Isometries = symmetries of manifold + metric

$(M, g_{\mu\nu}) \to (M, g_{\mu\nu})$ map of M onto itself, with $g_{\mu\nu}$ unchanged.

Two reasons to study:
* isometry $\to$ conserved quantity
* more symmetry $\to$ easier to solve for $g_{\mu\nu}$

Examples 1) R^n $\quad ds^2 = \sum_{i=1}^{n} dx_i^2$ has:

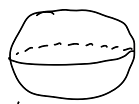
$$\left.\begin{array}{lll} x^\mu \to x^\mu + a^\mu & n & \text{translations} \\ x^\mu \to \Lambda^\mu{}_\nu x^\nu & \frac{n(n-1)}{2} & \text{rotations} \end{array}\right\} \frac{n(n+1)}{2} \text{ isometries}$$

(= maximum possible)

2) $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ has same isometries.

4 translations, 3 rotations, 3 boosts

3) S^2 $\quad ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$



3 rotations about x, y, z axes in embedded space.

4) A generic metric $g_{\mu\nu}(x) \to$ no isometries

Math fact Appendix B

Each infinitesimal isometry $\Longleftrightarrow$ vector field K^μ not necessarily constant

K^μ = Killing field. Satisfies Killing's equation

$$\boxed{\nabla_{(\mu} K_{\nu)} = 0} \iff \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$$

For example, suppose $x^{\rho_*} \to x^{\rho_*} + a^{\rho_*}$ is an isometry.
some particular value

Then $\partial_{\rho_*} g_{\mu\nu} = 0$ (metric is unchanged), and $K = \partial_{\rho^*}$ is the Killing vector.

It has components $K^\mu = (\partial_{\rho_*})^\mu = \delta^\mu_{\rho^*}$

Killing vector $\longleftrightarrow$ conserved quantities $\begin{cases} \text{for individual particles on geodesics} \\ \text{for entire spacetime, if } K^\mu \text{ timelike } (K^\mu K_\mu < 0) \end{cases}$

Recall geodesic eqn for $p^\mu = m U^\mu$ = 4-momentum on timelike geodesic

$$p^\nu \nabla_\nu p_\mu = 0 \quad \Rightarrow \quad \underbrace{p^\nu \partial_\nu p_\mu}_{\underbrace{m \frac{dx^\nu}{d\tau} \frac{\partial}{\partial x^\nu} p_\mu}_{m \frac{d}{d\tau} p_\mu}} - \underbrace{\Gamma^\rho_{\sigma\mu} p^\sigma p_\rho}_{\frac{1}{2} \partial_\mu g_{\nu\sigma} p^\nu p^\sigma} = 0 .$$

So $m \frac{d}{d\tau} p_\mu = \frac{1}{2} (\partial_\mu g_{\nu\sigma}) p^\nu p^\sigma$. If translation is an isometry, $\partial_{\mu^*} g_{\nu\sigma} = 0$ for a particular μ^*, then $m \frac{d}{d\tau} p_{\mu^*} = 0$, corresponding 4-momentum component is conserved.

(In Minkowski, all 4.)

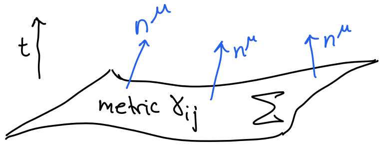
More generally, if $\nabla_{(\mu} K_{\nu)} = 0$, then

$$p^\mu \nabla_\mu (K_\nu p^\nu) = \underbrace{p^\mu p^\nu \nabla_\mu K_\nu}_{\substack{= p^\mu p^\nu (\nabla_{(\mu} K_{\nu)}) \\ = 0 \ \text{Killing}}} + K_\nu \underbrace{p^\mu \nabla_\mu p^\nu}_{=0, \text{ geodesic}} = 0, \text{ so } K_\nu p^\nu \text{ is conserved.}$$

Timelike Killing $\Leftrightarrow$ Conserved energy for whole spacetime + all contents.

Consider $J^\mu_T = K_\nu T^{\mu\nu}$ = conserved current

Then $\nabla_\mu J^\mu_T = \underbrace{T^{\mu\nu}}_{\text{symmetric}} \underbrace{\nabla_\mu K_\nu}_{0 \text{ (Killing)}} + K_\nu \underbrace{\nabla_\mu T^{\mu\nu}}_{0} = 0.$ Now consider a spacelike surface Σ.



n^μ = normal to Σ, in timelike direction $n_\mu n^\mu = -1$.

$$\text{Energy} = \int_\Sigma d^3\vec{x} \sqrt{\gamma}\, n_\mu J^\mu_T = \text{same for every } \Sigma.$$

Proof: Divergence thm.

$$0 = \int d^4x \sqrt{-g}\, \nabla_\mu J^\mu_T = \int_{\Sigma_2} - \int_{\Sigma_1} = \text{Energy 2} - \text{Energy 1}$$

