4-velocity Consider a particle worldline $x^\mu(\tau)$ Use τ = parameter λ

Then $U^\mu = \frac{dx^\mu}{d\tau}$ = 4-velocity. Recall $d\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$, so

$$U^\mu U^\nu \eta_{\mu\nu} = -\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1 \quad \text{(always)}$$

4-momentum $p^\mu = m U^\mu = (\underbrace{E}_{p^0}, \vec{p})$. If in rest frame, $p^\mu = (m, 0, 0, 0)$.

In inertial frame with speed $v = \frac{dx}{dt}$ along x-axis,

$$p^{\mu'} = \underbrace{\begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda^{\mu'}{}_{\nu}} \underbrace{\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{p^\nu} = \begin{pmatrix} m\gamma \\ mv\gamma \\ 0 \\ 0 \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$

Note: $U^x = v\gamma$ can exceed 1 $(=c)$
$\neq v$

(For small v, $p_x = mv$, $E = m + \frac{1}{2}mv^2$.)

$$p^\mu p_\mu = m^2 U^\mu U_\mu = -m^2 \Rightarrow E = \sqrt{m^2 + \vec{p}\cdot\vec{p}}$$

Fluids = collection of many particles
locally, has 4-momentum, mass density, energy, pressure

Describe with $T^{\mu\nu}$ = stress-energy tensor, rank (2,0)
= flux of p^μ across surface of constant x^ν.

In rest frame of fluid, T^{00} = energy density $\rho = \frac{\text{mass}}{\text{volume}}$

$T^{0i} = T^{i0}$ = momentum density $= \frac{(\vec{p})^i}{\text{volume}}$ $(i = 1, 2, 3)$

T^{ij} = stress $\begin{cases} T^{ii} = \text{pressure in direction } i = p^i \\ T^{ij} \ (i \neq j) = \text{shear} \end{cases}$

Example: Dust = matter. Number flux 4-vector $N^\mu = \underbrace{n}_{\frac{\text{particles}}{\text{volume}}} \underbrace{U^\mu}_{\text{4-velocity of particles}}$

$$T^{\mu\nu} = \underbrace{n}_{\frac{\text{particles}}{\text{volume}}} \underbrace{m}_{\text{mass/particle}} U^\mu U^\nu = \rho U^\mu U^\nu$$

Perfect fluid At each point: $T^{\mu\nu} = (\rho + P)\, U^\mu U^\nu + P\,\eta^{\mu\nu}$

pressure

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \quad \text{in rest frame.}$$

Dust = special case $P = 0$

Radiation = " " $P = \rho/3$

Vacuum energy = " " $P = -\rho$

More generally, $\partial_\mu T^{\mu\nu} = 0$ (trivial if ρ, P constant).

General Relativity

Weak Equivalence Principle

$\vec{F} = m_i \vec{a}$ $\qquad$ $\vec{F}_{grav} = -m_g \vec{\nabla}\Phi$

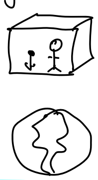
inertial mass $\qquad$ gravitational mass

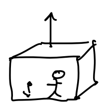
$$\boxed{m_i = m_g}$$

So $\vec{a} = -\vec{\nabla}\Phi$, for any mass.

Motion of freely falling particles same in:

Uniform gravitational field vs Uniformly accelerated frame:





Einstein Equivalence Principle In any small enough region of spacetime, laws of physics same as for SR.

Can't detect existence of gravitational field, can always define locally inertial (non-accelerated) frame.

Gravity $\leftrightarrow$ curvature of spacetime.

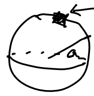
No proof! This is an assumption, to be tested by experiment.

Spacetime = differentiable manifold with metric

Manifold = set that locally = $\mathbb{R}^n$ ← dimension

Examples: 1) $\mathbb{R}^2 = xy$ plane

2) surface of sphere

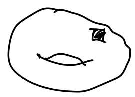


looks like $\mathbb{R}^2$, for distances $\ll a$.

