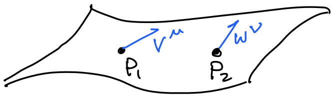
Parallel Transport

Given two points P_1 and P_2, can we compare vectors?

add, subtract, dot product.

No! Not uniquely.



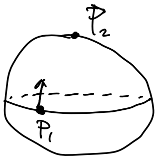
For example dot product: $V^\mu W^\nu \underbrace{g_{\mu\nu}}_{\text{evaluate at } P_1 \text{ or } P_2?}$

Can we move V^μ from P_1 to P_2?

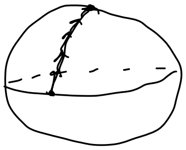
On flat manifold, trivial: keep components same.

On general (curved) manifold, depends on choice of path.

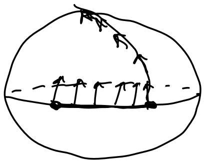
Classic example: S^2



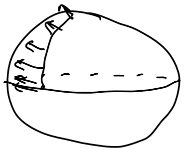
Choose path:



Choose another path

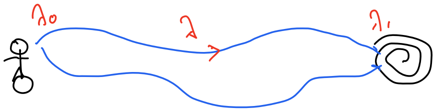


Don't agree! Back to start



Can't compare velocities on Earth to velocity of distant galaxy in unique way.

Must specify a path $x^\mu(\lambda)$



In flat space, components of tensor kept constant = parallel transport

$$\frac{d}{d\lambda} T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_\ell} = 0 \Rightarrow \frac{dx^\rho}{d\lambda}\frac{\partial}{\partial x^\rho} T^{\cdots}{}_{\cdots} = 0$$

on curved manifold, change to ∇_ρ.

$$\boxed{\frac{D}{d\lambda}\left(T^{\cdots}{}_{\cdots}\right) \equiv \frac{dx^\rho}{d\lambda}\nabla_\rho\left(T^{\cdots}{}_{\cdots}\right) = 0}$$

Given a tensor at P_1, solve this diff eq to parallel transport it to P_2.

For a vector V^μ, $\frac{dx^\rho}{d\lambda}\left(\partial_\rho V^\mu + \Gamma^\mu_{\rho\sigma} V^\sigma\right) = 0$, or

$$\boxed{\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda} V^\sigma = 0}$$

For a dual vector ω_μ, $\boxed{\frac{d\omega_\mu}{d\lambda} - \Gamma^\sigma_{\mu\rho}\frac{dx^\rho}{d\lambda}\omega_\sigma = 0}$

For the metric: $0 = \frac{D}{d\lambda} g_{\mu\nu} = \frac{dx^\rho}{d\lambda} \underbrace{\nabla_\rho g_{\mu\nu}}_{=0} = 0.$

if metric compatible connection

For an inner product of vectors:

$$\frac{D}{d\lambda}\left(g_{\mu\nu} V^\mu W^\nu\right) = \underbrace{\frac{D g_{\mu\nu}}{d\lambda}}_{0} V^\mu W^\nu + g_{\mu\nu} \underbrace{\left(\frac{D V^\mu}{d\lambda}\right)}_{0} W^\nu + g_{\mu\nu} V^\mu \underbrace{\left(\frac{D W^\nu}{d\lambda}\right)}_{0} = 0$$

So orthogonality of vectors is preserved by parallel transport.

Geodesics = "straight line" on general manifold
= path that parallel transports own tangent vector

Recall for path $x^\mu(\lambda)$, tangent vector is $\frac{dx^\mu}{d\lambda}$.

Parallel transport: $\frac{D}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = 0 \Rightarrow \boxed{\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0}$

Geodesic equation

Generalizes $\frac{d^2 x^\mu}{d\lambda^2} = 0 \Longrightarrow x^\mu = \lambda a^\mu + b^\mu$ in flat space.

constant vectors.

Example: geodesic in R^2 (flat plane) in polar coordinates, (r, θ).

