conserved E definitions exist.

Arnowitt - Deser - Misner energy: For asymtotically flat spacetimes

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$ small as $r \to \infty$

Then $$E_{ADM} = \frac{1}{4\pi G} \int_{i^0} d^2\vec{x} \sqrt{\gamma^{(2)}} \; \underbrace{\sigma^i \left(\partial_j h^j{}_i - \partial_i h^j{}_j \right)}$$

indices raised + lowered with $\eta_{\mu\nu} \to \delta_{ij}$

Agrees with Komar mass <u>if</u> $h_{\mu\nu}$ doesn't depend on t at i^0, so $K^\mu = (1,0,0,0)$ at ∞.

For spacetimes obeying technical (but reasonable) assumptions,

non-singular, asymtotically flat, "dominant energy condition" on $T_{\mu\nu}$

$E_{ADM} \geq 0$.

For $E_{ADM} = 0 \iff$ Minkowski

Angular Momentum Suppose $K = \partial_\phi$ is a Killing vector.

(May be 1, 2, or 3).

Then $J^\mu = K_\nu R^{\mu\nu}$ is conserved (just like Komar energy).

$$J_\phi = -\frac{1}{8\pi G} \int_{\partial\Sigma} d^2\vec{x} \sqrt{\gamma^{(2)}} \; n_\mu \sigma_\nu \nabla^\mu K^\nu$$

label, not index

Same form as E_{Komar}, but with $K^\nu \to -\frac{1}{2} K^\nu$

timelike $\qquad \partial\phi$

Charged Black Holes For EM field strength

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$T_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$ ($\neq 0$ outside of BH)

$$F_{rt} = -F_{tr} = -\frac{Q}{r^2}$$ (units $4\pi\epsilon_0 = 1$)

$$F_{\theta\phi} = F_{\phi\theta} = \cdots = 0$$

Einstein: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$

Simultaneous solution is Reissner–Nordstrom
~1916

Define $\Delta = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}$ $\qquad$ $M =$ Komar mass $\quad$ $Q =$ charge

$ds^2 = -\Delta\, dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2$ $\qquad$ Generalizes Schwarzschild

- Four Killing vectors (one timelike)
- True (curvature) singularity at $r=0$
- Horizon(s) where $\Delta = 0$ $\quad \left(\frac{dt}{dr} = \infty \text{ for null radial paths}\right)$

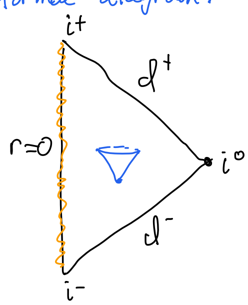
  This is a fake coordinate singularity only.

  Solve for horizon: $r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2}$

  - No horizon if $GM^2 < Q^2$
  - Two horizons if $GM^2 > Q^2$ $\quad$ "realistic" case
  - One horizon if $GM^2 = Q^2$

No horizon case $\quad$ "Too much" charge/mass $\Rightarrow$ naked singularity at $r=0$, timelike.

Conformal diagram: like Minkowski, but $r=0$ is singular.



Timelike geodesics never reach $r=0$

Observer can get to $r=0$, but not on a geodesic.

Can return to $\mathscr{I}^+$ from $r=0$.

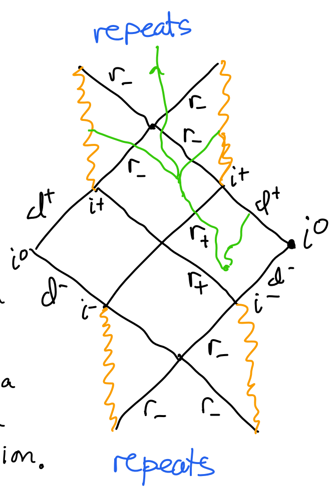
Two horizon case

$r > r_+$ $\quad \partial_r$ is spacelike

For $r_- < r < r_+$, $\quad \partial_r$ is timelike

$r < r_-$ $\quad \partial_r$ is spacelike

Observer falling into r_+ horizon need not die at $r=0$, singularity is a place, not a time. May re-emerge in a new asymptotically flat region.



(maximally extended!)

Sadly, not a realistic product of collapse.

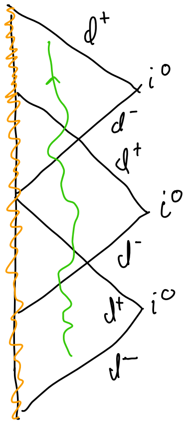
<u>One horizon case</u> $GM^2 = Q^2$ Horizon $r = 2GM$.

Called "extremal" black hole. Unstable, dump in a little more mass → two horizons. Mass, charge balanced.

