Binary inspiral How do R, ω, change as masses spiral in?

$$\omega = \sqrt{\frac{GM}{4R^3}} \quad \text{and} \quad R = \left(\frac{GM}{4\omega^2}\right)^{1/3}$$

Gravity waves will also have this frequency, but changes with R.

Energy of stars = potential + kinetic $= -\frac{GM^2}{2R} + 2\left(\frac{1}{2}M\underset{(\omega R)}{v^2}\right)$

$$= -\frac{GM^2}{2R} + M\omega^2 R = -\frac{GM^2}{4R}.$$

So $\frac{d}{dt}(\text{Energy}) = -\text{Power} = -\frac{2}{5}\frac{G^4M^5}{c^5R^5}$. So to find $\frac{dR}{dt}$,

$$\underbrace{\frac{d}{dt}\left(-\frac{GM^2}{4R}\right)}_{\frac{GM^2}{4R}\frac{dR}{dt}} = -\frac{2}{5}\frac{G^4M^5}{c^5R^5} \Rightarrow \boxed{\frac{dR}{dt} = -\frac{8}{5}\frac{G^3M^3}{c^5R^3}}$$

Use this to find how ω changes:

$$\frac{d\omega}{dt} = \frac{d}{dt}\left(\sqrt{\frac{GM}{4R^3}}\right) = -\frac{3}{4}\sqrt{GM}\,R^{-5/2}\frac{dR}{dt} = \frac{6}{5}\frac{(GM)^{7/2}}{c^5 R^{11/2}}$$

$$= \frac{6}{5}\frac{(GM)^{7/2}}{c^5\left(\frac{GM}{4\omega^2}\right)^{11/6}}, \quad \text{or}$$

$$\boxed{\frac{d\omega}{dt} = \frac{96}{5}\left(\frac{GM}{2^{1/5}c^3}\right)^{5/3}\omega^{11/3}}$$

Redo with circular orbits but $M_1 \neq M_2$:

$$\frac{d\omega}{dt} = \frac{96}{5}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}\omega^{11/3}, \quad \text{where} \quad \mathcal{M} = \frac{(M_1M_2)^{3/5}}{(M_1+M_2)^{1/5}} = \text{"chirp mass"}$$

Can plug in to check Hulse–Taylor observation (indirect) or to measure directly for grav. waves. (Fails for large ω.)

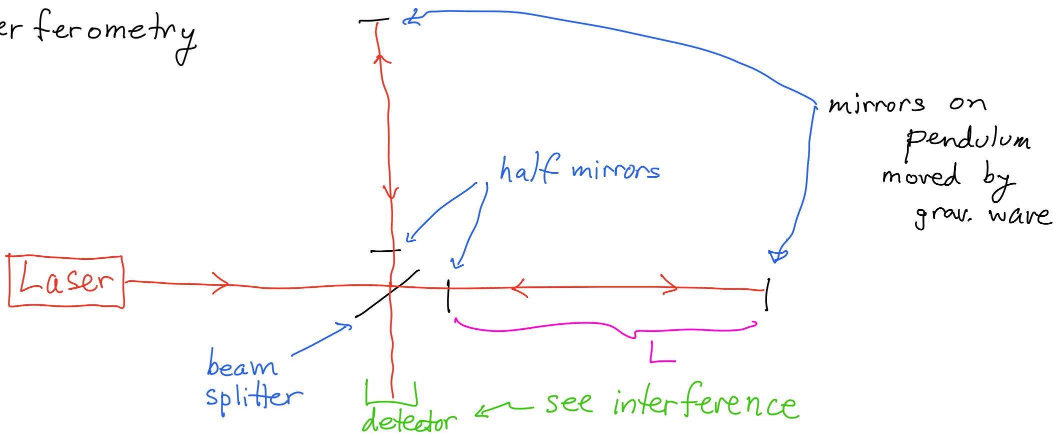
Direct detection at LIGO

Strain $h \lesssim 10^{-21}$ from BH–BH mergers

$$\frac{\delta L}{L} \sim h \sim 10^{-16}\left(\frac{h}{10^{-21}}\right)\left(\frac{L}{\text{km}}\right)\ \text{cm}$$

Recall atom $\sim 10^{-8}$ cm, nucleus $\sim 10^{-13}$ cm.

