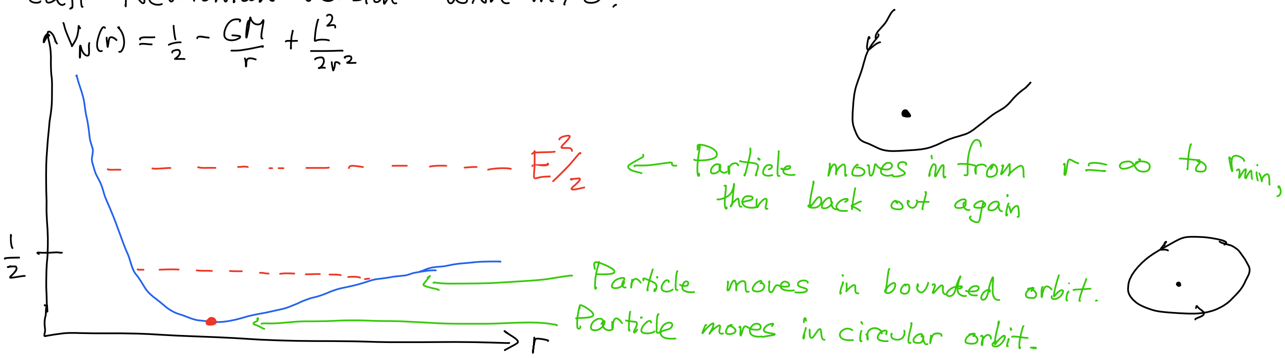
Potential for geodesics in Schwarzschild (continued)

$$V(r) = \frac{1}{2}\left(1 - \frac{2GM}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right)$$

$$= \underbrace{\frac{1}{2}\epsilon}_{\text{constant}} \underbrace{- \frac{\epsilon GM}{r} + \frac{L^2}{2r^2}}_{\text{Newton}} \underbrace{- \frac{GML^2}{r^3}}_{\text{GR}} \quad \text{GR = extra attractive term}$$

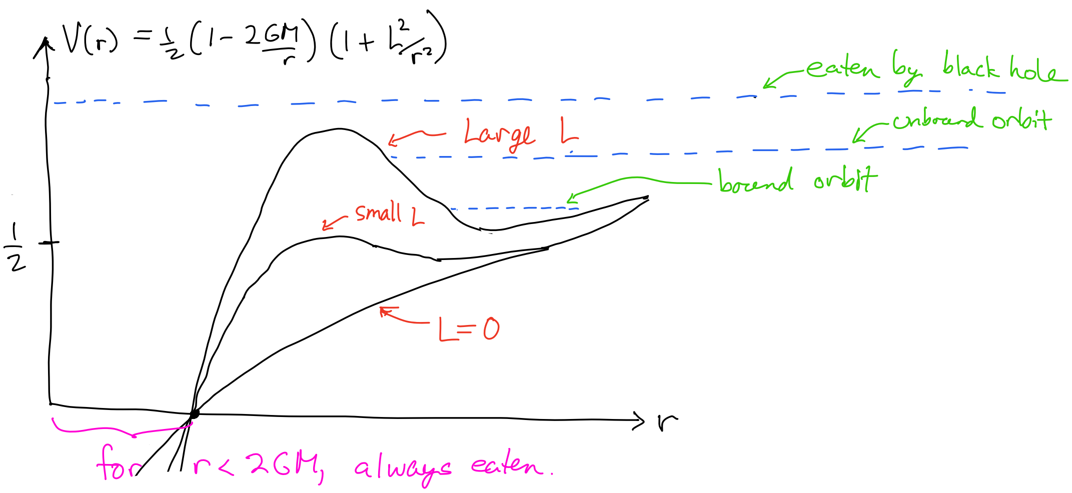
Now solve for $r(\lambda)$, then can use $\begin{cases} E \to t(\lambda) \\ L \to \phi(\lambda) \end{cases}$

Recall Newtonian version with $m \neq 0$.