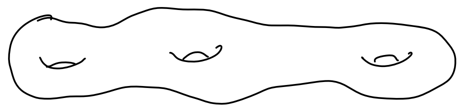
3) disk

← manifold with boundary (edge)
won't come up much

4) torus



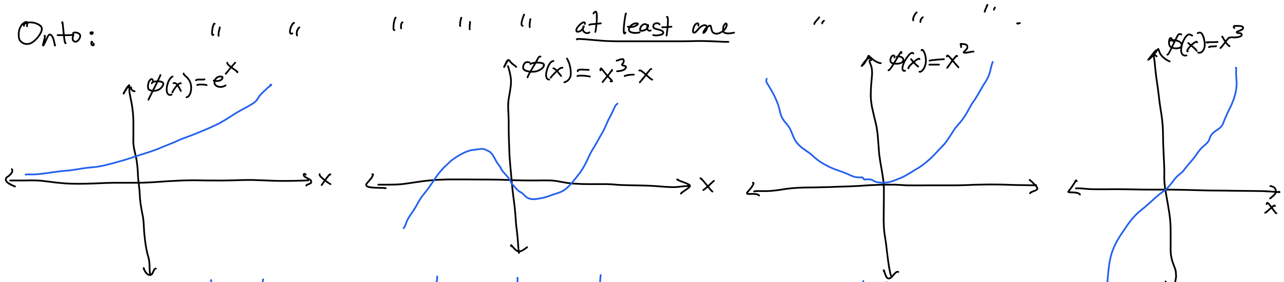
5) genus g surface



Drawn as embedded in 3-d space, but not necessa

Maps Given two sets M, N, a map $\phi: M \to N$ assigns exactly one element of N to each element of M.

One-to-one: each element of N has at most one element of M.
Onto: " " " " " at least one " " ".



one-to-one, not onto — onto, not one-to-one — neither — both

If both, then the map is invertible. $\Longrightarrow$ inverse map is $\phi^{-1}: N \to M$.

Smooth map: $\phi: \mathbb{R}^m \to \mathbb{R}^n$ if continuous and C^∞ ← all derivatives exist

Open set: = "solid chunk not including boundary"

Examples: 1) open ball in $\mathbb{R}^n$ = set of points x such that $\sum_i (x_i - y)^2 < r^2$ for some y

2) any union of open balls
(e.g. interior of solid cube)

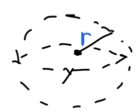
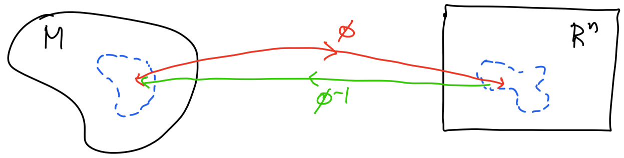


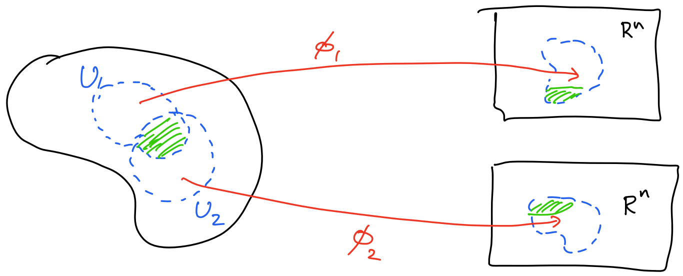
Chart = coordinate system Given M: onto, 1-1 map ϕ from $U \to \mathbb{R}^n$,
open subset of M

where $\phi(U)$ = open subset of $\mathbb{R}^n$ Chart = (U, ϕ)



ϕ = chart.