Laser interferometry



Technical problems: noise $\begin{cases}\text{thermal}\\ \text{seismic}\\ \text{passing cars}\end{cases}$

LIGO: detectors in WA, LA (3000 km apart)

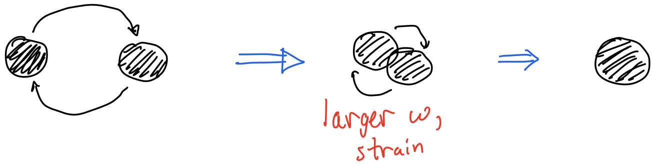
laser interferometry gravitational observatory

Each detector has $L = 4$ km

First signal: 09:51 UTC 9/14/2015 GW150914

2 initial black holes $36^{+5}_{-4}\, M_\odot$ and $29^{+4}_{-4}\, M_\odot$

solar mass



No Hair Theorem says settles down to Kerr black hole, M, J

no wave

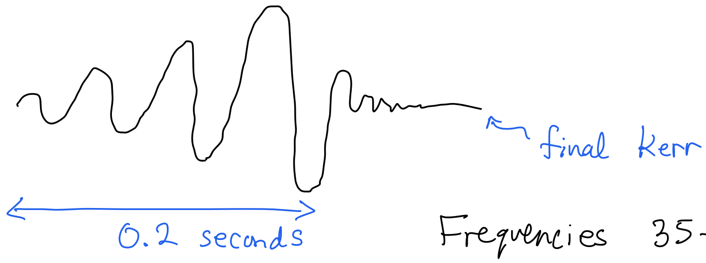
Final state $M = 62^{+4}_{-4}\, M_\odot$. So, $3^{+4}_{-4}\, M_\odot c^2$ = energy radiated in gravity waves

Event was at redshift $z = 0.09^{+0.03}_{-0.04} \sim 400$ Mpc

(recall Andromeda galaxy at 780 kpc)

Max strain $\sim 1.0 \times 10^{-21}$

Chirp



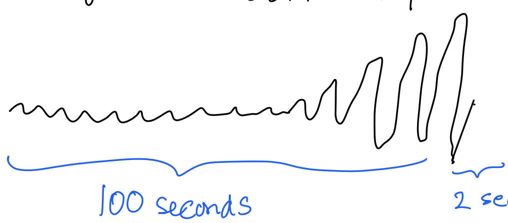
Frequencies 35–250 Hz (audible)

Initial claims of γ-ray burst now discounted.

So far:

Merger type	Number
BH-BH	~ 210 ← largest mass $260 M_\odot$
BH-neutron star	6
NS-NS	2 ← one with EM confirmation. Much longer chirp, because smaller masses

NS NS merger GW170817 chirp ~ 1.6 and ~ 1 $M_\odot$

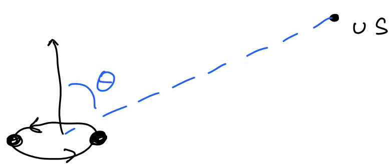


2 seconds of γ, X-ray, UV, visible, IR, radio, seen by 70 different groups

Host galaxy NGC 4993, at 40 Mpc.

Eventual final state = BH.

Polarizations + or ×:



Peak strains are

$$h_+ = \frac{2c}{\omega r}\left(\frac{G\mathcal{M}\omega}{c^3}\right)^{5/3}(1+\cos^2\theta)$$

$$h_\times = \frac{4c}{\omega r}\left(\frac{G\mathcal{M}\omega}{c^3}\right)^{5/3}\cos\theta$$

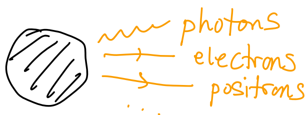
Measure ω, h_+, $h_\times$ $\Rightarrow$ fit $\mathcal{M}$, θ, r ← distance!

Determining pre-merger masses: deviations from simple picture above.

Gravity wave astronomy: use these as "standard sirens" to relate redshift $\leftrightarrow$ distance. Precision $a(t)$.

