Metrics

Recall in SR, $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ was a <u>constant</u> (0,2) tensor.

On a general manifold, $g_{\mu\nu}$ = metric $\neq$ constant, = rank (0,2) symmetric tensor.

If $\det(g_{\mu\nu}) \equiv g \neq 0$ then inverse metric exists, and $g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho}$.

Metric = line element $\equiv \underset{\text{name}}{ds^2} = \underbrace{g_{\mu\nu}}_{\text{components}} \underbrace{dx^{\mu} dx^{\nu}}_{\text{basis dual vectors}}$

could call "g", but don't

= $(\text{distance})^2$ between nearby points. $\quad g_{\mu\nu} = g_{\nu\mu}$ (symmetric).

In Euclidean $\mathbb{R}^3$, $ds^2 = (dx)^2 + (dy)^2 + (dz)^2$ (rectangular coordinate chart)

$= (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta \, d\phi^2$ (spherical chart)

Metric maps $(\text{vector}) \otimes (\text{vector}) \rightarrow$ number. $\quad ds^2(V, W) = g_{\mu\nu} V^{\mu} W^{\nu}$

<u>Example</u>: Consider Euclidean metric on $\mathbb{R}^2$: $\quad ds^2 = dx^2 + dy^2$

Components: $\quad g_{xx} = 1 \quad g_{yy} = 1 \quad g_{xy} = g_{yx} = 0.$

Now change coordinates: $\begin{cases} x' = x + ay^3 \\ y' = y \end{cases}$. Then $\begin{cases} dx' = dx + 3ay^2 dy \\ dy' = dy \end{cases}$, so

$\begin{cases} dx = dx' - 3ay'^2 dy' \\ dy = dy' \end{cases}$. So: $\quad ds^2 = (dx' - 3ay'^2 dy')^2 + (dy')^2$

$= dx'^2 - 3ay'^2 (dx'dy' + dy'dx') + (1 + 9a^2 y'^4) dy'^2$

$g_{x'x'} = 1 \quad g_{x'y'} = g_{y'x'} = -3ay'^2 \quad g_{y'y'} = 1 + 9a^2 y'^4.$

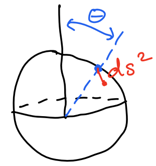
$g_{\mu'\nu'} = \begin{pmatrix} 1 & -3ay'^2 \\ -3ay'^2 & 1 + 9a^2 y'^4 \end{pmatrix}$ $\leftarrow$ Doesn't "look" like flat Euclidean space, but it is. Not curved.

Inverse metric: $\quad g^{\mu'\nu'} = \begin{pmatrix} 1 + 9a^2 y'^4 & +3ay'^2 \\ +3ay'^2 & 1 \end{pmatrix}$

$g^{x'x'} = 1 + 9a^2 y'^4 \quad g^{y'y'} = 1 \quad g^{x'y'} = g^{y'x'} = 3ay'^2.$

For S^2 = 2-sphere = non-Euclidean metric manifold,

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$



$$g_{\mu\nu} = \begin{array}{cc} \begin{array}{cc} \theta & \phi \end{array} & \\ \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} & \begin{array}{c} \theta \\ \phi \end{array} \end{array}$$

$g_{\theta\theta} = 1 \qquad g_{\theta\phi} = g_{\phi\theta} = 0 \qquad g_{\phi\phi} = \sin^2\theta.$ ← Not invertible at $\theta = 0$.

Canonical form of metric at a point P: can choose a chart so that at P:

$$g_{\mu\nu}(P) = \mathrm{diag}(\underbrace{-1,\dots,-1}_{\#\text{ of } -1\text{'s}}, \underbrace{+1,\dots,+1}_{\#\text{ of } 1\text{'s}}, \underbrace{0,\dots,0}_{\#\text{ of } 0\text{'s}})$$ ← only if non-invertible.