$x^\mu = r$

$$\frac{d^2 r}{d\lambda^2} + \underbrace{\cancel{\Gamma^r_{rr}}}_{0} \left(\frac{dr}{d\lambda}\right)^2 + 2 \underbrace{\cancel{\Gamma^r_{r\theta}}}_{0} \left(\frac{dr}{d\lambda}\right)\left(\frac{d\theta}{d\lambda}\right) + \underbrace{\Gamma^r_{\theta\theta}}_{-r} \left(\frac{d\theta}{d\lambda}\right)^2 = 0$$

So $\boxed{\frac{d^2 r}{d\lambda^2} - r \left(\frac{d\theta}{d\lambda}\right)^2 = 0}$

$x^\mu = \theta$

$$\frac{d^2 \theta}{d\lambda^2} + \underbrace{\cancel{\Gamma^\theta_{rr}}}_{0} \left(\frac{dr}{d\lambda}\right)^2 + 2 \underbrace{\Gamma^\theta_{r\theta}}_{\frac{1}{r}} \frac{dr}{d\lambda} \frac{d\theta}{d\lambda} + \overset{0}{\cancel{\Gamma^\theta_{\theta\theta}}} \left(\frac{d\theta}{d\lambda}\right)^2 = 0$$

So $\boxed{\frac{d^2 \theta}{d\lambda^2} + \frac{2}{r} \frac{dr}{d\lambda} \frac{d\theta}{d\lambda} = 0}$

2nd order, nonlinear, coupled diff. eq.

Try for some solutions: 1) $\theta = \text{constant} \Rightarrow r = a\lambda + b$ (radial lines)

2) $r = \text{constant} \Rightarrow \theta = \text{constant}$ (point)

3) $\begin{cases} x = r\cos\theta = a\lambda + b \\ y = r\sin\theta = c\lambda + e \end{cases}$ Is this a solution?

$$0 = \frac{d^2}{d\lambda^2}(r\cos\theta) = \frac{d}{d\lambda}\left(\frac{dr}{d\lambda}\cos\theta - r\sin\theta\frac{d\theta}{d\lambda}\right) = \frac{d^2r}{d\lambda^2}\cos\theta - 2\sin\theta\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - r\cos\theta\left(\frac{d\theta}{d\lambda}\right)^2 - r\sin\theta\frac{d^2\theta}{d\lambda^2} \quad *$$

$$0 = \frac{d^2}{d\lambda^2} r\sin\theta = \frac{d^2r}{d\lambda^2}\sin\theta + 2\cos\theta\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - r\sin\theta\left(\frac{d\theta}{d\lambda}\right)^2 + r\cos\theta\frac{d^2\theta}{d\lambda^2} \quad **$$

Add $(*)\cos\theta + (**)\sin\theta = 0 \Rightarrow \frac{d^2r}{d\lambda^2} - r\left(\frac{d\theta}{d\lambda}\right)^2 = 0$ ✓

Add $-\frac{\sin\theta}{r}(*) + \frac{\cos\theta}{r}(**) = 0 \Rightarrow \frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} = 0$ ✓

Don't need a metric	Do need a metric
Define a manifold (charts on open sets)	Distance, volume
Vectors, dual vectors, tensors	Causality (what affects you, what you can affect, light cones)
Contract indices (1 raised, 1 lowered)	Raise and lower tensor indices
Connection (not unique) * covariant derivative	Unique connection = Christoffel
* parallel transport	
* <curvature tensor>	⋮
* geodesics	geodesic = path that maximizes proper time (or minimizes proper distance, if spacelike)

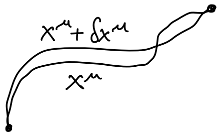
In G.R., we always have a metric.

Consider time-like paths $x^\mu(\lambda)$ $\lambda_0 < \lambda < \lambda_1$

$$T = \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} \qquad (g_{\mu\nu} \text{ replaces } \eta_{\mu\nu} \text{ in SR})$$

T = elapsed proper time = functional of paths
(given fn $x^\mu(\lambda)$, get number T)

Use calculus of variations to find max T.



For
$$\begin{cases} x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda) \\ g_{\mu\nu}(x^\mu(\lambda)) \to g_{\mu\nu}(x^\mu(\lambda) + \delta x^\rho(\lambda)\, \partial_\rho g_{\mu\nu} \\ \delta x^\mu(\lambda_0) = \delta x^\mu(\lambda_1) = 0 \end{cases}$$

Claim: the geodesic is path for which $\frac{\delta(\Delta T)}{\delta x^\mu} = 0$.

Trick: geodesic will also extremize $I = \frac{1}{2}\int_{\tau_0}^{\tau_1} d\tau \; g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$

Notes: no $\sqrt{}$, and use τ to parameterize path.