Future: Upgraded LIGO's.
Pulsar timing arrays $\Leftrightarrow$ evidence for stochastic grav. wave background
Space-based LISA (2030's) arms $10^6 \times$ LIGO, will see mergers in our own galaxy, waves from early universe.

Hawking radiation The $\underbrace{\text{vacuum}}_{\text{lowest energy}}$ state outside a black hole is not empty.



Temperature $k_B T = \dfrac{\hbar c^3}{8\pi G M}$

Described by quantum fields in curved space.

Each momentum mode $\longleftrightarrow$ harmonic oscillator.

$H = \hbar\omega(a^\dagger a + \tfrac{1}{2})$ with $[a, a^\dagger] = 1$, and $a = \dfrac{1}{\sqrt{2\hbar\omega m}}(m\omega X + iP)$

$a^\dagger =$ " $($" $X - iP)$

Energy e-states: $|n\rangle = \dfrac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$ ← ground state

$H|n\rangle = \hbar\omega(n+\tfrac{1}{2})$

Scalar field $\phi(t, \vec{x}) = \phi(x^\mu)$

Classically, a number for each $t, \vec{x}$
QM, an operator " " ".

Note: position is not an operator.

Action $= \int d^4x\, \mathcal{L}$, where $\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2$

$= \frac{1}{2}(\dot{\phi}^2 + |\vec{\nabla}\phi|^2 - m^2\phi^2)$ (flat space)

Momentum conjugate to ϕ (<u>not</u> a vector) is

$$\pi(t,\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

Hamiltonian density: $\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}|\vec{\nabla}\phi|^2 + \frac{1}{2}m^2\phi^2$

$H = \int d^3\vec{x}\ \mathcal{H}$. Then $|\Psi(t)\rangle = e^{-iHt/\hbar}\ |\Psi(0)\rangle$

Commutation relations:

$$[\phi(t,\vec{x}),\ \pi(t,\vec{x}')] = i\hbar\ \delta^{(3)}(\vec{x}-\vec{x}')$$
$$[\phi(t,\vec{x}),\ \phi(t,\vec{x}')] = 0$$
$$[\pi(t,\vec{x}),\ \pi(t,\vec{x}')] = 0$$

$\omega = \sqrt{|\vec{k}|^2 + m^2}$

Define $a_k^\dagger = \dfrac{1}{\sqrt{(2\pi)^3 2\omega}} \displaystyle\int d^3\vec{x}\ e^{-i\vec{k}\cdot\vec{x}}\ [\omega\phi(t,\vec{x}) - i\pi(t,\vec{x})]$ ← creates a particle with momentum $\hbar\vec{k}$

$a_k =$ " " $e^{i\vec{k}\cdot\vec{x}}\ [\omega\phi(\text{"}) + i\pi(\text{"})]$

Then
$$\left.\begin{aligned} [a_k, a_{k'}^\dagger] &= \delta^{(3)}(\vec{k}-\vec{k}') \\ [a_k, a_{k'}] &= 0 \\ [a_k^\dagger, a_{k'}^\dagger] &= 0 \end{aligned}\right\}$$
Independent HO for each $\vec{k}$.

Vacuum state $|0\rangle$ obeys $a_k|0\rangle = 0$ for all $\vec{k}$.

$a_k^\dagger|0\rangle = |\vec{k}\rangle$ = state with one particle

$a_{k_1}^\dagger a_{k_2}^\dagger|0\rangle = |\vec{k}_1, \vec{k}_2\rangle$ = state with two particles

might be same

Fock states: $|n_{k_1}, n_{k_2}, \ldots\rangle$

number of particles with $\vec{p} = \hbar\vec{k}_1$

$$H = \int d^3\vec{k}\ \hbar\omega\ \frac{1}{2}\left[a_k^\dagger a_k + a_k a_k^\dagger\right] = \int d^3\vec{k}\ \hbar\omega\left[\underbrace{a_k^\dagger a_k}_{=0 \text{ acting on } |0\rangle} + \underbrace{\tfrac{1}{2}\delta^{(3)}(0)}_{\text{infinite vacuum energy}}\right]$$

Cancel it by subtracting ∞ constant

Energy of state is $\displaystyle\int d^3\vec{k}\ \hbar\omega\left(n_k + \tfrac{1}{2}\right)$

Particle interpretation of state is <u>observer-dependent</u>.

