PHYS 751 General Relativity

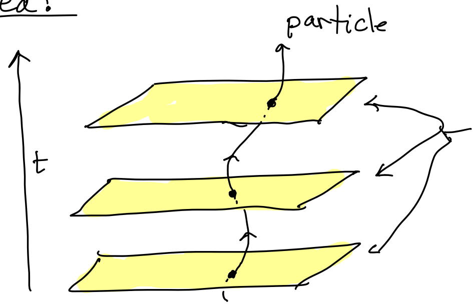
Spacetime = 4-dimensional set (3 space & 1 time)

Individual point = <u>event</u> (a firecracker explodes)

Path of a particle = <u>worldline</u>

<u>What worldlines are allowed?</u>

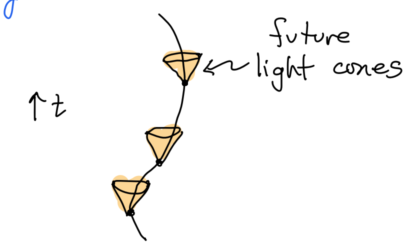
Pre-Einstein <1905



- Any world line allowed, if continuous, passes through each surface once.
- Simultaneity unambiguous.

<u>Causal structure</u> defined by constant t surfaces.

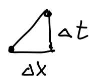
Einstein >1905



- All particle velocities $\leq c$
- Light cone interiors → allowed world lines through each point
- No preferred time slicing → simultaneity ambiguous.

<u>Spacetime interval</u> between 2 points: $(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$

Comments: • No special role pre-Einstein



- Can be positive "spacelike"
  negative "timelike"
  real "null" or "lightlike"
- $c = 2.99792458 \times 10^8$ meters/sec (exactly, since 1983, by choice of units)

  Equally good units choice: $c = 1$

<u>Inertial frame</u> = coordinates t, x, y, z so that any observer at fixed x, y, z is unaccelerated.

Consider another inertial frame t', x', y', z'. Then, for same 2 events:

$$\underbrace{(\Delta s)^2}_{\text{same}} = \underbrace{-(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2}_{\text{different}}$$

Spacetime with these properties = <u>Minkowski</u> space

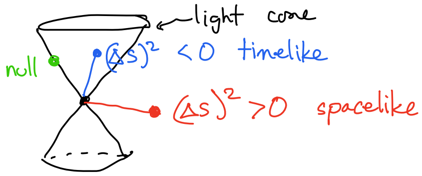
Write $x^\mu = \begin{cases} x^0 = ct \\ x^1 = x \\ x^2 = y \\ x^3 = z \end{cases}$ or $\mu, \nu, \rho, \sigma, \lambda = 0,1,2,3$

If we want only space coordinates, use $i, j, k = 1,2,3$ $x^i = \begin{cases} x \\ y \\ z \end{cases}$.

Define the metric $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$ Note $c=1$ from now on. Then $(\Delta s)^2 = \eta_{\mu\nu}\,\Delta x^\mu \Delta x^\nu$.

Note μ, ν repeated $\Leftrightarrow$ summed over! "dummy indices"
unless fixed by context (for example on both sides of an equation)

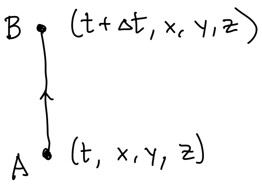
$(\Delta s)^2 = \eta_{\rho\sigma}\,\Delta x^\rho \Delta x^\sigma$ = same thing



Proper time $(\Delta \tau)^2 = -(\Delta s)^2 = -\eta_{\mu\nu}\,\Delta x^\mu \Delta x^\nu$

$\Delta\tau$ = time measured by an observer moving on straight line between 2 events (>0).

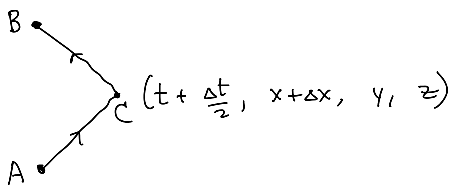
Proper time = coordinate time only for observer at rest in inertial frame.



$(\Delta\tau)^2 = -\eta_{\mu\nu}\,\Delta x^\mu \Delta x^\nu = -\left[(-1)(\Delta t)^2 + 0 + 0 + 0\right]$

$= \Delta t^2$ So $\Delta\tau_{AB} = \Delta t$

Consider a different worldline between same two events:



Observer moves at speed v, $\Delta x = v\left(\frac{\Delta t}{2}\right)$

Elapsed proper time: $\Delta\tau_{ACB} = \Delta\tau_{AC} + \Delta\tau_{CB}$.