Proof: Let $f = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$, so $T = \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{-f}$, and

$$\delta(T) = \int_{\lambda_0}^{\lambda_1} d\lambda \; \delta(\sqrt{-f}) = -\frac{1}{2}\int_{\lambda_0}^{\lambda_1} d\lambda \frac{\delta f}{\sqrt{-f}}.$$

Now choose $\lambda = \tau$ = proper time along path

Then $f = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = g_{\mu\nu}U^\mu U^\nu = -1$, so $\sqrt{-f} = +1$

$$\text{So } \delta(T) = -\frac{1}{2}\int_{\tau_0}^{\tau_1} d\tau \; \delta f = -\delta\Big(\underbrace{\frac{1}{2}\int_{\tau_1}^{\tau_0} d\tau \; g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}_{I}\Big)$$

Maximizing T (proper time) same as minimizing I.

$$\delta I = \frac{1}{2}\int_{\tau_0}^{\tau_1} d\tau \left[\delta x^\rho \partial_\rho g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \underbrace{2 g_{\mu\nu}\frac{d(\delta x^\mu)}{d\tau}\frac{dx^\nu}{d\tau}}_{\substack{\text{integrate by parts, boundary terms} \\ \text{vanish } \delta x^\mu = 0 \text{ at endpoints}}} \right] = 0$$

So $$\delta I = \frac{1}{2}\int_{\tau_0}^{\tau_1} d\tau \left[\delta x^\rho \partial_\rho g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - 2\,\delta x^\mu \frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right)\right]$$

relabel $\mu \to \rho$

$$= \frac{1}{2}\int_{\tau_0}^{\tau} d\tau\ \delta x^\rho \left[\partial_\rho g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - 2\underbrace{\frac{d}{d\tau}\left(g_{\rho\nu}\frac{dx^\nu}{d\tau}\right)}_{g_{\rho\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}\partial_\mu g_{\rho\nu}}\right]$$

So, if $\delta I = 0$ for any δx^ρ, need

$$-2\,g_{\rho\nu}\frac{d^2x^\nu}{d\tau^2} + \left[\partial_\rho g_{\mu\nu} - 2\,\partial_\mu g_{\rho\nu}\right]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0.$$

Multiply by $-\frac{1}{2}g^{\rho\sigma}$:

$$\frac{d^2x^\sigma}{d\tau^2} + \underbrace{\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\left(\frac{1}{2}g^{\sigma\rho}\left[2\underbrace{\partial_\mu g_{\rho\nu}}_{\text{replace by } \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu},} - \partial_\rho g_{\mu\nu}\right]\right) = 0$$

because this is symmetric under $\mu \leftrightarrow \nu$

So, $$\underbrace{\frac{d^2x^\sigma}{d\tau^2} + \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\left(\underbrace{\frac{1}{2}g^{\sigma\rho}\left[\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right]}_{\Gamma^\sigma_{\mu\nu} \,=\, \text{Christoffel}}\right)}_{\text{geodesic equation.}} = 0$$

Can rescale $\tau \to \underbrace{\lambda = a\tau + b}_{\text{affine parameter}}$, so that $\frac{\partial}{\partial\tau} = a\frac{\partial}{\partial\lambda}$.

Summary: Of all paths from x_0^μ to x_1^μ, the geodesic $x^\mu(\lambda)$ maximizes τ and minimizes I, and parallel transports own tangent vector.

Other ways of writing geodesic equation:

1) Use $U^\mu = \frac{dx^\mu}{d\tau}$, then $\boxed{U^\nu \nabla_\nu U^\mu = 0} \Rightarrow \frac{dx^\nu}{d\tau}\left(\partial_\nu \frac{dx^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}\frac{dx^\rho}{d\tau}\right) = 0.$

2) Use 4-momentum $p^\mu = m U^\mu$ (if $m \neq 0$) $\Rightarrow \boxed{p^\nu \nabla_\nu p^\mu = 0}$

Null geodesics (light)

$$\boxed{\frac{d^2x^\rho}{d\lambda^2} + \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\Gamma^\rho_{\mu\nu} = 0}$$

$\lambda =$ affine parameter
$\neq a\tau + b$
($\tau = 0$ for whole path!)

Can choose λ so that $p^\mu = \frac{dx^\mu}{d\lambda}$ = 4-momentum = $(E, \vec{p})$, and $p^\mu p_\mu = p^2 = 0$.

Energy measured by observer with 4-velocity U^μ is $E = -p_\mu U^\mu$

Any geodesic is completely $\underset{-1}{\text{timelike}}$, $\underset{+1}{\text{spacelike}}$, or $\underset{0}{\text{null}}$ along whole path.

(Proof: inner products preserved, so $U^\mu U_\mu$ doesn't change.)

Geodesics in expanding universe

$$ds^2 = \underbrace{-dt^2}_{(dx^0)^2} + [a(t)]^2 \underbrace{(dx^2 + dy^2 + dz^2)}_{\delta_{ij}\,dx^i dx^j}$$

(Assume this metric for now, will prove later.)