Example: do a LT. Then $\vec{k} \to \vec{k}'$.

$$|n_k, 0, 0, 0\ 0, 0, 0 \ldots\rangle \longrightarrow |0, 0, 0, n_{k'}, 0, 0, \ldots\rangle$$

$$\text{Energy} \quad \hbar\omega = \hbar\sqrt{|\vec{k}|^2 + m^2} \quad \to \quad \hbar\omega' = \hbar\sqrt{|\vec{k}'|^2 + m^2}$$

Particles in general metric

$$\mathcal{L} = \sqrt{-g}\left(-\tfrac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \tfrac{1}{2} m \phi^2 - \xi R \phi^2 + \ldots\right)$$

$$[\phi(t, \vec{x}), \ \pi(t, \vec{x}')] = \frac{i\hbar}{\sqrt{-g}} \delta^{(3)}(\vec{x} - \vec{x}')$$

Particles described by <u>different</u> ladder operators.

Bogoliubov transformation:

trade $a_k, a_k^\dagger$ for $b_k, b_k^\dagger$

Define
$$b_k^\dagger = \int d^3\vec{k}' \left(\alpha_{kk'} a_{k'}^\dagger - \beta_{kk'} a_{k'}\right)$$

complex coefficients

$$b_k = \int d^3\vec{k}' \left(\alpha_{kk'}^* a_{k'} - \beta_{kk'}^* a_{k'}^\dagger\right)$$

Demand
$$[b_k, b_{k'}^\dagger] = \delta^{(3)}(\vec{k} - \vec{k}')$$
$$[b_k, b_{k'}] = 0 \qquad [b_k^\dagger, b_{k'}^\dagger] = 0$$

$$\Rightarrow \int d^3\vec{q} \left(\alpha_{kq} \alpha_{qk'}^* - \beta_{kq} \beta_{qk'}^*\right) = \delta^{(3)}(\vec{k} - \vec{k}')$$
$$\int d^3\vec{q} \left(\alpha_{kq} \beta_{qk'} - \beta_{kq} \alpha_{qk'}\right) = 0$$

Freely falling observer detects particles described by $a, a^\dagger$.

Another observer might " " " " $b, b^\dagger$.

Suppose the particle state is $|0\rangle_a$ (defined by $a_k|0\rangle = 0$ for all $\vec{k}$).

A freely falling observer says number of particles with momentum $\hbar \vec{k}$ is $\;{}_a\langle 0|\, a^\dagger_k a_k \,|0\rangle_a = 0$. A boosted observer agrees.

But a general observer sees:

$$ {}_a\langle 0|\, b^\dagger_k b_k \,|0\rangle_a = \int d^3\vec{k}' \int d^3\vec{k}'' \; {}_a\langle 0| \left(\alpha^*_{kk'} \cancel{a^\dagger_{k'}} - \beta^*_{kk'} a_{k'}\right)\left(\alpha_{kk''} \cancel{a_{k''}} - \beta_{kk''} a^\dagger_{k''}\right) |0\rangle_a $$

$$ = \int d^3\vec{k}' \int d^3\vec{k}'' \; \beta^*_{kk'} \beta_{kk''} \; {}_a\langle 0|\, \underbrace{a_{k'} a^\dagger_{k''}}_{a^\dagger_{k''}\cancel{a_{k'}} \,+\, \delta^{(3)}(\vec{k}' - \vec{k}'')} \,|0\rangle_a $$

$$ = \int d^3\vec{k}' \; |\beta_{kk'}|^2 \neq 0. $$

Finding $\alpha_{k,k'}$ and $\beta_{k,k'}$ for an observer is non-trivial for

1) Arbitrary motion in flat space
2) Curved space

Detector carried by observer uses $b, b^\dagger$ determined by proper time τ, not coordinate t.