Circular orbit: $\left.\frac{\partial V}{\partial r}\right|_{r=r_c} \Longrightarrow \frac{GM}{r_c^2} - \frac{L^2}{r_c^3} = 0 \rightarrow r_c = \frac{L^2}{GM}$

GR version:



Circular orbits? $\left.\frac{\partial V}{\partial r}\right|_{r=r_c} = 0 \Rightarrow \frac{GM}{r_c^2} - \frac{L^2}{r_c^3} + \frac{3GML^2}{r_c^4} = 0$

So $r_c = \dfrac{L^2 \pm \sqrt{L^4 - 12L^2G^2M^2}}{2GM}$ $+$ = stable, $-$ = unstable

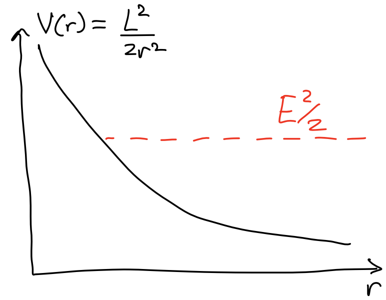
$= \dfrac{L^2}{2GM}\left(1 \pm \sqrt{1 - \dfrac{12G^2M^2}{L^2}}\right)$

For large $\dfrac{L}{GM}$: $r_c = \dfrac{L^2}{2GM}\left(1 \pm \left[1 - \dfrac{6G^2M^2}{L^2} + \dots\right]\right) \Rightarrow$

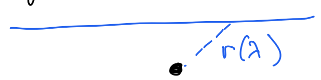
$\underbrace{r_c = \dfrac{L^2}{GM} - 3GM}_{\text{stable: Newtonian result}}$ or $\underbrace{r_c = 3GM}_{\text{unstable}} = \frac{3}{2}R_s$

- For $L < \sqrt{12}\,GM$, no circular orbits ($\sqrt{\ }$ = imaginary)
- For $L = \sqrt{12}\,GM$, $r_c = \dfrac{L^2}{2GM} = 6GM = 3R_s$ (unstable)
- Need $r_c > 6GM = 3R_s$ for stable circular orbits
  $\frac{3}{2}R_s < r_c < 3R_s$ for unstable " " .

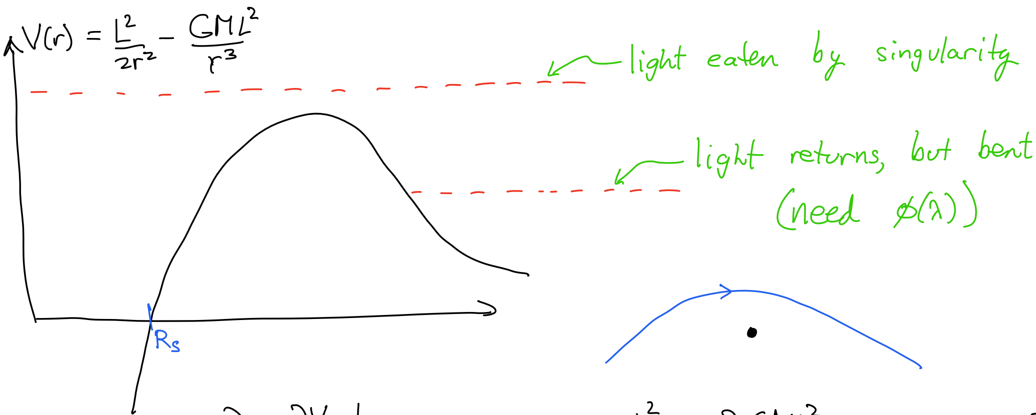
Geodesics for light ($m=0$, $\epsilon = 0$) Newtonian case



all light rays return (in fact, straight lines)



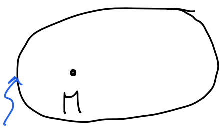
GR case



Circular orbit for light? $\left.\dfrac{\partial V}{\partial r}\right|_{r=r_c} = 0 \Rightarrow -\dfrac{L^2}{r_c^3} + \dfrac{3GML^2}{r_c^4} = 0 \rightarrow r_c = 3GM$ (unstable)

Precession of perihelia = classic test of GR. For $m \neq 0$:

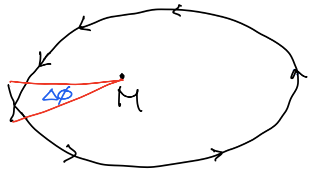
Newton



$r(\phi)$ has period 2π

closest approach

GR



Closest approach

$\phi_2 - \phi_1 = 2\pi + \Delta\phi$

Need to solve for $r(\phi)$ $\quad \left(\frac{d\phi}{d\tau}\right)^2 = \frac{L^2}{r^4} \qquad \left(\frac{dr}{d\tau}\right)^2 = E^2 - \underbrace{\left(1 - \frac{2GM}{r}\right)\left(1 + \frac{L^2}{r^2}\right)}_{2V(r)}$