Killing vectors for Minkowski

$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$$\underbrace{T_x = \partial_x \qquad T_y = \partial_y \qquad T_z = \partial_z}_{\text{translations}} \qquad \underbrace{T_t = \partial_t}_{\text{time translations}}$$

$$\underbrace{R_x = y\,\partial_z - z\,\partial_x \qquad R_y = z\,\partial_x - x\,\partial_z \qquad R_z = x\,\partial_y - y\,\partial_x}_{\text{rotations}}$$

$$\underbrace{B_x = x\,\partial_t + t\,\partial_x \qquad B_y = y\,\partial_t + t\,\partial_y \qquad B_z = z\,\partial_t + t\,\partial_z}_{\text{boosts}}$$

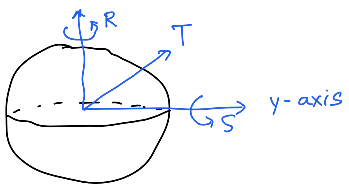
for example $(B_z)^0 = z \qquad (B_z)^1 = 0$

$(B_z)^2 = 0 \qquad (B_z)^3 = t$

Killing vectors for S^2:

$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

$R = \partial_\phi \qquad S = \cos\phi\, \partial_\theta - \cot\theta \sin\phi\, \partial_\phi \qquad T = -\sin\phi\, \partial_\theta - \cot\theta \cos\phi\, \partial_\phi$



These are maximally symmetric spaces $= \frac{n(n+1)}{2}$ isometries $=$ R same everywhere (necessary, not sufficient for $n \geq 3$).

$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$$

$$R_{\nu\sigma} = \frac{R}{n} g_{\nu\sigma}$$

Examples: R^n with flat metric $(R=0)$

S^n n-sphere $R>0$

H^n hyperboloid $R<0$

Expanding universe

$ds^2 = -dt^2 + a(t)^2\left[dx^2 + dy^2 + dz^2\right]$. For generic $a(t)$,

has 6 isometries (not maximally symmetric)

3 translations $\Rightarrow$ p_x, p_y, p_z conserved

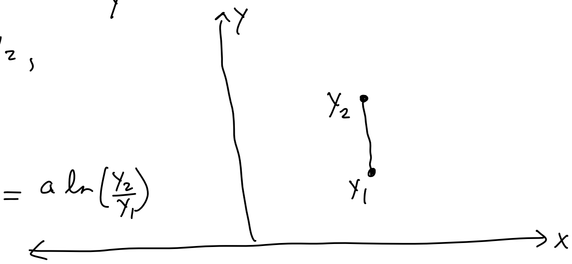
3 rotations $\Rightarrow$ L_x, L_y, L_z conserved

no timelike Killing vector, energy not conserved

no boosts

H^2 = Poincaré "half plane" $ds^2 = \frac{a^2}{y^2}(dx^2+dy^2)$ $(-\infty < x < \infty,\ y>0)$

Consider this path: y from y_1 to y_2, x constant



$$\Delta S = \int_{y_1}^{y_2} dy \sqrt{g_{\mu\nu}\frac{dx^\mu}{dy}\frac{dx^\nu}{dy}} = a\int_{y_1}^{y_2}\frac{dy}{y} = a\ln\left(\frac{y_2}{y_1}\right)$$

y = coordinate and parameter

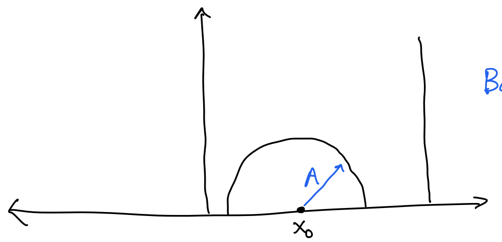
Any path that goes to $y=0$ has infinite length. Coordinates deceive!

Geometry: $\Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}$ $\Gamma^y_{xx} = \frac{1}{y}$ $\Gamma^y_{yy} = -1/y$ others 0.

$R^x{}_{yxy} = -\frac{1}{y^2}$ $R_{xyxy} = -\frac{1}{a^2}$ $R_{xx} = R_{yy} = -\frac{1}{y^2}$ $R = -\frac{2}{a^2}$ (constant negative curvature)

Geodesics: $(x-x_0)^2 + y^2 = A^2$ or $x = x_0 = \text{constant}$

constants



Both have ∞ length.

Proof: Let s = affine parameter along path = proper distance

$$ds^2 = \frac{a^2}{y^2}(dx^2+dy^2) \implies \boxed{\frac{y^2}{a^2} = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} \quad (*)$$

Geodesic eqns: $\frac{d^2x}{ds^2} + 2\underbrace{\Gamma^x_{xy}}_{-1/y}\frac{dx}{ds}\frac{dy}{ds} = 0 \implies \boxed{\frac{d^2x}{ds^2} - \frac{2}{y}\frac{dx}{ds}\frac{dy}{ds}} \quad (**)$

$$\frac{d^2y}{ds^2} + \underbrace{\Gamma^y_{xx}}_{1/y}\left(\frac{dx}{ds}\right)^2 + \underbrace{\Gamma^y_{yy}}_{-1/y}\left(\frac{dy}{ds}\right)^2 = 0 \Rightarrow \frac{d^2y}{ds^2} + \frac{1}{y}\left(\frac{dx}{ds}\right)^2 - \frac{1}{y}\left(\frac{dy}{ds}\right)^2 = 0$$