Unruh effect: Accelerated observer in Minkowski space

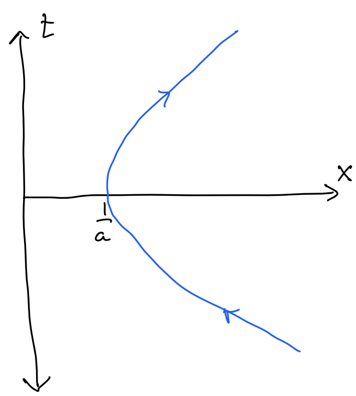
$$ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 $$

For an observer with constant x acceleration:

$$ t(\tau) = \frac{1}{a}\sinh(a\tau) \qquad \text{Accel: } a^\mu = \frac{d^2 x^\mu}{d\tau^2} = a\left(\sinh(a\tau), \cosh(a\tau), 0, 0\right) $$

$$ x(\tau) = \frac{1}{a}\cosh(a\tau) $$

$$ \text{So } \; a^\mu a_\mu = a^2 \big(\underbrace{-\sinh^2(a\tau) + \cosh^2(a\tau)}_{1} + 0 + 0\big) $$
$$ = a^2 $$

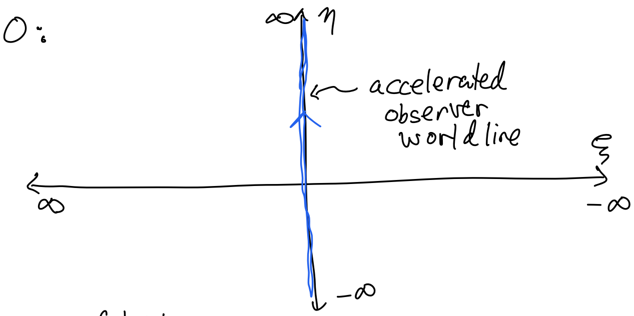


Define new coords (η, ξ) for which observer is co-moving:

$$ t = \frac{1}{a} e^{a\xi} \sinh(a\eta) $$

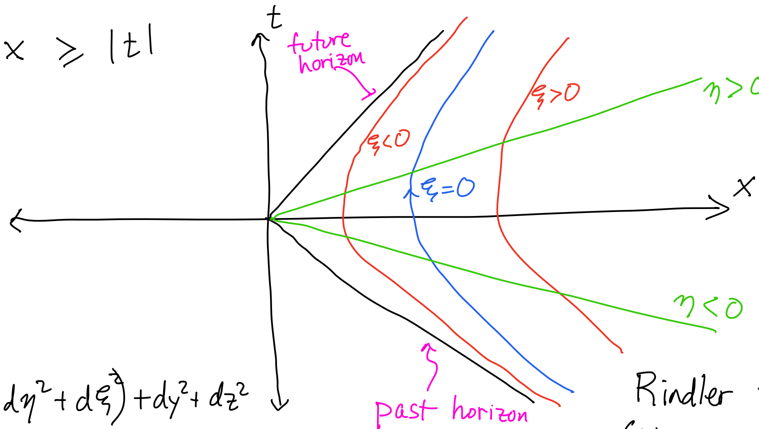
$$ x = \frac{1}{a} e^{a\xi} \cosh(a\eta) $$

Observer has $\eta = \tau$, $\xi = 0$:



But, (η, ξ) don't cover whole manifold!

They only cover $x \geq |t|$



Rindler wedge
(like region $r > 2GM$ in Schwarzschild)
= part of flat spacetime

Metric $= ds^2 = e^{2a\xi}\left(-d\eta^2 + d\xi^2\right) + dy^2 + dz^2$

Quantize w.r.t. η = "time",
ξ = "space".

Glossing over important details, for massless particles, with no y, z momentum (so $\omega = k$)

$$\beta_{\omega\omega'} = \frac{e^{-\pi\omega/2a}}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}}\,\delta(\omega - \omega')$$

So

$$\langle 0 | N_\omega | 0 \rangle = \frac{\left| e^{-\pi\omega/2a} \right|^2}{2\sinh\left(\frac{\pi\omega}{a}\right)} = \frac{e^{-\pi\omega/a}}{e^{\pi\omega/a} - e^{-\pi\omega/a}} = \frac{1}{e^{2\pi\omega c/a} - 1}.$$

put back c

This has the same form as Planck blackbody radiation,

$$\frac{1}{e^{\hbar\omega/k_B T} - 1}.$$

So the accelerated detector sees a thermal distribution,

with $k_B T = \frac{\hbar a}{2\pi c}$.