Christoffels:
$$\left\{\begin{array}{l} \Gamma^0_{ij} = a\dot{a}\,\delta_{ij} \\ \Gamma^i_{j0} = \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j \\ \text{all others } 0 \end{array}\right\}$$

Carroll finds by variation of I, p.114 (could also use explicit formula).

Geodesic eqns:
$$\left.\begin{array}{l} \frac{d^2t}{d\lambda^2} + a\dot{a}\,\delta_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0 \\ \frac{d^2x^i}{d\lambda^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\lambda}\frac{dx^i}{d\lambda} = 0 \end{array}\right\}$$

Solve for coordinates of path
$t(\lambda)$ $x(\lambda)$ $y(\lambda)$ $z(\lambda)$
$\lambda =$ affine parameter

Solve for null geodesics

with $y = z =$ constant, along increasing x.

Necessary but not sufficient: $dx = \frac{1}{a}dt \Rightarrow \frac{dx}{d\lambda} = \frac{1}{a}\frac{dt}{d\lambda}$. So:

$$\frac{d^2t}{d\lambda^2} + a\dot{a}\left(\frac{1}{a}\frac{dt}{d\lambda}\right)^2 = 0 \quad \text{(1st equation)} \Rightarrow \frac{d^2t}{d\lambda^2} + \frac{\dot{a}}{a}\left(\frac{dt}{d\lambda}\right)^2 = 0.$$

2nd equation gives nothing new.

Solution: $\frac{dt}{d\lambda} = \frac{\omega_0}{a(t)}$ $\qquad \omega_0 =$ constant

Check: $\frac{d^2t}{d\lambda^2} = \frac{d}{d\lambda}\left(\frac{\omega_0}{a}\right) = \frac{dt}{d\lambda}\frac{d}{dt}\left(\frac{\omega_0}{a}\right) = -\frac{dt}{d\lambda}\,\omega_0\frac{\dot a}{a^2} = -\left(\frac{dt}{d\lambda}\right)^2\frac{\dot a}{a}$ ✓

Also, $\frac{dx}{d\lambda} = \frac{1}{a}\frac{dt}{d\lambda} = \frac{\omega_0}{a(t)^2}$. If $a(t)$ is increasing, $\frac{dx}{dt} = \frac{\frac{dx}{d\lambda}}{\frac{dt}{d\lambda}} = \frac{1}{a}$ is decreasing

coordinate speed

$\neq 1 = c$

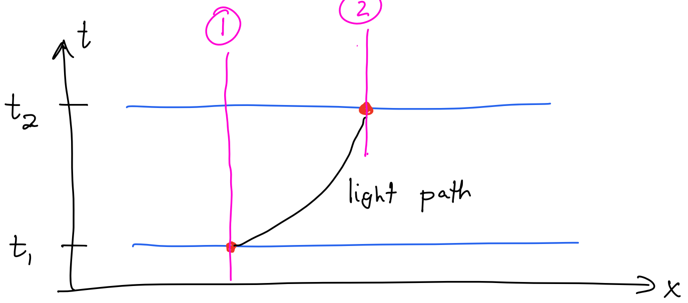
Comoving observer (at constant x,y,z) has $U^\mu = (U^0,0,0,0)$

Recall $-1 = U^\mu U^\nu g_{\mu\nu} = (U^0)^2 g_{00} = -(U^0)^2$, so $U^0 = 1$.

$$E = \hbar\omega = -\underset{\text{observer}}{U^\mu}\,\underset{\text{photon}}{p_\mu} = -U^\mu g_{\mu\nu}\frac{dx^\nu}{d\lambda} = \frac{dt}{d\lambda} = \frac{\hbar\omega_0}{a}$$

So, $\omega_0 = \omega$ for photon when $a(t) = 1$.

Consider two co-moving observers $\underset{\text{emitter}}{1}$, $\underset{\text{receiver}}{2}$ $\quad E_1 = \frac{\omega_0}{a(t_1)} \quad E_2 = \frac{\omega_0}{a(t_2)}$



So $\frac{E_2}{E_1} = \frac{a(t_1)}{a(t_2)}$,

energy of photons is redshifted away as universe expands $a(t_2) > a(t_1)$.