Redundant with (*) (**).

$$(**) \implies y^2\frac{d}{ds}\left(\frac{1}{y^2}\frac{dx}{ds}\right) = 0 \implies \frac{1}{y^2}\frac{dx}{ds} = \underset{\text{constant}}{b} \implies \boxed{\frac{dx}{ds} = by^2} \quad (***)$$

Plug into (*):

$$\frac{y^2}{a^2} = b^2y^4 + \left(\frac{dy}{ds}\right)^2 \Rightarrow \frac{dy}{ds} = y\sqrt{\frac{1}{a^2} - b^2y^2}.$$ Combine with (***),

$$dx = \frac{b\,y}{\sqrt{\frac{1}{a^2} - b^2y^2}}\,dy \Rightarrow dx = \frac{y\,dy}{\sqrt{A^2-y^2}}$$ where $A = \frac{1}{a\,b}$. ⟵ assumes $b \neq 0$.

Integrate both sides: $\sqrt{A^2-y^2} = x - x_0 \Rightarrow (x-x_0)^2 + y^2 = A^2$

If $b=0$, then $\frac{dx}{ds} = 0 \Rightarrow x = x_0 = \text{constant}$

$$\frac{dy}{ds} = \frac{y}{a} \Rightarrow d(\ln y) = \frac{ds}{a} \Rightarrow \ln y = \frac{s}{a} + \ln(y_0)$$

Killing's equations: $\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \Rightarrow \boxed{\partial_\mu K_\nu + \partial_\nu K_\mu - 2\Gamma^\rho_{\mu\nu} K_\rho = 0}$

$$\left.\begin{aligned}
&\partial_x K_x - \Gamma^\rho_{xx} K_\rho = 0 \Rightarrow \partial_x K_x - \tfrac{1}{y} K_y = 0\\
&\partial_y K_y - \Gamma^\rho_{yy} K_\rho = 0 \Rightarrow \partial_y K_y + \tfrac{1}{y} K_y = 0\\
&\partial_x K_y + \partial_y K_x - 2\Gamma^\rho_{xy} K_\rho \Rightarrow \partial_x K_y + \partial_y K_x + \tfrac{2}{y} K_x = 0.
\end{aligned}\right\}$$

Nicer to write in terms of $\left.\begin{aligned} K^x &= y^2 K_x \\ K^y &= y^2 K_y \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} K_x &= K^x/y^2 \\ K_y &= K^y/y^2 \end{aligned}\right.$

Then $\left\{\begin{aligned}
&\partial_x K^x - \tfrac{1}{y} K^y = 0\\
&\partial_y K^y - \tfrac{1}{y} K^y = 0\\
&\partial_x K^y + \partial_y K^x = 0
\end{aligned}\right.$

Try: $\partial_x K^x = f(x) = \frac{1}{y} K^y \Rightarrow K^y = y\,f(x)$ ✓

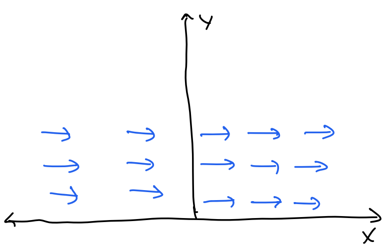
$\Rightarrow y\,f'(x) + \partial_y K^x = 0$

So $K^x = -\frac{y^2}{2} f'(x) + g(x) \Rightarrow -\frac{y^2}{2} f''(x) + g'(x) = f(x)$

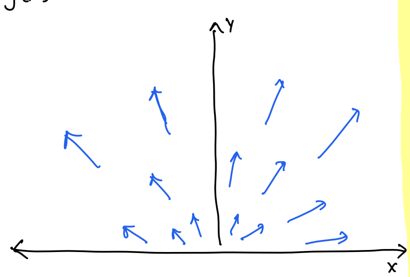
no y dependence $\Rightarrow f''(x) = 0$

So $\boxed{f(x) = ax+b.}$ Then $g'(x) = ax+b \Rightarrow \boxed{g(x) = \frac{ax^2}{2} + bx + c}$

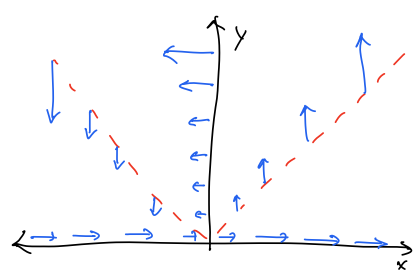
$g(x) = 1 \Rightarrow K^x = 1 \quad K^y = 0$



$g(x) = x \quad K^x = x \quad K^y = y$



$g(x) = x^2 \rightarrow K^x = (x^2 - y^2) \quad K^y = 2xy$



Gravity (Chapter 4)

1) Matter = source for fields
2) Fields = effect on matter

EM

1) $\partial_\mu F^{\mu\nu} = J^\nu$
2) $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Newton's gravity

$\nabla^2 \Phi = 4\pi G_N \rho$

$\vec{F} = -m \vec{\nabla} \Phi = m \vec{a}$

Einstein's gravity

$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$

$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0$

To check Einstein:

- Mathematical consistency involves tensors consistently $\left.\begin{matrix} \nabla_\mu G^{\mu\nu} = 0 \\ \nabla_\mu T^{\mu\nu} = 0 \end{matrix}\right\} \checkmark$
- Get Newton's in limit of slow motion, weak Φ
- Predict new experimental results

  Three classics: 1) Precession of perihelion of Mercury
  2) Deflection of light by Sun
  3) Gravitational redshift

Time delays of light, radar
Binary pulsar
Expansion of universe
Growth of structure

Black holes
Clocks (GPS)
Gravitational lensing
Gravitational waves

Newtonian limit

$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}$ for x^i = spacelike. Then:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{00} \left(\frac{dt}{d\tau}\right)^2 = 0.$$

Consider a gravitational field that is

Static: $\partial_t g_{\mu\nu} = 0.$

Weak: $g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{h_{\mu\nu}}_{\text{small}}$ $\quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, where

$$h^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\sigma} h_{\rho\sigma}$$

Evaluate Christoffels to <u>linear order</u> in $h_{\mu\nu}$ ...

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\rho}\left(\cancel{\partial_0 g_{\rho 0}} + \cancel{\partial_0 g_{0\rho}} - \partial_\rho g_{00}\right) = -\frac{1}{2} g^{\mu\rho}\partial_\rho g_{00} = -\frac{1}{2}\eta^{\mu\rho}\partial_\rho h_{00}$$

No t dependence

$$\text{So } \frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\rho}\partial_\rho h_{00}\underbrace{\left(\frac{dt}{d\tau}\right)^2}_{1+\text{small}} \qquad \begin{cases} \dfrac{d^2t}{d\tau^2} = \dfrac{1}{2}\cancel{\partial_t h_{00}}^{\,0} = 0 \\[2ex] \dfrac{d^2x^i}{d\tau^2} = \dfrac{1}{2}\partial_i h_{00} \end{cases}$$

Newton says: $\vec{a} = \dfrac{d^2x^i}{dt^2} = -\vec{\nabla}\Phi$

Einstein geodesic: $\vec{a} = \dfrac{1}{2}\vec{\nabla} h_{00}$

So $h_{00} = -2\Phi \implies$ $\boxed{g_{00} = -(1+2\Phi)}$

Einstein's equation $\quad R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \kappa T_{\mu\nu}$ $\qquad \kappa = 8\pi G_{\text{Newton}}$

How do we know?

Contract both sides with $g^{\mu\nu}$:

$$R - \frac{1}{2}R\underbrace{g^{\mu\nu}g_{\mu\nu}}_{4} = \kappa \underbrace{T}_{g^{\mu\nu}T_{\mu\nu}} \implies R = -\kappa T \implies R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

Newtonian limit for dust: $T_{\mu\nu} = (\rho + \cancel{P}^{\,0})\underbrace{U_\mu U_\nu}_{-\delta_{\mu 0}\delta_{\nu 0}} + \cancel{P}^{\,0} g_{\mu\nu}$

So $T_{00} = \rho$, all other components 0.

$T = g^{00}T_{00} \approx -T_{00} = -\rho$

So $R_{00} = \kappa\left(\rho - \frac{1}{2}(-\rho)(-1)\right) = \dfrac{\kappa\rho}{2}$.

Meanwhile, in terms of the metric $(g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu})$:

$$R_{00} = R^\mu{}_{0\mu 0} = R^i{}_{0i0} = \partial_i\Gamma^i_{00} - \underbrace{\cancel{\partial_0}\Gamma^i_{i0}}_{\text{static}} + \underbrace{\Gamma^i_{i\mu}\Gamma^\mu_{00} - \Gamma^i_{0\mu}\Gamma^\mu_{i0}}_{\text{neglect, } \mathcal{O}(h^2)}$$

$$= \partial_i\left[\frac{1}{2}g^{i\mu}\left(\underbrace{\cancel{\partial_0 g_{\mu 0}} + \cancel{\partial_0 g_{0\mu}}}_{\text{static}} - \partial_\mu g_{00}\right)\right]$$

$$= -\frac{1}{2}\partial_i\partial^i h_{00} = -\frac{1}{2}\nabla^2 h_{00} = -\frac{1}{2}\nabla^2(-2\Phi)$$

from geodesic $\Leftrightarrow$ $\vec{F} = m\vec{a}$

So $\dfrac{\kappa\rho}{2} = \nabla^2\Phi = 4\pi G_N \rho$ $\qquad$ So $\boxed{\kappa = 8\pi G_N}$

Poisson equation for gravitational potential.

In vacuum (outside star, planet, black hole; includes gravitational waves)

$T_{\mu\nu} = 0$, so $\boxed{R_{\mu\nu} = 0}$ Einstein's eqns are 2nd order, coupled, non-linear equations for metric.

Solving requires either 1) isometries
2) approximations

Next: derive Einstein's eqns from action principle.