Two extremal black holes w/ same charge sign will neither attract nor repel.

M_1 M_2 $F_{grav} = -\frac{GM_1M_2}{r^2}$ $F_{EM} = +\frac{Q_1Q_2}{r^2}$

Very unrealistic, but fun toys. Exact solutions with <u>many</u> same-sign extremal black holes. Surprising because GR is non-linear.



Observer can visit infinitely many asymptotically flat universes.

Rotating Black Holes Kerr 1963

No more spherical symmetry. Two Killing vectors: $K^i = \partial_t$ and ∂_ϕ.

Stationary, but not static. $R_{\mu\nu} = 0$.

To simplify writing, define:

$$\left.\begin{aligned} \Delta(r) &= r^2 - 2GMr + a^2 \\ \rho^2(r,\theta) &= r^2 + a^2\cos^2\theta \end{aligned}\right\}$$

M, $a = \frac{J}{M}$ are constants

M = Komar mass J = angular momentum

For no naked singularity, need $a < 1$. From collapse, $a < 0.998$.

Evidence for $a \neq 0$ from gravity waves

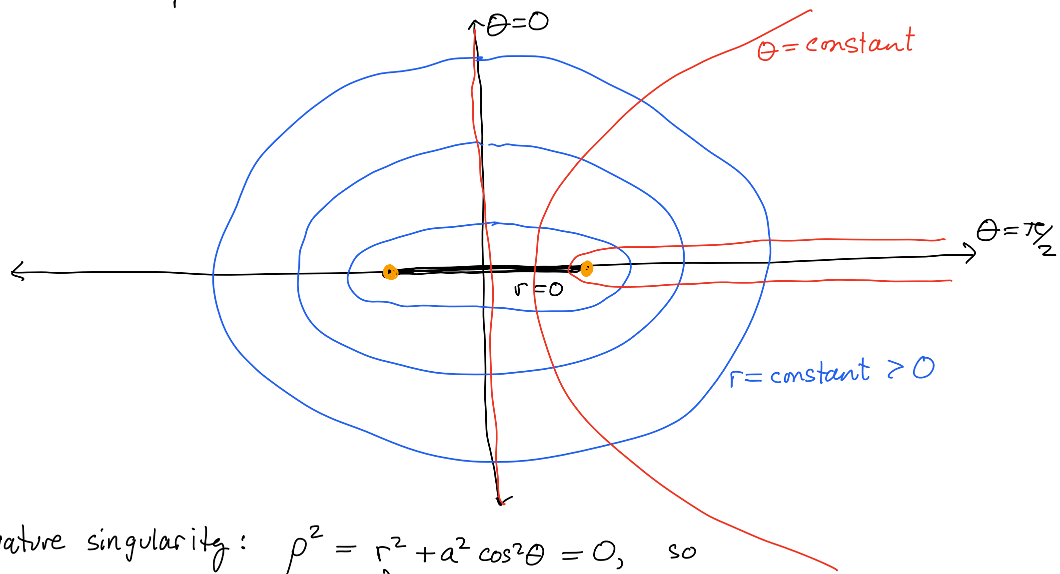
Then
$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi^2 - \frac{2GMar}{\rho^2}\underbrace{(dt\,d\phi + d\phi\,dt)}_{\text{cross terms}}$$

→ not static.

Recover Schwarzschild for $J = 0$.

Proper distance between $(r=0, \theta=0)$ and $(r=0, \theta=\frac{\pi}{2})$ is $a=\frac{J}{m}$

Coordinate map of Kerr (non-extended) in (r,θ) at $t=0$.



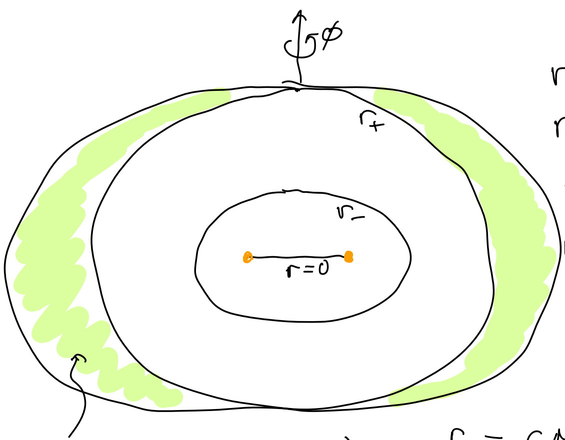
Curvature singularity: $\rho^2 = r^2 + a^2\cos^2\theta = 0$, so $\left(r=0 \text{ and } \theta=\frac{\pi}{2}\right)$ (ring)

$r=0$, $\theta=\frac{\pi}{2}$ is coordinate (fake) singularity

Event horizons where $g^{rr}=0 \Rightarrow \Delta = 0 \Rightarrow r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$ coordinate singularity

Can be extended to $r<0$ (not shown).