So $\boxed{\left(\frac{dr}{d\phi}\right)^2 = \frac{(E^2-1)}{L^2} r^4 + 2GMr - r^2 + \frac{2GM}{L^2} r^3}$

Trick 1: define $x = \frac{L^2}{GMr} \rightarrow \left(\frac{dx}{d\phi}\right)^2 = \frac{(E^2-1)L^2}{G^2M^2} + 2x - x^2 + \frac{2G^2M^2}{L^2} x^3$

Trick 2: Take $\frac{d}{d\phi}$, divide by $\frac{dx}{d\phi} \rightarrow \frac{d^2x}{d\phi^2} = \underbrace{1 - x}_{\text{Newton}} + \underbrace{\frac{3G^2M^2}{L^2} x^2}_{\text{GR correction}}$

Newtonian solution: $x = 1 + e\cos(\phi - \phi_0)$ solves $\frac{dx}{d\phi} = 1 - x$

eccentricity of ellipse

choose $\phi_0 = 0$

GR solution: try $x = 1 + e\cos\phi + \delta$ ← Carroll calls this x_1

Plug in, drop δ^2 and higher. $\quad \frac{d^2\delta}{d\phi^2} + \delta = \frac{3G^2M^2}{L^2}(1 + e\cos\phi)^2.$

Solution: $\delta = \frac{3G^2M^2}{L^2}\left[\underbrace{\left(1 + \tfrac{1}{2}e^2\right)}_{\text{constant}} - \underbrace{\tfrac{1}{6}e^2\cos(2\phi)}_{\text{oscillates, averages to 0, ignore}} + \underbrace{e\phi\sin\phi}_{\text{grows over many orbits}}\right]$

So $x = \frac{L^2}{GMr} = 1 + e\left(\cos\phi + \underbrace{\frac{3G^2M^2}{L^2}}_{\alpha}\phi\sin\phi\right) + \ldots$ ← small, doesn't accumulate

So $x = 1 + e(\cos\phi + \alpha\,\phi\sin\phi) + \dots \approx 1 + e\cos[(1-\alpha)\phi] + \mathcal{O}(\alpha^2)$

for small α

Perihelion at $\phi = 0$, $\phi = \frac{2\pi n}{1-\alpha} = 2\pi n + 2\pi n\alpha + \dots$

So $\frac{\Delta\phi}{n} = 2\pi\alpha = \frac{6\pi G^2 M^2}{L^2}$. For a nearly circular orbit, $L^2 \approx GM(1-e^2)a$

Semi-major axis

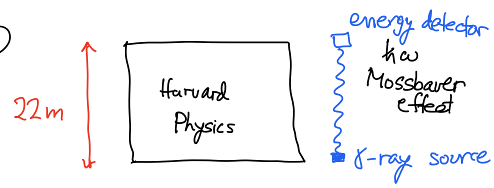
So $$\boxed{\frac{\Delta\phi}{\text{orbit}} = \frac{6\pi GM}{a(1-e^2)c^2}}.$$

For Mercury, $\begin{cases} M_{sun} = 1.99\times10^{30}\ \text{kg} \\ a = 57.9\times10^6\ \text{km} \\ e = 0.2056 \end{cases}$ prediction: $\frac{\Delta\phi}{\text{orbit}} = \frac{0.103''}{\text{orbit}} = \frac{43.0''}{\text{century}}$

Experiment: $\frac{\Delta\phi}{\text{orbit}} = 5601 - \underbrace{(5025)}_{\text{Earth's frame rotates}} - \underbrace{532}_{\text{other planets perturb}} = 43$

Gravitational Redshift

Pound & Rebka, 1960



Observer at fixed (r,θ,ϕ) has 4-velocity

$U^\mu = (U^0, 0, 0, 0)$ with $-1 = U^\mu U^\nu g_{\mu\nu} = U^0 U^0\left(1 - \frac{2GM}{r}\right)$, so

$U^0 = \left(1 - \frac{2GM}{r}\right)^{-1/2}$. Now $\hbar\omega = -p_\mu U^\mu$ ← velocity of observer

p_μ ← 4-momentum of photon

or $\hbar\omega = -U^\mu g_{\nu\mu}\frac{dx^\nu}{d\lambda} = -U^0 g_{00}\frac{dt}{d\lambda} = -\left(1-\frac{2GM}{r}\right)^{-1/2}\underbrace{\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda}}_{-K_\mu \frac{dx^\mu}{d\lambda} = E = \text{constant}}$

So $\hbar\omega = \left(1 - \frac{2GM}{r}\right)^{-1/2} E$ (Check: $r\to\infty$ usual formula).