Warm-up: Action for EM in SR

Maxwell's eqns $\begin{cases} \partial_\nu F^{\mu\nu} = J^\mu & \text{(Coulomb, Ampere)} \\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu & \text{(Faraday, no magnetic monopoles)} \end{cases}$

$\vec{B} = \vec{\nabla} \times \vec{A}$ $\quad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V$

Action $S = \int d^4x\, \mathcal{L}$ $\quad \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu J^\mu.$

Require $\delta S = 0$ for any small δA_μ.

$$\delta S = \int d^4x \; \delta A_\mu J^\mu + \underbrace{\tfrac{1}{4}\delta\left([\partial_\nu A_\mu - \partial_\mu A_\nu][\partial_\rho A_\sigma - \partial_\sigma A_\rho]\, \eta^{\mu\rho} \eta^{\nu\sigma} \right)}_{\frac{1}{4} \partial_\nu (\delta A_\mu) F^{\mu\nu} (4)}$$

4 identical terms

$$\overset{\text{integrate by parts}}{=} \int d^4x \; \delta A_\mu \underbrace{\left(J^\mu - \partial_\nu F^{\mu\nu} \right)}_{\text{must } = 0} \quad \text{if } \delta S = 0 \text{ for } \underline{\text{any}}\ \delta A_\mu.$$

(Same from Euler-Lagrange eqns: $\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) = 0.$)

Einstein-Hilbert action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} R + \mathcal{L}_{matter}(g^{\mu\nu}, \phi_i) \right]$$

ϕ_i: fields describing EM, matter, etc.

$g_{\mu\nu}$ = dynamical field

$\delta S = 0$ for any variation $\delta g^{\mu\nu}$ ← vary <u>inverse</u> metric, simpler

with $\delta g^{\mu\nu} = 0$ on boundary of region.

Goal: write $\delta S = \int d^4x \sqrt{-g}\; \delta g^{\mu\nu}\, (\text{something})_{\mu\nu}$. Then $(\text{something})_{\mu\nu} = 0$ is our eqn of motion

$$S_H = \frac{1}{2K}\int d^4x\, \sqrt{-g}\, \underbrace{g^{\mu\nu} R_{\mu\nu}}_{R} \qquad \text{(Include matter later...)}$$

$$\delta S_H = \frac{1}{2K}\int d^4x \left[\underbrace{\sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu}}_{1} + \underbrace{\sqrt{-g}\, \delta g^{\mu\nu} R_{\mu\nu}}_{2} + \underbrace{\delta(\sqrt{-g})\, R}_{3} \right]$$

Already in desired form

Claim: $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu}$

Proof: For any matrix M_{ij}, varied by $M_{ij} \to M_{ij} + \delta M_{ij}$,

$$\ln(\det M) = \mathrm{Tr}\,(\underbrace{\ln M}_{\text{defined by}}) \longrightarrow \frac{1}{\det M}\,\delta(\det M) = \mathrm{Tr}[M^{-1}\delta M]$$

defined by $M = e^{\ln M} = 1 + \ln M + \frac{1}{2}(\ln M)^2 + \frac{1}{6}(\ln M)^3 + \dots$

So $\frac{1}{g}\delta g = g^{\mu\nu}\,\delta g_{\mu\nu} = -g_{\mu\nu}\,\delta g^{\mu\nu}$, so $\delta g = -g\, g_{\mu\nu}\,\delta g^{\mu\nu}$, and

$$\delta(\sqrt{-g}) = -\frac{1}{2\sqrt{-g}}\,\delta g = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\,\delta g^{\mu\nu} \quad ✓$$

So $3 = \frac{1}{2K}\int d^4x\,\sqrt{-g}\;\delta g^{\mu\nu}\left(-\frac{1}{2} g_{\mu\nu} R\right)$

Finally $1 = 0$ (p. 161–162)

So $\delta S_H = \int d^4x\,\sqrt{-g}\,\left[R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu}\right]\delta g^{\mu\nu}$

To include matter, $S = \underbrace{\frac{1}{16\pi G_N}}_{\frac{1}{2K}} S_H + S_{matter}[g^{\mu\nu}, \phi]$.

Now define $T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}} \Rightarrow \delta S = \int d^4x\,\sqrt{-g}\,\delta g^{\mu\nu}\Big[\underbrace{\frac{1}{2K}(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}) - \frac{1}{2}T_{\mu\nu}}_{=0,\ \text{Einstein's equation.}}\Big]$

A special case: $S_\Lambda = -\frac{1}{K}\int d^4x\,\sqrt{-g}\,\Lambda$ = vacuum energy term

constant

a.k.a. cosmological constant

Can call this part of gravity or part of matter or neither.

What does S_Λ contribute to eqns of motion?

$$\delta S_\Lambda = -\frac{1}{\kappa}\int d^4x\, \delta(\sqrt{-g})\,\Lambda = -\frac{\Lambda}{\kappa}\int d^4x \left(-\tfrac{1}{2}\sqrt{-g}\, g_{\mu\nu}\right)\delta g^{\mu\nu} = \frac{\Lambda}{16\pi G_N}\int d^4x \sqrt{-g}\, g_{\mu\nu}\,\delta g^{\mu\nu}$$

So, arrive at:

$$\boxed{R_{\mu\nu} - \tfrac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}}$$

from expansion rate of scale factor of universe

from experiments with masses, planets, etc.