If energy of observed photon known (standard atomic or molecular emission or absorption lines)

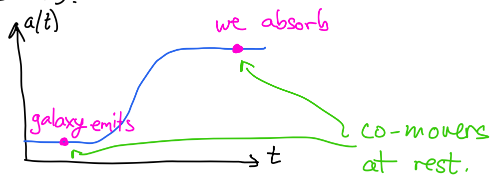
then use this to measure scale factor ratios.

Define $z = \text{redshift} = \frac{E_1 - E_2}{E_2} = \frac{\omega_1 - \omega_2}{\omega_2} = \frac{a(t_1)}{a(t_2)} - 1$

$z = 0$ if no expansion (source very close)

$z =$ large if source very far (galaxies, quasars).

Not a Doppler shift. For example, expansion of the universe could be:



Perfect fluids in expanding universe

Recall in SR:

$$\partial_\mu T^{\mu\nu} = 0 \qquad \text{(energy-momentum conservation)}$$

$$T^{\mu\nu} = (\rho + P)\, U^\mu U^\nu + P\, \eta^{\mu\nu}$$

energy density

4-velocity field of fluid

pressure

For GR, replace $\eta^{\mu\nu} \to g^{\mu\nu}$ $\quad \partial_\mu \to \nabla_\mu$. So:

$$\boxed{\begin{aligned} &\nabla_\mu T^{\mu\nu} = 0 \\ &T^{\mu\nu} = (\rho + P)\, U^\mu U^\nu + P g^{\mu\nu} \end{aligned}} \longleftarrow \text{general fluid}$$

For expanding universe

$$\left.\begin{aligned} g^{\mu\nu} &= \begin{pmatrix} -1 & & & 0 \\ & \frac{1}{a^2} & & \\ & & \frac{1}{a^2} & \\ 0 & & & \frac{1}{a^2} \end{pmatrix} \\ U^\mu &= (1, 0, 0, 0) \end{aligned}\right\} \Rightarrow T^{\mu\nu} = \begin{pmatrix} \rho & & & 0 \\ & \frac{P}{a^2} & & \\ & & P/a^2 & \\ 0 & & & P/a^2 \end{pmatrix}$$

Now $\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\rho} T^{\rho\nu} + \Gamma^\nu_{\mu\rho} T^{\mu\rho} = 0$ for all ν.

Look at $\nu = 0$:

$$\underbrace{\partial_\mu T^{\mu 0}}_{①} + \underbrace{\Gamma^\mu_{\mu\rho} T^{\rho 0}}_{②} + \underbrace{\Gamma^0_{\mu\rho} T^{\mu\rho}}_{③} = 0$$

$$\left.\begin{aligned} ① &= \partial_0 T^{00} = \dot\rho \\ ② &= \Gamma^\mu_{\mu 0}\, \rho = \Gamma^i_{i0}\, \rho = 3 \frac{\dot a}{a} \rho \\ ③ &= \underset{0}{\cancel{\Gamma^0_{00}}}\, T^{00} + \Gamma^0_{ii}\, T^{ii} = 3 (\dot a a) \left(\frac{P}{a^2}\right) = 3 \frac{\dot a}{a} P \end{aligned}\right\} \quad \boxed{\dot\rho = -3 \frac{\dot a}{a} (\rho + P)}$$

Look at $\nu = i = 1, 2, 3$:

$$\underbrace{\partial_\mu T^{\mu i}}_{①} + \underbrace{\Gamma^\mu_{\mu\rho} T^{\rho i}}_{②} + \underbrace{\Gamma^i_{\mu\rho} T^{\mu\rho}}_{③} = 0$$

$$\left.\begin{aligned} ① &= \partial_i \left(\frac{P}{a^2}\right) = \frac{1}{a^2} \partial_i P \\ ② &= \underset{0}{\cancel{\Gamma^\mu_{\mu i}}} \left(\frac{P}{a^2}\right) = 0 \\ ③ &= \Gamma^i_{j0}\, \underset{0}{\cancel{T^{j0}}} + \Gamma^i_{0j}\, \underset{0}{\cancel{T^{0j}}} = 0 \end{aligned}\right\} \quad \boxed{\partial_i P = 0} \longleftarrow$$

fluid motionless in co-moving chart, so pressure constant in space.