$\hbar\omega$: energy for atomic transitions in emitter, detector

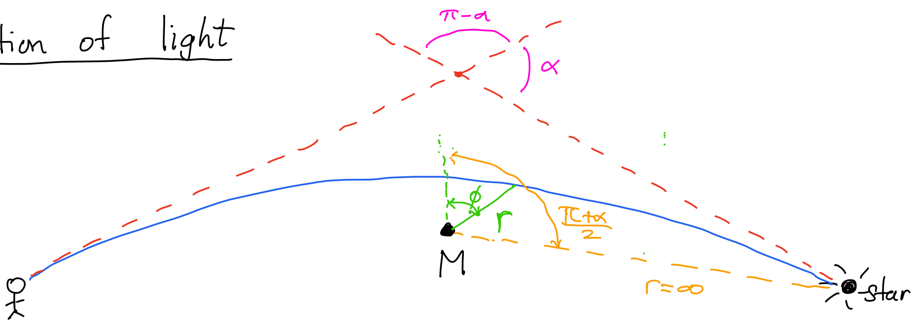
E: constant energy, conserved from timelike Killing vector

Taking the ratio at two different heights,

$$\frac{\hbar\omega_1}{\hbar\omega_2} = \frac{\left(1-\frac{2GM}{r_2}\right)^{1/2}}{\left(1-\frac{2GM}{r_1}\right)^{1/2}} \approx 1 + GM\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \approx 1 + GM\frac{(r_2-r_1)}{r_1 r_2}$$

For $r_2 > r_1$, $\frac{\hbar\omega_1}{\hbar\omega_2} > 1$ or $\hbar\omega_2 < \hbar\omega_1$. Photon loses energy as it climbs out of potential well.

Deflection of light



Need to solve for null geodesic. Recall $\left(\frac{d\phi}{d\lambda}\right)^2 = \frac{L^2}{r^4}$ and

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{L^2}{r^2}\left(1-\frac{2GM}{r}\right). \quad \text{so}$$

$$\boxed{\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4E^2}{L^2} - r^2\left(1-\frac{2GM}{r}\right) \quad \text{null geodesic}}$$

Let $u = \frac{GM}{r}$

$$\left(\frac{du}{d\phi}\right)^2 = \frac{G^2M^2E^2}{L^2} - u^2 + 2u^3.$$

Take $\frac{d}{d\phi}$, divide by $2u'$.

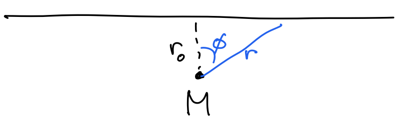
$$u'' = -u + 3u^2$$

GR correction.

Consider Newtonian limit first: $u'' = -u \Rightarrow u = C_1\cos(\phi-\phi_0)$.

When $\phi = 0$, $r = r_0 =$ minimum, so u is maximum. $\phi_0 = 0$, $C_1 = \frac{GM}{r_0}$.

So $u = \frac{GM}{r_0}\cos(\phi) = \frac{GM}{r} \implies r\cos\phi = r_0$ (straight line).



For GR, try $u = u_N + \delta$, where $u_N = \frac{GM}{r_0}\cos\phi$, $\delta \ll u_N$

$$\underbrace{u_N'' + \delta'' = -u_N}_{0} - \delta + 3(u_N+\delta)^2, \quad \text{so} \quad \delta'' = -\delta + 3u_N^2 = -\delta + \frac{3G^2M^2}{r_0^2}\cos^2\phi.$$

Solution: $\delta = \frac{G^2M^2}{2r_0^2}\left(4-\cos^2\phi\right) = \frac{G^2M^2}{2r_0^2}\left(3-\cos(2\phi)\right)$, so

$$u = \frac{GM}{r} = \frac{GM}{r_0}\left[\cos\phi + \frac{GM}{2r_0}\left(3-\cos(2\phi)\right) + \dots\right].$$