Recall $T_{\mu\nu} = (\rho + P)U_\mu U_\nu + P g_{\mu\nu}$ for fluid with density ρ, pressure P.

Can interpret Λ as part of this, so

$$T_{\mu\nu} = \underbrace{T^{(\text{dust})}_{\mu\nu} + T^{(\text{rad})}_{\mu\nu}}_{T^{(\text{matter})}_{\mu\nu}} + T^{\Lambda}_{\mu\nu}$$

$$T^{\Lambda}_{\mu\nu} = -\frac{\Lambda}{8\pi G_N} g_{\mu\nu} \Rightarrow \rho + P = 0, \quad \text{so} \quad P = \frac{-\Lambda}{8\pi G_N} = -\rho,$$

and $\rho_{\text{vacuum}} = \dfrac{\Lambda}{8\pi G_N}$ ← could have either sign, experimentally positive.

Summary:

$$\boxed{\begin{aligned} S &= \int d^4x \sqrt{-g}\left[\frac{1}{16\pi G_N}(R - 2\Lambda) + \hat{\mathcal{L}}_{\text{matter}}\right] \\ R_{\mu\nu} - \tfrac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} &= 8\pi G_N T^{(\text{matter})}_{\mu\nu} \end{aligned}}$$

Dimensional analysis: ($c = 1$, distance = time)

$G_N = 6.67\times10^{-11}\ \dfrac{m^3}{kg\cdot s^2}$ $\left[\dfrac{\text{distance}}{\text{mass}}\right]$

$g_{\mu\nu}$ dimensionless

(Recall $\eta_{\mu\nu} = (-1, 1, 1, 1)$)

$T_{\mu\nu}$ = energy density $\left[\dfrac{\text{mass}}{(\text{distance})^3}\right]$

Λ $\left[\dfrac{1}{(\text{distance})^2}\right]$ $R_{\mu\nu}$, R $\left[\dfrac{1}{(\text{distance})^2}\right]$

Experimentally (from $a(t)$ = scale factor),

$$\Lambda = 1.2 \times 10^{-52} \text{ meters}^{-2} = (9.1 \times 10^{25} \text{ meters})^{-2} = (9.6 \times 10^{9} \text{ light-years})^{-2}$$

Negligible on distance scales $\ll 10^9$ light-years. No effect on solar system or even galactic scales.

$$\rho_{vac} = \frac{\Lambda}{8\pi G_N} = 5.9 \times 10^{-10} \frac{J}{m^3} = 5.9 \times 10^{-9} \frac{\text{erg}}{\text{cm}^3} = \frac{3.7 \text{ GeV}}{(\text{meters})^3} \sim \text{4 proton masses in 1 } m^3 \text{ box.}$$

Natural units $\hbar = 1$ and $c = 1$, can convert

1 meter $= 3.00 \times 10^8$ sec $= 5.07 \times 10^{15}$ GeV^{-1}.

So $\rho_{vac} = (2.3 \times 10^{-12} \text{ GeV})^4 = (2.3 \times 10^{-3} \text{ eV})^4$.

What should we have expected? ρ_{vac} has quantum corrections.

Each virtual particle type $\Rightarrow$ $\frac{1}{2}\hbar\omega_{\vec{p}}$ = zero point energy for modes with momentum $\vec{p}$.

$$\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}, \quad \text{so} \quad \Delta\rho_{vac} = \sum_{\substack{\text{particle} \\ \text{type } j}} \int d^3\vec{p} \sqrt{\vec{p}^2 + m_j^2} = \infty$$

(diverges like $\int^\infty dp\, p^3$)

Maybe should only integrate up to some large value $|\vec{p}|_{max}$.
Momentum cutoff related to quantum gravity.

Estimate: $m_p = \left(\frac{\hbar c}{G_N}\right)^{1/2} = 2.18 \times 10^{-5}$ g $= 1.22 \times 10^{19}$ GeV = Planck mass

(Aside: length scale $= \left(\frac{\hbar G_N}{c^3}\right)^{1/2} = 1.62 \times 10^{-35}$ meters = Planck length

$$\text{So } \Delta\rho_{vac} \sim \int^{|\vec{p}|_{max}} dp\, p^3 = (|\vec{p}|_{max})^4 \sim 10^{76} \text{ GeV}^4$$

$$\rho_{vac}^{observed} = 2.8 \times 10^{-47} \text{ GeV}$$

oops!

Cosmological constant problem Why is ρ_{vac} so small?

$$\rho_{vac}^{observed} = \underbrace{\rho_{vac}^{classical} + \Delta\rho_{vac}}_{\text{tuned to 1 part in } 10^{120}\text{!}}$$

Possible resolutions:

- Maybe zero point energy doesn't contribute?
  Hard to make a consistent theory
- Supersymmetry cancels fermions with bosons in $\Delta\rho_{vac}$
  Only reduces tuning to 10^{60}
- There are 10^{500} possible universes, we got "lucky".
  Anthropic principle: if ρ_{vac} weren't small, we wouldn't be here to observe it!
  - If $\rho_{vac} \gg$ observed, universe expands too fast, no stars
  - If $\rho_{vac} < 0$, universe collapses [$a(t)$ decreases with t].