Signature =

$-1, 1, 1, 1$ = Minkowski = Lorentzian = SR

$+1, +1, +1, +1$ = Euclidean = Riemannian

If you move away from P, metric changes. Can also choose $\partial_\rho g_{\mu\nu} = 0$ at P.

But $\partial_\rho \partial_\sigma g_{\mu\nu} \neq 0$ in general.

Chart choice with $\begin{cases} g_{\hat\mu\hat\nu} = \eta_{\hat\mu\hat\nu} \\ \partial_{\hat\mu} g_{\hat\nu\hat\rho} = 0 \end{cases}$ at P = locally inertial coordinates.

Basis vectors = local Lorentz frame ⇒ hats on indices.

Einstein Equivalence Principle Any metric looks like $\eta_{\mu\nu}$ to first order at P.

Small enough region ≈ flat Minkowski space.

Example: expanding universe. Suppose $ds^2 = -dt^2 + [a(t)]^2 (dx^2 + dy^2 + dz^2)$.

$a(t)$ = scale factor. If $a(t) = 1$, Minkowski space.

If $a(t) \neq 1$, not Minkowski, but has Minkowski signature.

Later, we will show that $a(t) = a_0 t^q$ where $q = 2/3$ if filled with uniform matter
$q = 1/2$ " " " " light

As $t \to 0$, metric becomes non-invertible.

Maybe this is just a problem with our chart at $t=0$? Pick a better chart?

No! Can't find a good chart at $t=0$. $0 < t < \infty$

↖ Big bang singularity.

Comoving points = points at fixed (x,y,z)

Proper distance between comoving points increases with t:

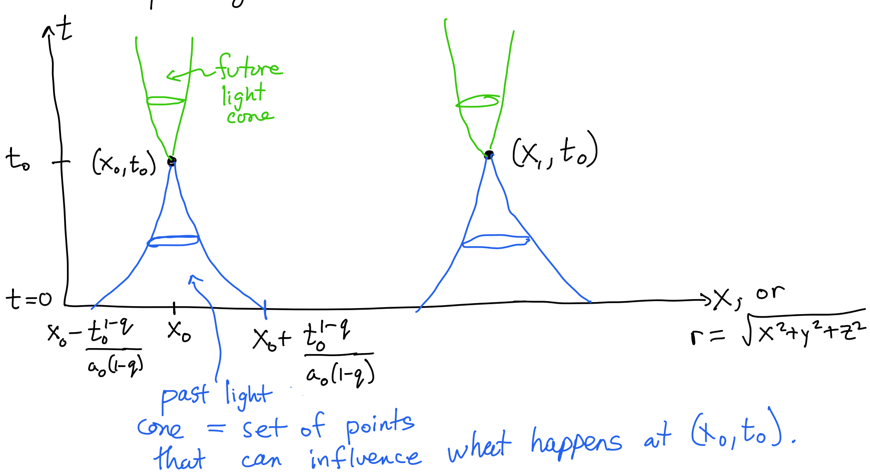
$ds^2 = a(t)^2(\Delta x)^2$

Light cones = points for which $ds=0$ from a given point.
= null curves

For $y=z=$ constant: $\quad 0 = -dt^2 + a_0^2 t^{2q} dx^2 \Rightarrow \dfrac{dx}{dt} = \pm \dfrac{1}{a_0 t^q}$

Solve: $\quad x = x_0 \pm \dfrac{(t^{1-q} - t_0^{1-q})}{(1-q)\,a_0} \iff t = \left[t_0^{1-q} \pm (1-q)a_0(x-x_0)\right]^{1/(1-q)}$

For an expanding universe, $0<q<1$:



Past light cones of two events may or may not intersect.

Horizon of an event = boundary of past light cone.

Points outside of horizon can't affect the event.

Events whose past light cones (or horizons) don't intersect are <u>causally disconnected</u>.