Need $\phi = \frac{\pi}{2}+\frac{\alpha}{2}$ when $r \to \infty$, which means $u \to 0$.

$$0 = \frac{GM}{\infty} = \frac{GM}{r_0}\left[\underbrace{\cos\left(\frac{\pi}{2}+\frac{\alpha}{2}\right)}_{-\sin\left(\frac{\alpha}{2}\right)} + \frac{GM}{2r_0}\underbrace{\left(4-\cos^2\left(\frac{\pi}{2}+\frac{\alpha}{2}\right)\right)}_{4+\mathcal{O}(\alpha^2)}\right]$$

So $\sin\left(\frac{\alpha}{2}\right) = \frac{2GM}{r_0} \Rightarrow \alpha = \frac{4GM}{r_0} = \frac{2R_s}{r_0}$.

For the Sun, with $r_0 \approx R_{sun}$ and $M = M_{sun}$,

$$\alpha = \frac{2(2.95\text{ km})}{6.96\times10^5\text{ km}} = 8.48\times10^{-6}\text{ radians} = 4.86\times10^{-4}\text{ degrees} = 1.75\underbrace{\text{ arc sec}}_{\frac{1}{(60)^2}\text{ degrees}}$$

Einstein got wrong in 1905, corrected 1915.

Verified by solar eclipse, 1919, to 10% accuracy.

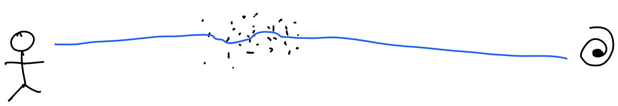
" " quasars to < 1%.

More generally, for a weak metric perturbation,

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)\left[dr^2 + r^2 d\Omega^2\right] \qquad \Phi \ll 1,$$

then $\alpha = 2\int_{\text{path}} ds\, (\vec{\nabla}\Phi)_\perp$ ← perpendicular to path.

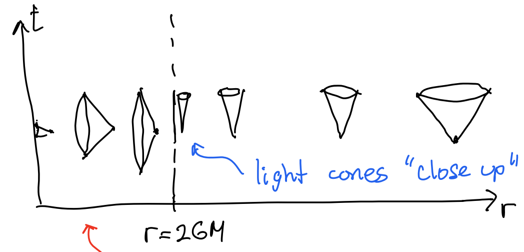
(Carroll section 7.3)



Light cones + new charts for Schwarzschild

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + \cancel{r^2 d\Omega^2}$$ consider radial null curves

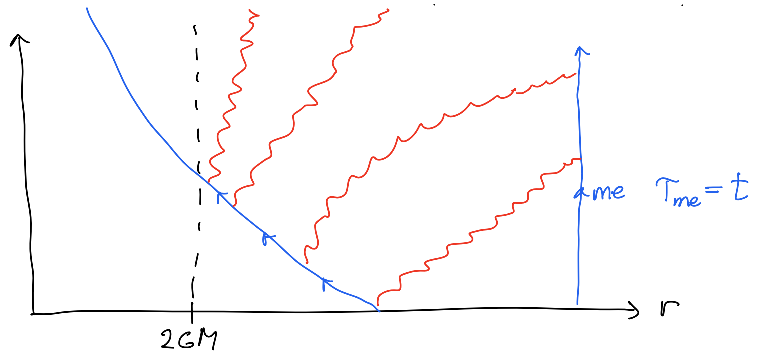
$$\frac{dt}{dr} = \pm\underbrace{\left(1-\frac{2GM}{r}\right)^{-1}}$$ $\to \infty$ for $r \to R_s$; $\to 1$ for $r \to \infty$



For $r < 2GM$, ∂_r is timelike, ∂_t is spacelike.

$$\frac{dr}{dt} = \pm\left(\frac{2GM}{r} - 1\right)$$

A distant observer (me) watches you fall in:



As you get closer to $r = 2GM$, I have to wait longer to see your light signals (redshifted).

I never see the light signal from $r = 2GM$, even though you get there in finite proper time τ_{you}.

Use better coordinates
- tortoise ← waystation on path to enlightenment
- Eddington-Finkelstein ← useful for problems
- Kruskal ← reveals secrets!