CC problem 2: why is $\frac{1}{\sqrt{\Lambda}} = 9.6 \times 10^9$ years so close to $t_{universe} \overset{?}{=} 13.7 \times 10^9$ years

Skim sections 4.6, 4.7, skip 4.8 alternatives.

Chapter 5 Schwarzschild Metric

Describes black holes, region outside spherical massive bodies.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \underbrace{d\Omega^2}_{d\theta^2 + \sin^2\theta\, d\phi^2}$$

This is the unique spherically symmetric vacuum solution, exact.

$$R_{\mu\nu} = 0 \quad \text{(but not at } r = 0\text{)}.$$

For $M = 0 \rightarrow$ Minkowski space, SR.

For $r \gg GM \rightarrow$ Newton's laws (HW4, problem 1)

Derivation: Assume
- static (no time dependence of source or metric components)
- spherical symmetry (no θ, ϕ dependence " " " ")

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + e^{2\gamma(r)} r^2 d\Omega^2$$

write as exponentials to force >0, ease taking derivatives

Redefine coordinates $(r \to e^{-\gamma(r)}\, r)$ to eliminate γ (set $\gamma = 0$).

Christoffel symbols: (note metric diagonal!)

$\Gamma^t_{tr} = \partial_r \alpha = \alpha'$ $\quad \Gamma^r_{tt} = e^{2(\alpha-\beta)}\,\partial_r \alpha = e^{2(\alpha-\beta)}\,\alpha'$ $\quad \Gamma^r_{rr} = \partial_r \beta = \beta'$

$\Gamma^\theta_{r\theta} = \frac{1}{r}$ $\quad \Gamma^r_{\theta\theta} = -r\,e^{-2\beta}$ $\quad \Gamma^\phi_{r\phi} = \frac{1}{r}$

$\Gamma^r_{\phi\phi} = -r\,e^{-2\beta}\sin^2\theta$ $\quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$ $\quad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$ (others 0).

Riemann tensor has 6 independent components

Ricci tensor has 3 " " :

$$\left.\begin{aligned} R_{tt} &= e^{2(\alpha-\beta)}\left[\alpha'' + \alpha'^2 - \alpha'\beta' + 2\alpha'/r\right] \\ R_{rr} &= -\alpha'' - \alpha'^2 + \alpha'\beta' + 2\beta'/r \\ R_{\theta\theta} &= 1 + e^{-2\beta}\left[r(\beta' - \alpha') - 1\right] \\ R_{\phi\phi} &= \sin^2\theta\, R_{\theta\theta} \end{aligned}\right\} \text{set all } = 0$$

From R_{tt}, R_{rr} : $\alpha' = -\beta' \implies \alpha = -\beta + c$. So

$$ds^2 = -e^{-2\beta+2c}\,dt^2 + e^{2\beta}\,dr^2 + r^2\,d\Omega^2$$

Now redefine $t \to e^{-c}t$, eliminates c. So $c=0$, $\beta = -\alpha$, and

$$ds^2 = -e^{2\alpha}\,dt^2 + e^{-2\alpha}\,dr^2 + r^2\,d\Omega^2.$$

From $R_{\theta\theta} = 0$, $\underbrace{e^{2\alpha}(2r\alpha' + 1)}_{(re^{2\alpha})'} = 1$

So $re^{2\alpha} = r - R_s$ ← constant of integration

So $e^{2\alpha} = 1 - \frac{R_s}{r}$. What is R_s? Compare with Newton for large r,

$R_s = 2GM$ $\quad M$ = total mass, observer at large r.

Birkhoff's Theorem This is the unique solution with $\underbrace{\text{spherical symmetry}}$

Has 3 Killing vectors R, S, T with $\left.\begin{array}{l}[R,S]=T \\ [S,T]=R \\ [T,R]=S\end{array}\right\}$ angular momentum conserved

Don't need to assume static! Automatically has a time-like Killing vector.

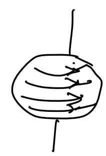
$K = \partial_t$ (so energy conserved).

Schwarzschild solution is not only stationary (invariant under $t \to t + t_0$)

but static (also invariant under $t \to -t$).

A spinning top is stationary but not static

keeps doing same thing — doesn't do anything



$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2$$

singularity at $r = 2GM = R_s$ = Schwarzschild radius

$$\left\{\begin{array}{ll} g_{tt} = 0 & g_{rr} = \infty \\ g^{tt} = \infty & g^{rr} = 0 \end{array}\right.$$

Looks bad, but this is an artifact of bad coordinates.

Coordinate singularity = not a physical singularity

Recall coordinates deceive!

Example: polar coordinates $ds^2 = dr^2 + \underbrace{r^2 d\phi^2}$

$g_{\phi\phi} \to 0, \quad g^{\phi\phi} \to \infty \quad$ at $r=0$

But curvature is 0 at $r=0$. Better chart at $r=0$: $ds^2 = dx^2 + dy^2$

How do we tell a true singularity?