How is energy conserved? Detector also emits particles.

Hawking effect (Thermal emission by Schwarzschild)

Consider a static observer (r, θ, ϕ = fixed) near horizon.

Compute proper acceleration. Start with proper velocity:

$$U^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{c\,dt}{d\tau}, 0, 0, 0\right) = \frac{c}{\sqrt{1-\frac{2GM}{r}}}$$

Proper acceleration $a^\mu = U^\nu \nabla_\nu U^\mu = \cancel{\underbrace{U^\nu}_{\text{only non-zero for } \nu = t} \underbrace{\partial_\nu}_{\text{only non-zero for } \nu = r} U^\mu} + \underbrace{\Gamma^\mu_{\nu\sigma} U^\nu U^\sigma}_{\underbrace{\Gamma^\mu_{tt}}_{-\delta^{\mu r}\frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)} \left(\frac{c^2}{1-\frac{2GM}{c^2 r}}\right)}$

$$= -\delta^{\mu r}\frac{GM}{r^2}$$

So $a = \sqrt{a^\mu a_\mu} = \sqrt{\left(\frac{GM}{r^2}\right)^2 \underbrace{g_{rr}}_{\left(1-\frac{2GM}{c^2 r}\right)}}$

So $\boxed{a = \frac{GM}{r^2}\left(1-\frac{2GM}{c^2 r}\right)^{-1/2}}$ Note this blows up as $r \to \frac{2GM}{c^2} = R_s$

← Carroll (9.170) is wrong.

For r sufficiently close to $\frac{2GM}{c^2}$, $a \gg \left|R_{\mu\nu\rho\sigma}\right|^{1/2}$, which has two nice features:

1) Just like accelerated observer in Minkowski, so can apply Unruh's result from Rindler.
2) Mass of particle becomes negligible

only works for $r \approx R_s$

So the static detector near the horizon sees a thermal spectrum, with

$$k_B T_{\text{horizon}} = \frac{\hbar a}{2\pi c} = \frac{\hbar}{2\pi c}\frac{GM}{r^2}\left(1-\frac{2GM}{c^2 r}\right)^{-1/2}$$

$$\approx \frac{\hbar}{2\pi c}\frac{GM}{\left(\frac{2GM}{c^2}\right)}\left(1-\frac{2GM}{c^2 r}\right)^{-1/2} \approx \frac{\hbar c^3}{8\pi GM}\left(1-\frac{2GM}{c^2 r}\right)^{-1/2}$$

Note this blows up for $r \to R_s = \frac{2GM}{c^2}$.

(A freely falling observer sees no particles, nothing special at horizon.)

For a detector far away, take into account redshift factor for each particle, $\left(1 - \frac{2GM}{c^2 r}\right)^{1/2}$.

So $$k_B T_\infty = \frac{\hbar c^3}{8\pi G M} = \text{Hawking temperature.}$$

Hawking's original paper (1974) got this by computing the Bogoliubov coefficients. Black holes lose mass, evaporate!

<u>If black holes have temperature, do they also have entropy?</u>

Yes! Bekenstein 1972 (<u>before</u> Hawking)

$$S_{BH} = \frac{k_B c^3}{4\hbar G} A \quad \text{area of horizon} = 4\pi R_s^2 = 4\pi \left(\frac{2GM}{c^2}\right)^2$$

Throw in more mass, increases entropy.

<u>Does expanding universe create particles out of nothing?</u>

Yes! (Need violation of time-translation; no everywhere-timelike Killing vector.)

For de Sitter space (relevant for inflation)

$$\frac{\text{rate}}{\text{volume}} \sim H^4 \left(\frac{m}{H}\right)^3 e^{-2\pi m/H}$$