$(\Delta\tau_{AC})^2 = -\eta_{\mu\nu}\left(\frac{\Delta t}{2}, \Delta x, 0, 0\right)^\mu \left(\frac{\Delta t}{2}, \Delta x, 0, 0\right)^\nu = \left(\frac{\Delta t}{2}\right)^2 - (\Delta x)^2$ (positive iff $v<c$)

So $\Delta\tau_{AC} = \frac{\Delta t}{2}\sqrt{1-v^2}$. Same way $\Delta\tau_{CB}$ = same thing.

So $\Delta\tau_{ACB} = \Delta t\sqrt{1-v^2}$ always less than $\Delta\tau_{AB}$

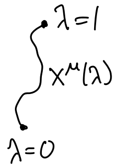
Twin "paradox". (Nothing paradoxical. No problem dealing with acceleration in special relativity.) The twin who accelerates is younger.

For infinitesimal $(\Delta t, \Delta x, \Delta y, \Delta z) = (dt, dx, dy, dz)$,

$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$ = <u>line element</u>.

Parameterized path through spacetime

$x^\mu(\lambda)$, for, say, $0 \leq \lambda \leq 1$.



timelike (could be a particle worldline)

spacelike (can't be a worldline)

$$\Delta\tau = \int_0^1 d\lambda \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

= elapsed proper time

$$\Delta s = \int_0^1 d\lambda \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

(Note: could choose $\lambda = \tau$ = proper time.)

Coordinate transformations

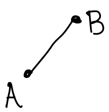
Translations $x^\mu \longrightarrow x^{\mu'} = \delta^{\mu'}_{\mu}(x^\mu + a^\mu)$ $\quad \delta^{\mu'}_{\mu} = \begin{cases} 1 & \text{if } \mu = \mu' \\ 0 & \text{if } \mu \neq \mu' \end{cases}$

Rotations/boosts $x^\mu = x^{\mu'} = \Lambda^{\mu'}{}_{\nu}\, x^\nu$ index order, height matters

Matrix form: $x' = \Lambda x$

Require $(\Delta s)^2$ unchanged.

For translations: $(\Delta s)^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu \stackrel{?}{=} \eta_{\mu'\nu'}\Delta x^{\mu'}\Delta x^{\nu'}$



$$\Delta x^\mu = x^\mu_B - x^\mu_A$$

$$\Delta x^{\mu'} = x^{\mu'}_B - x^{\mu'}_A = \delta^{\mu'}_{\mu}(x^\mu_B + a^\mu) - \delta^{\mu'}_{\mu}(x^\mu_A + a^\mu) = \delta^{\mu'}_{\mu}(x^\mu_A - x^\mu_B)$$

So, need $\eta_{\mu\nu} = \delta^{\mu'}_{\mu}\delta^{\nu'}_{\nu}\eta_{\mu'\nu'}$. $\Longrightarrow$ η and η' have same matrix form.

For rotations/boosts: $(\Delta s)^2 \underset{\text{matrix form}}{=} (\Delta x)^T \eta\, \Delta x \stackrel{?}{=} \underbrace{(\Delta x')^T}_{\Delta x^T \Lambda^T} \underbrace{\eta'}_{\eta} \underbrace{\Delta x'}_{\Lambda \Delta x}$

So, need $\eta = \Lambda^T \eta \Lambda$ (matrix form), or

$$\boxed{\eta_{\rho\sigma} = \Lambda^{\mu'}{}_{\rho}\, \eta_{\mu'\nu'}\, \Lambda^{\nu'}{}_{\sigma} = \Lambda^{\mu'}{}_{\rho}\, \Lambda^{\nu'}{}_{\sigma}\, \eta_{\mu'\nu'}}$$

$\Lambda^{\mu'}{}_{\nu}$ = Lorentz transformation. Set of all = Lorentz group = $O(3,1)$

Similar to rotation group $O(3) \Longleftrightarrow$ 3x3 matrices R with $I = R^T I R$ (3x3 identity)

<u>Examples:</u> $\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$ = identity (do nothing)

Not continuously connected to identity

$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ = parity = space inversion (takes $x,y,z \to -x,-y,-z$).

$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ = time reversal $(t \to -t)$

<u>Proper orthochronous</u> or <u>restricted</u> Lorentz group has $\det \Lambda = 1$, $\Lambda^{0'}{}_{0} = 1$.