If R, or $R^{\mu\nu}R_{\mu\nu}$, or $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, or .... blow up

scalars = chart independent

None of these blow up at $r = 2GM$.

$$R = 0 \,, \qquad R^{\mu\nu}R_{\mu\nu} = 0, \qquad R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \underbrace{\frac{48G^2M^2}{r^6}}$$

blows up as $r \to 0$, but not $r \to 2GM$

So $r = 0$ is a true (curvature) singularity.

$r = 2GM = R_s$ is still interesting = "horizon".

$R_s^{sun} = 2G_N \dfrac{M_{sun}}{c^2} = 2.95$ km $\ll R_{sun} = 6.96 \times 10^5$ km

$R_s^{earth} = 8.85$ mm $\ll R_{earth} = 6.38 \times 10^3$ km

$R_s^{electron} = 1.35 \times 10^{-57}$ meters $\ll$ Compton wavelength $\ll$ Planck length

For $r < R_{object}$, can't use Schwarzschild solution (not vacuum, so $R_{\mu\nu} \neq 0$).

Geodesics in Schwarzschild

λ = affine parameter (τ if timelike, $m \neq 0$).

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \quad \Rightarrow$$

$$\begin{cases} \dfrac{d^2t}{d\lambda^2} + \dfrac{2GM}{r(r-2GM)}\dfrac{dr}{d\lambda}\dfrac{dt}{d\lambda} = 0 \\[2ex] \dfrac{d^2r}{d\lambda^2} + \dfrac{GM}{r^2}(r-2GM)\left(\dfrac{dt}{d\lambda}\right)^2 - \dfrac{GM}{r(r-2GM)}\left(\dfrac{dr}{d\lambda}\right)^2 - (r-2GM)\left[\left(\dfrac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\dfrac{d\phi}{d\lambda}\right)^2\right] = 0 \\[2ex] \dfrac{d^2\theta}{d\lambda^2} + \dfrac{2}{r}\dfrac{d\theta}{d\lambda}\dfrac{dr}{d\lambda} - \sin\theta\cos\theta\left(\dfrac{d\phi}{d\lambda}\right)^2 = 0 \\[2ex] \dfrac{d^2\phi}{d\lambda^2} + \dfrac{2}{r}\dfrac{d\phi}{d\lambda}\dfrac{dr}{d\lambda} + 2\dfrac{\cos\theta}{\sin\theta}\dfrac{d\theta}{d\lambda}\dfrac{d\phi}{d\lambda} = 0 \end{cases}$$

Solve indirectly using Killing vectors $\Longleftrightarrow$ conserved quantities

$$K_\mu \frac{dx^\mu}{d\lambda} = \text{constant}$$

Then, plug in to check.

Timelike $K^\mu = (\partial_t)^\mu = (1,0,0,0) \Rightarrow K_\mu = \left(-\left(1-\frac{2GM}{r}\right), 0, 0, 0\right)$

Conserved energy: $E = -K_\mu \frac{dx^\mu}{d\lambda} = \left(1-\frac{2GM}{r}\right)\frac{dt}{d\lambda} = \text{constant}$

R, S, T = rotations about $r=0$.

$R^\mu = (\partial_\phi)^\mu = (0,0,0,1) \Rightarrow R_\mu = (0,0,0,r^2\sin^2\theta)$

Consider orbits at fixed $\theta = \frac{\pi}{2}$ (equatorial) $\Rightarrow \frac{d\theta}{d\lambda} = 0$. Then

conserved $L = R_\mu \frac{dx^\mu}{d\lambda} = r^2\frac{d\phi}{d\lambda}$ = constant = angular momentum.

Kepler's 2nd law: equal "areas" $r^2 d\phi$ in equal "time" $d\lambda$.

$S_\mu \frac{dx^\mu}{d\lambda} = 0$ and $T_\mu \frac{dx^\mu}{d\lambda} = 0$ automatically, not useful.

For timelike ($m \neq 0$, $\lambda = \tau$) $E = \frac{\text{energy}}{\text{mass}}$, $L = \frac{\text{angular momentum}}{\text{mass}}$

For null ($m=0$) E = energy, L = ang. mom.

Also, $\begin{cases} \text{timelike:} & -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 1 & \lambda = \tau \\ \text{null:} & -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 & \lambda \neq \tau \end{cases}$ Let $\epsilon = 1$ for timelike, $\epsilon = 0$ for null.

Then $-\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = -\epsilon$

$\Longrightarrow -E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1-\frac{2GM}{r}\right)\left(\frac{L^2}{r^2}+\epsilon\right) = 0.$ So:

$$\boxed{\begin{aligned} &\tfrac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E}, \quad \text{where} \\ &\mathcal{E} = E^2/2 = \text{constant} \\ &V(r) = \tfrac{1}{2}\left(1-\frac{2GM}{r}\right)\left(\frac{L^2}{r^2}+\epsilon\right) \end{aligned}}$$

This is equivalent to a 1-d problem for $r(\lambda)$, "time" $= \lambda$ "position" $= r$ "mass" $= 1$

"kinetic energy" $= \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2$ "potential energy" $= V(r)$

"total energy" $= \mathcal{E} = E^2/2$.