Equation of State P = function of ρ. Usually:

$$P = \underset{\text{constant}}{w}\rho \qquad w = \begin{cases} 0 & \text{dust} \\ 1/3 & \text{radiation} \\ -1 & \text{vacuum energy} = \text{cosmological constant} \end{cases}$$

Then $\dot\rho = -3\frac{\dot a}{a}(\rho + w\rho)$, or $\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a}$, or $\frac{d}{dt}(\ln\rho) = \frac{d}{dt}\ln\left(a^{-3(1+w)}\right)$

Solution: $$\frac{\rho(t)}{\rho(t_0)} = \left[\frac{a(t)}{a(t_0)}\right]^{-3(1+w)}$$

For dust: $P = 0$, $\rho \propto \frac{1}{a^3}$. Each particle has mass m, $\rho = nm$, with $n = \frac{\#\text{ of particles}}{\text{co-moving volume}} = \frac{\text{constant}}{a^3}$

For radiation: $P = \frac{\rho}{3}$, $\rho \propto \frac{1}{a^4}$ Photons same as dust, <u>except</u> each photon energy decreases like $\frac{1}{a}$.

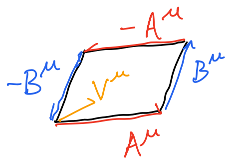
For vacuum energy: $P = -\rho$ $\rho = \text{constant}$

Energy <u>not</u> conserved in GR. $E = \int \rho\, a^3 d^3\vec{x}$ is constant <u>only</u> for pure dust.

Recall conservation of energy ⟺ time translation invariance

Not true in GR except in special cases.

Curvature Parallel transport a vector around closed, small path:



$$\delta V^\mu = \underbrace{R^\mu{}_{\rho\nu\sigma}} V^\rho A^\nu B^\sigma$$

Defines Riemann curvature tensor
(needs a connection, not a metric)

Equivalent definition: consider $[\nabla_\mu, \nabla_\nu] V^\rho = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho =$ rank (1,2) tensor

$$= \left[\partial_\mu (\nabla_\nu V^\rho) + \Gamma^\rho_{\mu\nu} \nabla_\nu V^\sigma - \Gamma^\sigma_{\mu\nu} \nabla_\sigma V^\rho\right] - (\mu \leftrightarrow \nu)$$

$$= \left[\partial_\mu (\partial_\nu V^\rho + \Gamma^\rho_{\nu\sigma} V^\sigma) + \Gamma^\rho_{\mu\sigma}(\partial_\nu V^\sigma + \Gamma^\sigma_{\nu\kappa} V^\kappa) - \Gamma^\sigma_{\mu\nu} \nabla_\sigma V^\rho\right] - (\mu \leftrightarrow \nu)$$

$$= \left[\cancel{\partial_\mu \partial_\nu V^\rho} + \partial_\mu \Gamma^\rho_{\nu\sigma} V^\sigma + \cancel{\Gamma^\rho_{\nu\sigma}\partial_\mu V^\sigma + \Gamma^\rho_{\mu\sigma}\partial_\nu V^\sigma} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\kappa} V^\kappa - \Gamma^\sigma_{\mu\nu} \nabla_\sigma V^\rho\right] - (\mu \leftrightarrow \nu)$$

$$= \underbrace{\left[\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right]}_{\equiv R^\rho{}_{\sigma\mu\nu}} V^\sigma - 2\Gamma^\sigma_{[\mu\nu]} \nabla_\sigma V^\rho$$

Note order of indices crucial. Books don't agree.

Comparison: $[\nabla_\mu, \nabla_\nu] V^\rho \equiv R^\rho{}_{\sigma\mu\nu} V^\sigma$ (Carroll)

$[\nabla_\mu, \nabla_\nu] V^\rho \equiv R_{\nu\mu\sigma}{}^\rho V^\sigma$ (Wald)

So $R_{\nu\mu\sigma}{}^\rho$ Wald $= R^\rho{}_{\sigma\mu\nu}$ Carroll. Lower an index with the metric:

$R_{\nu\mu\sigma\rho}$ Wald $= R_{\rho\sigma\mu\nu}$ Carroll.

Properties of Riemann tensor

0) It's a tensor

1) $R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu} = R^\rho{}_{\sigma[\mu\nu]} \equiv \frac{1}{2}\left(R^\rho{}_{\sigma\mu\nu} - R^\rho{}_{\sigma\nu\mu}\right)$

2) Can lower first index with a metric: $R_{\rho\sigma\mu\nu} = g_{\rho\kappa} R^\kappa{}_{\sigma\mu\nu}$

3) In locally inertial chart at P:

$\partial_{\hat\mu} g_{\hat\rho\hat\nu} = 0 \implies \Gamma^{\hat\mu}_{\hat\rho\hat\nu} = 0$, so

$$R_{\hat\rho\hat\sigma\hat\mu\hat\nu} = g_{\hat\rho\hat\kappa}\left(\partial_{\hat\mu}\Gamma^{\hat\kappa}_{\hat\nu\hat\sigma} - \partial_\nu \Gamma^{\hat\kappa}_{\hat\mu\hat\sigma}\right) = \frac{1}{2}\left(\partial_{\hat\mu}\partial_{\hat\sigma} g_{\hat\rho\hat\nu} - \partial_{\hat\mu}\partial_{\hat\rho} g_{\hat\nu\hat\sigma} - \partial_{\hat\nu}\partial_{\hat\sigma} g_{\hat\rho\hat\mu} + \partial_{\hat\nu}\partial_{\hat\rho} g_{\hat\mu\hat\sigma}\right)$$

4) $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} = R_{[\rho\sigma]\mu\nu} = R_{[\rho\sigma][\mu\nu]}$ (follows from 3, change chart)

5) $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$ (" " " " ")

6) $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$ (" " " " ")

7) $R_{\rho[\sigma\mu\nu]} = 0$

8) $R_{[\rho\sigma\mu\nu]} = 0$

9) How many independent components in n dimensions?

View as a symmetric matrix $R_{\underbrace{[\rho\sigma]}_{\substack{\text{first index}\\ N=\frac{n(n-1)}{2}}}\underbrace{[\mu\nu]}_{\substack{\text{second index}\\ N=\frac{n(n-1)}{2}}}}$.

Symmetric $N \times N$ matrix has $\frac{N(N+1)}{2} = \frac{1}{2}\frac{n(n-1)}{2}\left[\frac{n(n-1)}{2}+1\right] = \frac{1}{8}n(n-1)(n^2-n+2)$ independent components. But, there are $\frac{n(n-1)(n-2)(n-3)}{4!}$ constraints on this from 8)

So $n_R = \frac{1}{8}n(n-1)(n^2-n+2) - \frac{1}{24}n(n-1)(n-2)(n-3) = \frac{1}{12}n^2(n^2-1)$

manifold dimension	rank (0,4)	Riemann	
1	1	0	← no curvature on circle!
2	16	1	← curvature specified by 1 number at each point
3	81	6	
4	256	20	← " " " 20 numbers at each point (other 236 determined algebraicly)

10) Values of $R_{\rho\sigma\mu\nu}$ at <u>different</u> points constrained by:

$$\nabla_{[\kappa} R_{\rho\sigma]\mu\nu} = 0 \quad \text{or} \quad \nabla_\kappa R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\kappa\mu\nu} + \nabla_\sigma R_{\kappa\rho\mu\nu} = 0.$$

This is the Bianchi identity, follows from:

$$[[\nabla_\kappa, \nabla_\rho], \nabla_\sigma] + [[\nabla_\rho, \nabla_\sigma], \nabla_\kappa] + [[\nabla_\sigma, \nabla_\kappa], \nabla_\rho] = 0.$$

(write out all terms)

<u>Ricci tensor</u> $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ is symmetric $R_{\mu\nu} = R_{\nu\mu} = R_{(\mu\nu)}$

<u>Ricci scalar</u> $R = g^{\mu\nu} R_{\mu\nu}$

Non-zero $R_{\mu\nu}$ or R are sufficient, but not necessary, for curvature.

<u>Weyl tensor</u>: $C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho}\right) + \frac{2}{(n-1)(n-2)} g_{\rho[\mu} g_{\nu]\sigma} R$

Can check: $g^{\rho\mu} C_{\rho\sigma\mu\nu} = 0$, $\quad g^{\sigma\nu} C_{\rho\sigma\mu\nu} = 0$.

If $n<3$, undefined.

If $n=3$, $C_{\rho\sigma\mu\nu} = 0$.

Weyl tensor also known as **conformal tensor.**

Conformal transformation: $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ Ω = any function.

Then $C^{\rho}{}_{\sigma\mu\nu} \to C^{\rho}{}_{\sigma\mu\nu}$.

Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$. Satisfies $\nabla^{\mu} G_{\mu\nu} = 0$.

Proof: start with Bianchi identity:

$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$ Multiply by $g^{\nu\sigma}$:

$\nabla_\lambda R_{\rho\mu} - \nabla_\rho R_{\lambda\mu} + \nabla^{\nu} R_{\lambda\rho\mu\nu} = 0$ Multiply by $g^{\mu\lambda}$

$\nabla^{\mu} R_{\rho\mu} - \nabla_\rho R + \nabla^{\nu} R_{\rho\nu} = 0$. So

same

$$\nabla^{\mu} R_{\rho\mu} - \frac{1}{2}\nabla_\rho R = 0 \quad \to \quad \nabla^{\mu} \underbrace{\left(R_{\rho\mu} - \frac{1}{2} g_{\rho\mu} R\right)}_{G_{\rho\mu} = G_{\mu\rho}} = 0.$$

Note: $T^{\mu\nu} = T^{\nu\mu}$ $\nabla_\mu T^{\mu\nu} = 0$

$G^{\mu\nu} = G^{\nu\mu}$ $\nabla_\mu G^{\mu\nu} = 0$ } Coincidence? Could they be the same thing?