Redraw:



$r_+ = GM + \sqrt{G^2M^2-a^2}$ = outer horizon

$r_- = GM - \sqrt{G^2M^2-a^2}$ = inner horizon

$r=0$, $\theta=\frac{\pi}{2}$ = singularity ring

$r=0$, $\theta<\frac{\pi}{2}$ = portal to new region

$r=-\infty$ = two new asymptotically flat regions, repeats (doesn't occur in realistic collapse)

ergosphere $(r_+ < r < r_e)$

$r_e = GM + \sqrt{G^2M^2 - a^2\cos^2\theta}$ = stationary limit surface

$K^\mu = \partial_t$ is spacelike

light cones force all observers (not just geodesics) to move in $+\phi$ direction (can still escape to $r=\infty$)

Bizarre feature: for $r<0$, there are closed timelike curves.

Consider path with θ, t, r constant. Take $\theta = \frac{\pi}{2}$.

$$ds^2 = d\phi^2 \underbrace{\frac{1}{r^2}\left[(r^2+a^2 - a^2(a^2 - GMr + r^2)\right]}_{\approx \frac{2Ga^2}{r} \text{ for } |r| \ll a.}$$

For r small and negative, $ds^2 < 0$ on a path with $\theta = \frac{\pi}{2}$, $t = \text{const}$, ϕ changing. So, this is a timelike curve, and when you go from $\phi = 0$ to $\phi = 2\pi$, you meet your past self!

This is hidden behind the horizons r_+, r_-. Doesn't occur in realistic collapse.

However, the ergosphere is a feature of realistic collapse.

Geodesics in Kerr

Don't solve geodesic eqns, instead use Killing vectors.

$p^\mu = m \frac{dx^\mu}{d\tau}$. Let t, ϕ vary on geodesic

$$E = -K_\mu p^\mu = -K^\nu g_{\nu\mu} p^\mu = -g_{t\mu} p^\mu = -g_{tt}\, m \frac{dt}{d\tau} - g_{t\phi}\, m \frac{d\phi}{d\tau}$$

So $$E = m\left(1 - \frac{2GMr}{\rho^2}\right)\frac{dt}{d\tau} + \frac{2mGMar}{\rho^2}\sin^2\theta \frac{d\phi}{d\tau} = \text{const}$$

Also $$L = R_\mu p^\mu = R^\nu g_{\nu\mu} p^\mu = g_{\phi\mu} p^\mu = g_{\phi\phi} p^\phi + g_{\phi t} p^t$$

$$= m \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2) - a^2 \Delta \sin^2\theta\right]\frac{d\phi}{d\tau} - \frac{2mGar\sin^2\theta}{\rho^2}\frac{dt}{d\tau} = \text{const}$$

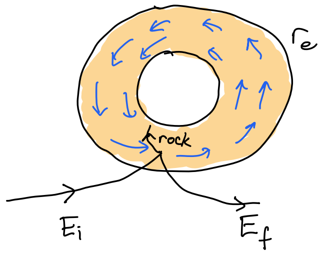
For large r, $\rho^2 \sim r^2$, so

$$E \approx m\left[\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} + \frac{2GMa}{r}\sin^2\theta \frac{d\phi}{d\tau}\right] > 0$$

Inside the ergosphere, K^μ is spacelike, $1 - \frac{2GMr}{\rho^2} < 0$, so can have $\underline{E<0}$.

Penrose process = "free" energy

Top view of Kerr at $\theta = \frac{\pi}{2}$ (equator)



1) Enter ergosphere on a geodesic, with rock. energy E_i
2) Throw rock into horizon, against rotation. $E_{rock} < 0$.
3) Exit ergosphere on geodesic with $E_f > E_i$

Energy comes from decreasing mass and slowing rotation of BH.
(Carroll section 6.7)

Start with (M, J) and slow rotation to 0, left with Schwarzschild black hole with mass M_{min}, where

$$M_{min} = \sqrt{\frac{1}{2}\left(M^2 + \sqrt{M^4 - J^2/G^2}\right)}$$

Large r limit of Kerr: (small a)

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 + \frac{2GM}{r}\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$- 2GJ\,\frac{\sin^2\theta}{r}\left(dt\, d\phi + d\phi\, dt\right)$$

Charged & rotating black holes

Take Kerr, replace $M \to M - \frac{Q^2}{2r}$ everywhere. Kerr-Newman

Also has an ergosphere.

This is the general M, J, Q case, realizes the No-Hair Theorem.