Tensor densities Levi-Civita symbol on an n-manifold

$$\tilde{\epsilon}_{\mu_1\ldots\mu_n} = \begin{cases} +1 & \text{if } \mu_1,\ldots,\mu_n = \text{even permutation of } 0,1,\ldots n-1 \\ -1 & \text{" " odd " " "} \\ 0 & \text{if any index repeated} \end{cases}$$

$\sim \leftarrow$ not a rank $(0,n)$ tensor

For any $n \times n$ matrix $M^{\mu}{}_{\mu'}$, can check that

$$\underbrace{\tilde{\epsilon}_{\mu_1 \cdots \mu_n} M^{\mu_1}{}_{\mu_1'} M^{\mu_2}{}_{\mu_2'} \cdots M^{\mu_n}{}_{\mu_n'}}_{\text{antisymmetric on interchange of } \mu_1' \ldots \mu_n'} = \tilde{\epsilon}_{\mu_1' \cdots \mu_n'} \underbrace{\det M}_{\text{book writes } |M|}$$

Now take $M^{\mu}{}_{\mu'} = \dfrac{\partial x^{\mu}}{\partial x^{\mu'}}$. Then $\tilde{\epsilon}_{\mu_1' \cdots \mu_n'} = \underbrace{\left| \dfrac{\partial x^{\mu'}}{\partial x^{\mu}} \right|}_{\text{extra!}} \underbrace{\dfrac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \dfrac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} \tilde{\epsilon}_{\mu_1 \cdots \mu_n}}_{\text{correct if } \tilde{\epsilon} = \text{tensor}}$

(=1 for L.T. in Minkowski space)

Another example: $g \equiv \det(g_{\mu\nu})$. Is this a rank $(0,0)$ tensor? No!

Proof: $g_{\mu'\nu'} = \dfrac{\partial x^{\mu}}{\partial x^{\mu'}} \dfrac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu}$. So:

$$\det(g_{\mu'\nu'}) = \det\left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right) \det\left(\frac{\partial x^{\nu}}{\partial x^{\nu'}}\right) \det(g_{\mu\nu}) = \underbrace{\det\left(\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right)^{-2}}_{\text{extra}} \det(g_{\mu\nu})$$

Used $\det(AB) = \det A \det B$, and $\det(A^{-1}) = 1/\det(A)$

A tensor density transforms like a tensor but with an extra power of $\left| \dfrac{\partial x^{\mu'}}{\partial x^{\mu}} \right|^{w}$.

w = "weight".

To make a tensor, multiply by $|g|^{w/2}$ to cancel the "extra" part.

So: $\epsilon_{\mu_1 \cdots \mu_n} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n}$ is a tensor.

not just 1, −1, 0   1, −1, 0

absolute values because $g = \det(g_{\mu\nu}) < 0$.

Raise all the indices: $\epsilon^{\mu_1 \cdots \mu_n} = \dfrac{1}{\sqrt{|g|}} \tilde{\epsilon}^{\mu_1 \cdots \mu_n}$ is a tensor

How to integrate volumes

$$d^n x = dx^0 dx^1 \cdots dx^n = \frac{1}{n!} \tilde{\epsilon}_{\mu_1 \cdots \mu_n} dx^{\mu_1} \cdots dx^{\mu_n}$$

= a tensor density of weight 1.

So $\sqrt{|g|}\, d^n x$ is a scalar (rank $(0,0)$ tensor)

To integrate a scalar function $\phi(x)$ properly (so that answer doesn't depend on chart),

$I = \int d^n x \sqrt{|g|}\, \phi(x^{\mu})$.   Abstract notation: $\int \phi(x)$ or $\int \phi(x)\, \tilde{\epsilon}$

Example: Spherical coordinates on $\mathbb{R}^3$, metric $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$

As a matrix: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \implies |g| = r^4 \sin^2\theta$

So $\sqrt{|g|} = r^2 \sin\theta$. $\implies \sqrt{|g|}\, d^3x = r^2 \sin\theta \, dr\, d\theta\, d\phi = dx\, dy\, dz$ ✓

Chapter 3: Curvature (depends on metric)

Covariant derivatives

Problem: $\partial_\mu(\text{tensor}) \neq \text{tensor}$ (unless rank 0,0)