Einstein's GR $G^{\mu\nu} = \kappa T^{\mu\nu}$

constant $\kappa = 8\pi \underbrace{G_{\text{Newton}}}_{\approx 6{,}6738 \times 10^{-11} \frac{m^3}{kg \cdot s^2}}$

Example: S^2 = 2-sphere w/ round metric $ds^2 = a^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$

- $0 < 0 < \pi$

$0 < \phi < 2\pi$

a = radius of sphere if embedded in $\mathbb{R}^3$ = constant = $\sqrt{\frac{\text{area of manifold}}{4\pi}}$

Christoffels: $\Gamma^{\theta}_{\phi\phi} = -\sin\theta \cos\theta$

$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$ } a cancels

Riemann tensor: $R^{\theta}{}_{\phi\theta\phi} = \partial_\theta \Gamma^{\theta}_{\phi\phi} - \cancel{\partial_\phi \Gamma^{\theta}_{\theta\phi}} + \cancel{\Gamma^{\theta}_{\theta\mu}} \Gamma^{\mu}_{\phi\phi} - \Gamma^{\theta}_{\phi\mu} \Gamma^{\mu}_{\theta\phi}$ only $\mu=\phi$ contributes

$$= \partial_\theta(-\sin\theta\cos\theta) - (-\sin\theta\cos\theta)(\cot\theta)$$

$$= -\cos^2\theta + \sin^2\theta + \cos^2\theta$$

$$= \sin^2\theta$$

Lower the index: $R_{\theta\phi\theta\phi} = \underbrace{g_{\theta\theta}}_{a^2} R^{\theta}{}_{\phi\theta\phi} = a^2\sin^2\theta.$

Use symmetries to get rest: $= -R_{\theta\phi\phi\theta} = -R_{\phi\theta\theta\phi} = R_{\phi\theta\phi\theta}.$

All others 0. $R_{\theta\theta\mu\nu} = 0$, $R_{\phi\phi\mu\nu} = 0$, $R_{\mu\nu\theta\theta} = 0$, $R_{\mu\nu\phi\phi} = 0.$

Recall 1 independent component of $R_{\mu\nu\rho\sigma}$ for $n=2$. ✓

Ricci tensor: $R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu} = g^{\theta\theta} R_{\theta\mu\theta\nu} + g^{\phi\phi} R_{\phi\mu\phi\nu}$

$$= \frac{1}{a^2} R_{\theta\mu\theta\nu} + \frac{1}{a^2\sin^2\theta} R_{\phi\mu\phi\nu}.$$

So $R_{\theta\theta} = \frac{1}{a^2}(0) + \frac{1}{a^2\sin^2\theta}(a^2\sin^2\theta) = 1$

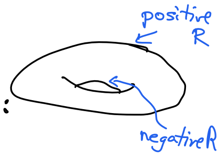
$R_{\phi\phi} = \frac{1}{a^2}(a^2\sin^2\theta) + \frac{1}{a^2\sin^2\theta}(0) = \sin^2\theta$

$R_{\theta\phi} = R_{\phi\theta} = 0$

Ricci scalar $R = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \frac{1}{a^2}(1) + \frac{1}{a^2\sin^2\theta}(\sin^2\theta)$

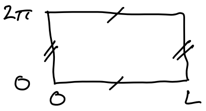
So $R = \frac{2}{a^2}$ = constant. This is a scalar (independent of chart). Not independent of metric.

Example: T^2 = 2-torus. Embedded in R^3, looks curved. Give it a metric where $R \neq$ constant:



But, can give it a different metric where $R_{\mu\nu\rho\sigma} = 0$ everywhere.

$ds^2 = dx^2 + d\phi^2$ $\quad 0 \leq \phi \leq 2\pi$, $0 \leq x \leq L$



identify sides.

In general, for a smooth 2-d manifold, orientable, compact: $\frac{1}{4\pi}\int d^2x \sqrt{g}\, R = 2-2g$

g = # of handles = genus. This is a topological invariant. (Gauss-Bonnet theorem.)