Compatible charts



Then $\phi_2 \circ \phi_1^{-1}$ maps $R^n \to R^n$ domain $= \phi_1(U_1 \cap U_2)$

$\phi_1 \circ \phi_2^{-1}$ " " " domain $= \phi_2(U_1 \cap U_2)$

Compatible means both are C^∞ functions. Always compatible if $U_1 \cap U_2 =$ empty

Manifold

(real, smooth, n-dimensional) is a set M together with a $\underbrace{\text{collection of charts}}_{\text{atlas}}$ (U_i, φ_i) with:

1) The charts cover all of M. ($M =$ union of U_i)
2) Any 2 charts compatible
3) Any chart compatible with all charts in the atlas is itself in the atlas. (technical point)

Examples

- $M = R^n$ is a manifold. (Let $U = M$ and ϕ = identity)
- S^2 = 2-sphere = $\{$points in R^3 with $x_1^2 + x_2^2 + x_3^2 = 1\}$ = M

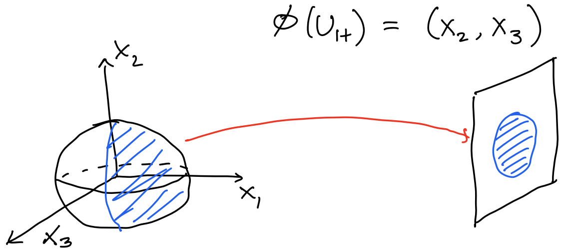
A chart: let U_{1+} = subset with $x_1 > 0$

$\phi(U_{1+}) = (x_2, x_3)$

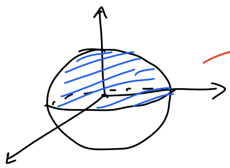
Another chart: let U_{1-} = subset with $x_1 < 0$

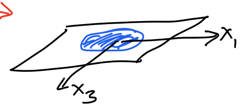
$\phi(U_{1-}) = (x_2, x_3)$

(trivially compatible; $U_{1+} \cap U_{1-}$ = empty)



Another:

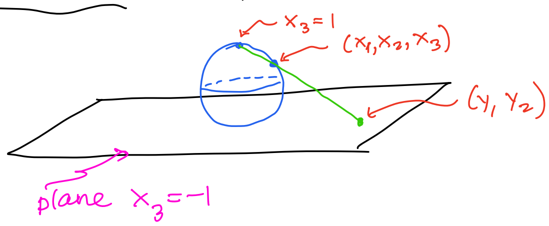




U_{2+} = subset with $x_2 > 0$

Another choice:

Chart 1: U_1 = all points except north pole $(0,0,1)$



$$\phi_1(x_1, x_2, x_3) = \left(\underbrace{\frac{2x_1}{1-x_3}}_{y_1}, \underbrace{\frac{2x_2}{1-x_3}}_{y_2} \right)$$

Chart 2: U_2 = all points except south pole $(0,0,-1)$

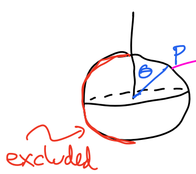
$$\phi_2(x_1, x_2, x_3) = \left(\underbrace{\frac{2x_1}{1+x_3}}_{z_1}, \underbrace{\frac{2x_2}{1+x_3}}_{z_2} \right)$$

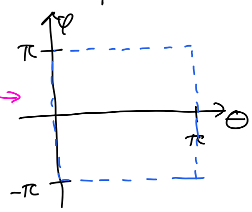
Together, charts cover whole sphere.

$\phi_2 \circ \phi_1^{-1}$ map is $(y_1, y_2) \to (z_1, z_2) = \left(\frac{4y_1}{y_1^2 + y_2^2}, \frac{4y_2}{y_1^2+y_2^2} \right)$ This is C^∞

for all points in intersection (excludes $(y_1, y_2) = (0,0)$ and $(z_1, z_2) = (0,0)$).

Another chart U_3 = all points described by $\theta \neq 0, \pi$ and $\varphi = \pm\pi$





By itself, doesn't cover all of S^2.
Sometimes, don't care.

Vectors on manifolds (general)

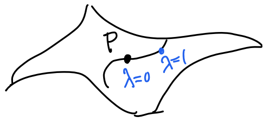
"Tangent space" on M = set of all vectors at each point p.