In SR, have equations like $\underbrace{\partial_\mu F^{\mu\nu}}_{\text{Not a tensor in GR}} = J^\nu$ ← tensor (Gauss, Ampere)

Need a "fixed up" derivative $\nabla_\mu = \partial_\mu + (\text{something})_\mu$

Properties we want:

1) $\nabla_\mu(\text{any tensor}) = \text{another tensor}$.
2) Linear $\nabla_\mu(T+S) = \nabla_\mu T + \nabla_\mu S$
3) Product rule $\nabla_\mu(TS) = (\nabla_\mu T)S + T(\nabla_\mu S)$

} Note: T, S carry other indices

Start with vectors:

$$\underbrace{\nabla_\mu V^\nu}_{} = \partial_\mu V^\nu + \underbrace{\Gamma^\nu_{\mu\rho}}_{\text{connection: not a tensor}} V^\rho$$

Need this to be a rank (1,1) tensor:

$$\boxed{\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu}$$ ← Choose connection to make it so

LHS: $\nabla_{\mu'} V^{\nu'} = \partial_{\mu'}(V^{\nu'}) + \Gamma^{\nu'}_{\mu'\rho'} V^{\rho'}$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu\right) + \Gamma^{\nu'}_{\mu'\rho'} \left(\frac{\partial x^{\rho'}}{\partial x^\rho} V^\rho\right)$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)\left(\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu}\right) V^\nu + \Gamma^{\nu'}_{\mu'\rho'} \frac{\partial x^{\rho'}}{\partial x^\rho} V^\rho$$

RHS: $\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho\right)$

Derivatives cancel, get algebraic equation:

$$V^\rho \left(\Gamma^{\nu'}_{\mu'\rho'} \frac{\partial x^{\rho'}}{\partial x^\rho} + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\rho} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\rho} \right) = 0$$

Must be true for any V^ρ so:

$$\Gamma^{\nu'}_{\mu'\rho'} = \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\rho}}_{\text{usual}} - \underbrace{\frac{\partial x^\mu}{\partial x^{\nu'}} \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\rho}}_{\text{"extra"}}$$

(3.10) in Carroll, with sign error

So $\Gamma^\nu_{\mu\rho}$ not a tensor, or a density

Same analysis for dual vector ω_μ gives:

$$\underbrace{\nabla_\mu \omega_\nu}_{\text{require this is a tensor}} = \underbrace{\partial_\mu \omega_\nu}_{\text{not a tensor}} + \underbrace{\tilde{\Gamma}^\rho_{\mu\nu}}_{\text{not a tensor}} \omega_\rho \implies \tilde{\Gamma}^{\nu'}_{\mu'\rho'} \text{ obeys (3.10) but with } - \to +$$

More requirements:

4) Commutes with contraction: $\nabla_\mu(\delta^\rho_\nu) = 0$, so that

$$\nabla_\mu \big(\underbrace{T^{\nu\sigma_1 \dots \sigma_{k-1}}_{\rho\lambda_1 \dots \lambda_{\ell-1}} \delta^\rho_\nu}_{T^{\rho\sigma_1 \dots \sigma_{k-1}}_{\rho\lambda_1 \dots \lambda_{\ell-1}}} \big) = \delta^\rho_\nu \nabla_\mu \left(T^{\nu\sigma_1 \dots \sigma_{k-1}}_{\rho\lambda_1 \dots \lambda_{\ell-1}} \right)$$

5) Reduces to ∂_μ when acting on scalar functions.