Define new coordinate $r^* =$ (tortoise) so that light cones always $45°$ drawn in (r^*, t).

$$\frac{dt}{dr^*} = \pm 1 \quad \to \quad \frac{dr^*}{dr} = \left(1-\frac{2GM}{r}\right)^{-1} \quad \to \quad r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$$

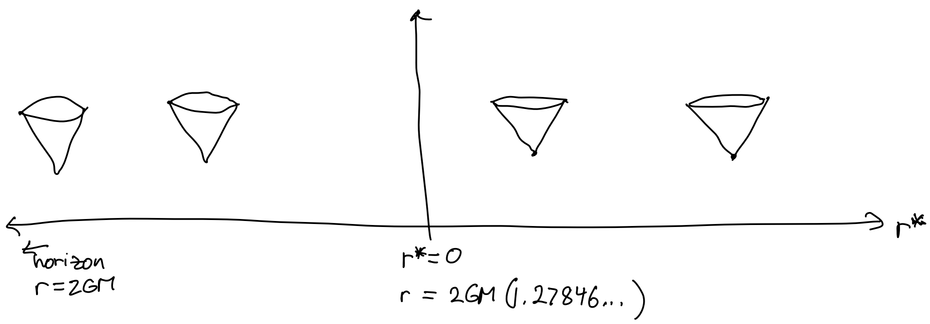
Then $ds^2 = \left(1-\frac{2GM}{r}\right)\left(-dt^2 + dr^{*2}\right) + r^2 d\Omega^2$

implicitly determined by

Note: $r = 2GM \leftrightarrow r^* = \ln(0) = -\infty$

$r = \text{large} \Leftrightarrow r^* = r$

$r < 2GM \Leftrightarrow r^*$ undefined (chart doesn't cover)



E-F coordinates

Let $u = t - r^*$ ← outgoing null radial geodesics $u = \text{const}$

$v = t + r^*$ ← ingoing " " " $v = \text{const}$

Choose (v, r, θ, ϕ) "ingoing EF coords"

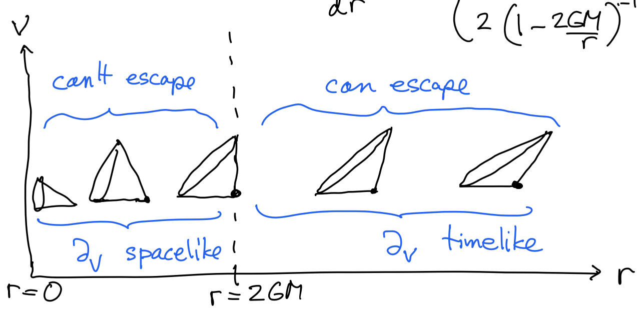
$$ds^2 = -\left(1-\frac{2GM}{r}\right)dr^2 + (dv\,dr + dr\,dv) + r^2 d\Omega^2$$

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2GM}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

invertible at $r = 2GM$

radial null curves

$$\frac{dv}{dr} = \begin{cases} 0 & (\text{in}) \\ 2\left(1-\frac{2GM}{r}\right)^{-1} & (\text{out}) \end{cases}$$



∂_r always null.

Kruskal coordinates (T, R, θ, ϕ) with

$$\boxed{\begin{aligned} T &= \left(\frac{r}{2GM}-1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right) \\ R &= \left(\frac{r}{2GM}-1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right) \end{aligned}}$$

Find dt, dr, plug in...

$$ds^2 = \frac{32\,G^3M^3}{r} e^{-r/2GM}\left(-dT^2 + dR^2\right) + r^2\, d\Omega^2$$

$r(T,R)$ defined by $T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}$

Radial null curves: $dT = \pm dR$ (light cones at 45° in T, R)

$r = 2GM$ at $T = \pm R$

$r = r_0 = \text{constant}$ $\underbrace{T^2 - R^2 = \text{constant}}_{\text{hyperbolas in } (T,R) \text{ plane}} = \left(1 - \frac{r_0}{2GM}\right) e^{r_0/2GM}$

$t = t_0 = \text{constant}$ at $\underbrace{\frac{T}{R} = \text{constant}}_{\text{lines in } (T,R) \text{ plane}} = \tanh\left(\frac{t_0}{4GM}\right)$

$r = r_0$ (singularity) at $T^2 - R^2 = 1$

Ranges: $-\infty < R < \infty$ $\quad -\sqrt{1+R^2} < T < \sqrt{1+R^2}$ $\quad 0 \leq \theta \leq \pi$ $\quad 0 \leq \phi \leq 2\pi$

not 0