For n-dim M, have n basis vectors $\hat{e}_\mu$, $\mu = 0, 1, 2, \dots, n-1$.

Problem: no preferred basis, no preferred chart, no preferred embedding into higher dim space.

Solution: $T_p \leftrightarrow$ space of directional derivatives $\frac{d}{d\lambda}$ along curves through P.

For smooth functions $f: M \to \mathbb{R}$, consider for each curve through P:

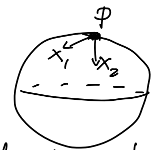


$\frac{d}{d\lambda}$ is a map $f \to \frac{df}{d\lambda}$. Intuitively:

vector along path through P $\longleftrightarrow$ $\frac{d}{d\lambda}$ for path through P

Claim 1 T_P = vector space. Can add, multiply by numbers:

$$a\frac{d}{d\lambda_1} + b\frac{d}{d\lambda_2} = \text{a derivative operator } \checkmark$$



Claim 2 T_P has dimension n. Let x^μ = coordinates at P

(need not be $\perp$, don't know what that means)

Then $\frac{\partial}{\partial x^\mu} = \partial_\mu$ = basis vectors for T_P.

$$\frac{df}{d\lambda} = \frac{dx^\mu(\lambda)}{d\lambda}\partial_\mu f$$

sum over n of these

Chain rule

So $\hat{e}_\mu = \partial_\mu$ = coordinate basis for T_P (not orthonormal)

Different chart $\to$ different basis.

$$\hat{e}_{\mu'} = \frac{\partial}{\partial x^{\mu'}} = \partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu = \frac{\partial x^\mu}{\partial x^{\mu'}}\hat{e}_\mu$$

old basis vectors

new basis vectors

A general vector $V = V^\mu \partial_\mu$

components

basis vectors (not components)

Often, just write equations involving components

Change basis: $V^\mu \partial_\mu = V^{\mu'}\partial_{\mu'}$ (V doesn't depend on basis choice)

$$= V^{\mu'}\frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu \quad \text{(Chain rule)}$$

So $V^\mu = V^{\mu'}\frac{\partial x^\mu}{\partial x^{\mu'}}$ or $\boxed{V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}V^\mu}$ Vector transformation law

Generalizes $V^{\mu'} = \Lambda^{\mu'}{}_\mu V^\mu$ for SR.

Vector field = vector for each P = derivative map from (fns on M) to (fns on M)

Example: If $M = \mathbb{R}^2$, $V = \underbrace{x^2 y}_{V_x} \underbrace{\frac{\partial}{\partial x}}_{\hat{e}_x} + \underbrace{x y^2}_{V_y} \underbrace{\frac{\partial}{\partial y}}_{\hat{e}_y}$ is a vector field.

Also, $V = V^r \underbrace{\frac{\partial}{\partial r}}_{\hat{e}_r} + V^\phi \underbrace{\frac{\partial}{\partial \phi}}_{\hat{e}_\phi}$, where $\begin{cases} V^r = \frac{\partial r}{\partial x} V^x + \frac{\partial r}{\partial y} V^y \\ V^\phi = \frac{\partial \phi}{\partial x} V^x + \frac{\partial \phi}{\partial y} V^y \end{cases}$

Given two vector fields V, W, can define commutator, by action of functions

$$[V, W](f) = V(W(f)) - W(V(f)).$$

In components:

$$[V, W]^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu.$$

Dual vectors on manifolds = $\underbrace{\text{linear maps from } T_P \to \mathbb{R}}_{T_P^* \,=\, \text{dual tangent space}}$

Dual vector $\equiv$ 1-form

If $\hat{e}_\mu = \partial_\mu$ is a basis for T_P, then
$\hat{\theta}^\mu = dx^\mu$ " " " " T_P^*, defined by $dx^\mu(\partial_\nu) \equiv \delta^\mu_\nu$

not multiplication

Example: gradient of $f(x^\mu) = df \equiv \underbrace{\frac{\partial f}{\partial x^\mu}}_{\text{components}} \; \underbrace{dx^\mu}_{\text{basis dual vectors}}$ = dual vector