So: $\nabla_\mu (A_\nu B^\nu) = (\nabla_\mu A_\nu) B^\nu + A_\nu (\nabla_\mu B^\nu)$

$= \left(\partial_\mu A_\nu + \tilde{\Gamma}^\rho_{\mu\nu} A_\rho \right) B^\nu + A_\nu \left(\partial_\mu B^\nu + \Gamma^\nu_{\mu\rho} B^\rho \right)$ (product rule)

$= \partial_\mu (A_\nu B^\nu) + A_\nu B^\rho \left(\tilde{\Gamma}^\nu_{\mu\rho} + \Gamma^\nu_{\mu\rho} \right)$ (relabelled indices)

So, need $\tilde{\Gamma}^\nu_{\mu\rho} = -\Gamma^\nu_{\mu\rho}$. So $\boxed{\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu} \omega_\rho}$

More generally, need:

$$\nabla_\rho T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_\ell} = \partial_\rho T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_\ell} + \Gamma^{\mu_1}_{\rho\sigma} T^{\sigma\mu_2 \dots \mu_k}{}_{\nu_1 \dots \nu_\ell} + \Gamma^{\mu_2}_{\rho\sigma} T^{\mu_1 \sigma \dots \mu_k}{}_{\nu_1 \dots \nu_\ell} + \dots - \Gamma^\sigma_{\rho\nu_1} T^{\mu_1 \dots \mu_k}{}_{\sigma\nu_2 \dots \nu_\ell} - \Gamma^\sigma_{\rho\nu_2} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \sigma \dots \nu_\ell} - \dots$$

General expression for covariant derivative, in terms of connection

Write connection as $\Gamma^{\rho}_{\mu\nu} = \underbrace{\Gamma^{\rho}_{(\mu\nu)}}_{\equiv \frac{1}{2}\left(\Gamma^{\rho}_{\mu\nu}+\Gamma^{\rho}_{\nu\mu}\right)} + \underbrace{\Gamma^{\rho}_{[\mu\nu]}}_{\equiv \frac{1}{2}\left(\Gamma^{\rho}_{\mu\nu}-\Gamma^{\rho}_{\nu\mu}\right)}$

symmetric $\qquad$ antisymmetric = torsion tensor

So far, lots of possible connections.

More requirements:

6) Torsion-free $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{(\mu\nu)} \iff \boxed{\Gamma^{\rho}_{[\mu\nu]} = 0}$

7) Metric compatible: $\boxed{\nabla_\mu g_{\nu\rho} = 0}$ $\qquad$ This implies: $\nabla_\mu g^{\nu\rho} = 0$

Now the connection is unique. Proof: solve for it

$$\nabla_\rho g_{\mu\nu} = 0 = \partial_\rho g_{\mu\nu} - \Gamma^{\sigma}_{\rho\mu} g_{\sigma\nu} - \Gamma^{\sigma}_{\rho\nu} g_{\mu\sigma} \qquad (1)$$

$$\nabla_\mu g_{\nu\rho} = 0 = \partial_\mu g_{\nu\rho} - \Gamma^{\sigma}_{\mu\nu} g_{\sigma\rho} - \Gamma^{\sigma}_{\mu\rho} g_{\nu\sigma} \qquad (2)$$

$$\nabla_\nu g_{\rho\mu} = 0 = \partial_\nu g_{\rho\mu} - \Gamma^{\sigma}_{\nu\rho} g_{\sigma\mu} - \Gamma^{\sigma}_{\nu\mu} g_{\sigma\rho} \qquad (3)$$

Take (1) − (2) − (3) $\Rightarrow$ $\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\mu\rho} + 2\Gamma^{\sigma}_{\mu\nu} g_{\sigma\rho} = 0$

Multiply by $g^{\rho\lambda}$: $\boxed{\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)}$ ← transforms like a connection

Levi-Civita connection = Christoffel symbol

Has n^3 components, but only $\frac{n^2(n+1)}{2}$ independent (40 for $n=4$)

Example: R^2 plane in polar chart; $ds^2 = dr^2 + r^2 d\theta^2$ (note = flat manifold, $ds^2 = dx^2 + dy^2$)

$g_{rr} = 1 \quad g_{\theta\theta} = r^2 \quad g_{r\theta} = g_{\theta r} = 0 \qquad g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$,