These are continuously connected to identity.

Rotation about z-axis: $\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $(0 \le \theta < 2\pi)$

Boost in z-direction: $\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \cosh\rho & 0 & 0 & -\sinh\rho \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\rho & 0 & 0 & \cosh\rho \end{pmatrix}$ $(-\infty < \rho < \infty)$

ρ = rapidity of boost.

$$\begin{cases} t' = t\cosh\rho - z\sinh\rho \\ z' = -t\sinh\rho + z\cosh\rho \end{cases}$$

A point with fixed z' moves in x^μ frame with speed $v = \frac{\sinh\rho}{\cosh\rho} = \tanh\rho$. So $\rho = \tanh^{-1}(v)$.

$\cosh\rho = \gamma = \frac{1}{\sqrt{1-v^2}}$

$\sinh\rho = v\gamma = \frac{v}{\sqrt{1-v^2}}$

$(v = \beta) \Longrightarrow$ $t' = \gamma(t - vz)$, $z' = \gamma(z - vt)$.

<u>Poincaré group</u> = translations, boosts, rotations

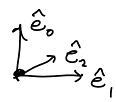
Defining feature: $(\Delta s)^2 = \eta_{\mu\nu}(\Delta x^\mu)(\Delta x^\nu)$ is unchanged. ($x'^\mu - x^\mu$)

Vectors

At each event, define a set of 4 basis vectors:

$\hat{e}_\mu$ — labels which basis vector
Note: <u>lowered</u>.

$\mu = 0,1,2,3$



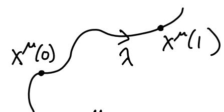
For example, $\hat{e}_0$ could be chosen to point along t direction, $\hat{e}_1$ " " " " " " x " , etc.

A vector A is: $A = A^\mu \hat{e}_\mu$ = independent of coordinate system, choice of basis

A^μ = components of vector A. (do depend on choice of basis).

Often, just talk about components, but the actual vector is A.

Example: let $x^\mu(\lambda)$ = path through spacetime.



then $V^\mu = \dfrac{dx^\mu}{d\lambda}$ = components of vector tangent to path.

Actual vector is $V = V^\mu \hat{e}_\mu$, where $\hat{e}_0$ = vector for path where only $x^0 = t$ changes
$\hat{e}_1$ = " " " " " $x^1 = x$ changes
etc.

Under a L.T., $V^\mu \to \dfrac{dx^{\mu'}}{d\lambda} = \dfrac{d}{d\lambda}\left(\Lambda^{\mu'}{}_\nu x^\nu\right) = \Lambda^{\mu'}{}_\nu \dfrac{dx^\nu}{d\lambda} = \Lambda^{\mu'}{}_\nu V^\nu$

So $V^\mu \longrightarrow V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu$

components of V in primed frame

component of V in original frame.

How do basis vectors transform?

V itself doesn't depend on coordinates used to describe it, so:

$$V = V^\nu \hat{e}_\nu = V^{\mu'} \hat{e}_{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu \hat{e}_{\mu'} = V^\nu \underbrace{\left(\Lambda^{\mu'}{}_\nu \hat{e}_{\mu'}\right)}_{\text{must be } \hat{e}_\nu}$$

So $\hat{e}_\nu = \Lambda^{\mu'}{}_\nu \hat{e}_{\mu'}$, or rewrite as $\hat{e}_\mu = \Lambda^{\nu'}{}_\mu \hat{e}_{\nu'}$.

Define the inverse L.T. $(\Lambda^{-1})^\mu{}_{\nu'}$ ← note primed index lowered, last. by

$\Lambda^{\nu'}{}_\mu (\Lambda^{-1})^\mu{}_{\rho'} = \delta^{\nu'}_{\rho'}$ (In matrix notation, $\Lambda \Lambda^{-1} = I$)

$(\Lambda^{-1})^\mu{}_{\nu'} \Lambda^{\nu'}{}_\rho = \delta^\mu_\rho$. Then $(\Lambda^{-1})^\mu{}_{\rho'} \hat{e}_\mu = \underbrace{\Lambda^{\nu'}{}_\mu (\Lambda^{-1})^\mu{}_{\rho'}}_{\delta^{\nu'}_{\rho'}} \hat{e}_{\nu'} = \hat{e}_{\rho'}$

So $\boxed{\hat{e}_{\rho'} = (\Lambda^{-1})^\mu{}_{\rho'} \hat{e}_\mu}$ ← How basis vectors transform.