Consider a path $x^\mu(\lambda)$ through P. Recall $\frac{d}{d\lambda}$ is a vector at P.

$$df\left(\frac{d}{d\lambda}\right) = \left(\frac{\partial f}{\partial x^\mu} dx^\mu\right)\left(\frac{\partial x^\nu}{\partial \lambda} \partial_\nu\right) = \frac{\partial f}{\partial x^\mu} \frac{\partial x^\nu}{\partial \lambda} \underbrace{dx^\mu(\partial_\nu)}_{\delta^\mu_\nu} = \frac{\partial f}{\partial x^\mu} \frac{\partial x^\mu}{\partial \lambda} = \frac{df}{d\lambda}$$

number.

df maps vectors $\frac{d}{d\lambda}$ to numbers. In particular, dx^μ maps $\frac{d}{d\lambda}$ to $\frac{dx^\mu}{d\lambda}$

A general dual vector: $\omega = \underbrace{\omega_\mu}_{\text{components}} \; \underbrace{dx^\mu}_{\text{basis (not components)}}$

Change coordinates: $dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} dx^{\mu}$ $\qquad \omega_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \omega_{\mu}$

basis of dual vectors, primed chart $\qquad$ " " " unprimed

Components depend on basis, ω does not.

Tensors on manifolds $\quad$ rank (k,ℓ) = map from $\begin{pmatrix} k \text{ dual vectors} \\ \ell \text{ vectors} \end{pmatrix} \Longrightarrow$ real numbers

Components $T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_\ell} = T(dx^{\mu_1}, \ldots dx^{\mu_k}, \partial_{\nu_1}, \ldots \partial_{\nu_\ell})$, $\quad$ then

$$T = T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_\ell}\, \partial_{\mu_1} \cdots \partial_{\mu_k}\, dx^{\nu_1} \cdots dx^{\nu_\ell}$$

Transformation law for components:

$$T^{\mu_1' \ldots \mu_k'}{}_{\nu_1' \ldots \nu_\ell'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \cdots \frac{\partial x^{\nu_\ell}}{\partial x^{\nu_\ell'}} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_\ell}$$

Can contract indices, just as in SR. $\quad T^{\mu\nu\rho}{}_{\rho}$ = rank $(2,0)$ tensor

$T^{\mu\nu\rho} S_{\nu\rho}$ = rank $(1,0)$ tensor = vector

Is partial derivative of a tensor a tensor? $\quad$ Only sometimes.

$\partial_\mu f = \frac{\partial f}{\partial x^\mu}$ is a $(0,1)$ tensor = dual vector.

But, $\partial_\mu A_\nu$ is not a $(0,2)$ tensor. (Even though correct indices!)

Proof: do a coordinate change: $\frac{\partial}{\partial x^{\mu'}} A_{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\nu}{\partial x^{\nu'}} A_\nu \right)$

$$= \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial A_\nu}{\partial x^\mu}}_{\checkmark} + \underbrace{\left[\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\nu}{\partial x^{\nu'}} \right) \right]}_{\text{extra} = \frac{\partial^2 (x^\nu)}{\partial x^{\mu'} \partial x^{\nu'}}} A_\nu$$

Note symmetric under $\mu' \leftrightarrow \nu'$

$\partial_\mu A_\nu - \partial_\nu A_\mu$ is a $(0,2)$ tensor (antisymmetric)

More generally, antisymmetric $(0,n)$ tensor = n-form

Examples: $\quad$ dual vector = 1-form

a 3-form: $\quad \partial_\mu \partial_\nu A_\rho - \partial_\mu \partial_\rho A_\nu + \partial_\nu \partial_\rho A_\mu - \partial_\nu \partial_\mu A_\rho + \partial_\rho \partial_\mu A_\nu - \partial_\rho \partial_\nu A_\mu$