$g^{rr} = 1 \quad g^{\theta\theta} = \frac{1}{r^2} \quad g^{r\theta} = g^{\theta r} = 0.$

$$\Gamma^{\theta}_{\theta\theta} = \frac{1}{2} g^{\theta\rho}\left(\partial_\theta g_{\theta\rho} + \partial_\theta g_{\theta\rho} - \partial_\rho g_{\theta\theta}\right)$$

$$= \frac{1}{2} g^{\theta\theta}\underbrace{\left(\partial_\theta g_{\theta\theta} + \partial_\theta g_{\theta\theta} - \partial_\theta g_{\theta\theta}\right)}_{0} + \frac{1}{2}\underbrace{g^{\theta r}}_{0}\left(\quad \cdots \quad \right)$$

$$= 0$$

Also can check: $\Gamma^r_{rr} = 0 \qquad \Gamma^r_{\theta r} = \Gamma^r_{r\theta} = 0 \qquad \Gamma^\theta_{rr} = 0.$

But: $\Gamma^\theta_{\theta r} = \frac{1}{2} g^{\theta\rho}\left(\partial_\theta g_{\rho r} + \partial_r g_{\rho\theta} - \partial_\rho \cancel{g_{\theta r}}^{\,0}\right)$

$$= \frac{1}{2} g^{\theta\theta}\left(\partial_\theta \cancel{g_{\theta r}}^{\,0} + \partial_r g_{\theta\theta}\right) = \frac{1}{2}\left(\frac{1}{r^2}\right)\partial_r (r^2) = \frac{1}{r}.$$

Also $\Gamma^r_{\theta\theta} = -r$ (see page 100).

By a choice of chart, can always make $\Gamma^\rho_{\mu\nu}(P) = 0$ at a single point P.
(1st derivs of metric vanish at P.)

Covariant Divergence of a vector

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\rho} V^\rho.$$

Claim: $\Gamma^\mu_{\mu\rho} = \frac{1}{\sqrt{|g|}} \partial_\rho\left(\sqrt{|g|}\right).$

Proof: Given a matrix A, depending on a variable x,

$$\mathrm{Tr}\left[A^{-1}\frac{dA}{dx}\right] = \frac{d}{dx}\left(\ln(\det A)\right) = \frac{1}{\det A}\frac{d}{dx}(\det A) = \frac{1}{|\det A|}\frac{d}{dx}|\det A|$$

$$= \frac{2}{\sqrt{|\det A|}}\frac{d}{dx}\sqrt{|\det A|}$$

Now let $A = g_{\mu\nu}$, $x = x^\rho$. Then $g^{\nu\rho}\partial_\rho g_{\mu\nu} = \frac{2}{\sqrt{|g|}}\partial_\rho\sqrt{|g|}$

Meanwhile, from (3.27) $\Gamma^\mu_{\mu\rho} = \frac{1}{2} g^{\mu\nu}\left(\partial_\mu g_{\rho\nu} + \partial_\rho g_{\mu\nu} - \partial_\nu g_{\mu\rho}\right)$

symmetric in $\mu \leftrightarrow \nu$ antisymmetric in $\mu \leftrightarrow \nu$

Each term in sum over μ,ν cancels with another

$$= \frac{1}{2} g^{\mu\nu}\partial_\rho g_{\mu\nu}$$

So $\Gamma^\mu_{\mu\rho} = \frac{1}{\sqrt{|g|}}\partial_\rho\sqrt{|g|}$. Thus: $\boxed{\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\,V^\mu\right).}$

Divergence Theorem in curved space

$$\int_{\text{volume}} d^n x \sqrt{|g|}\,\nabla_\mu V^\mu = \int d^n x\, \partial_\mu\left(\sqrt{|g|}\,V^\mu\right) \overset{\text{ordinary Divergence theorem}}{=} \int_{\text{Boundary}} d^{n-1}x \sqrt{|\gamma|}\, n_\mu V^\mu$$

γ_{ij} = "induced metric" = $g_{\mu\nu}$ restricted to boundary n_μ = normal vector out of boundary