Note: Carroll writes this as $\Lambda_\mu{}^{\rho'}$ (see p. 18)

Vectors can be added, multiplied by real numbers $\rightarrow$ new vector.

If V, W are vectors, a, b are real numbers, then $(a+b)(V+W) = aV + aW + bV + bW$
$= \text{vector}.$

But $a + V$ makes no sense.

Dual vectors = linear maps from vectors $\rightarrow$ numbers. $\omega(V) = \text{number}$

dual vector ω, vector V

Basis for dual vectors: $\hat{\theta}^{\mu}$ ← labels which dual vector, always raised.

defined by $\boxed{\hat{\theta}^{\mu}(\hat{e}_{\nu}) = \delta^{\mu}_{\ \nu}.}$ Then $\omega(V) = \omega_{\mu}\hat{\theta}^{\mu}(V^{\nu}\hat{e}_{\nu}) = \omega_{\mu}V^{\nu}\underbrace{\hat{\theta}^{\mu}(\hat{e}_{\nu})}_{\delta^{\mu}_{\ \nu}}$

$= \omega_{\mu}V^{\mu}$

Can check that under a L.T: $\boxed{\omega_{\mu'} = (\Lambda^{-1})^{\nu}_{\ \mu'}\,\omega_{\nu}}$ ← How dual vector components transform

$\boxed{\hat{\theta}^{\nu'} = \Lambda^{\nu'}_{\ \mu}\,\hat{\theta}^{\mu}}$ ← How dual vector basis transform.

Example: 4-d gradient of a function $\phi(x^{\mu})$ is a dual vector.
(not a vector!)

$$\underbrace{d\phi}_{\text{dual vector}} = \underbrace{\frac{\partial\phi}{\partial x^{\mu}}}_{\text{components}}\,\hat{\theta}^{\mu}$$

$\hat{\theta}^{\mu}$ ← basis dual vectors

Check: $\dfrac{\partial\phi}{\partial x^{\mu'}} = \dfrac{\partial x^{\nu}}{\partial x^{\mu'}}\dfrac{\partial\phi}{\partial x^{\nu}}$ (Chain rule)

$= (\Lambda^{-1})^{\nu}_{\ \mu'}\dfrac{\partial\phi}{\partial x^{\nu}}$ ✓

Notation: $\partial_{\mu}\phi = \dfrac{\partial\phi}{\partial x^{\mu}}$ = components of a dual vector.

μ in ∂_μ: lowered; μ in x^μ: raised

For a path $x^{\mu}(\lambda)$: $\partial_{\mu}\phi\,\dfrac{\partial x^{\mu}}{\partial\lambda} = \dfrac{d\phi}{d\lambda}$ (Chain rule)

$\partial_\mu\phi$: dual vector comp. $\dfrac{\partial x^{\mu}}{\partial\lambda}$: tangent vector to path comp. $\dfrac{d\phi}{d\lambda}$: number

Call space of vectors at each point P: T_P (tangent space)

" " " dual " " " P: T_P^*

Tensor of rank (k, ℓ) = map

$$\underbrace{T_P^* \times \cdots \times T_P^*}_{k \text{ times}} \times \underbrace{T_P \times \cdots \times T_P}_{\ell \text{ times}} \longrightarrow \mathbb{R}$$

$\mathbb{R}$ ← real numbers

If S is a tensor, then:

$$S = \underbrace{S^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}}_{\text{components, usually work with these.}} \underbrace{\hat{e}_{\mu_1} \cdots \hat{e}_{\mu_k} \hat{\theta}^{\nu_1} \cdots \hat{\theta}^{\nu_\ell}}_{\text{basis for tensors of rank } (k,\ell)}$$

(order matters, height matters, but doesn't have to be as shown)

Under a L.T.:

$$S^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_\ell'} = \Lambda^{\mu_1'}{}_{\mu_1} \cdots \Lambda^{\mu_k'}{}_{\mu_k} (\Lambda^{-1})^{\nu_1}{}_{\nu_1'} \cdots (\Lambda^{-1})^{\nu_\ell}{}_{\nu_\ell'} S^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

Example: metric $\eta_{\mu\nu} = (0,2)$ tensor (components)

$$\eta(V,W) = \eta_{\mu\nu} V^\mu W^\nu = V \cdot W = \text{"dot product"}$$

Inverse metric $\eta^{\mu\nu} = (2,0)$ tensor (components) defined by

$$\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho = (1,1) \text{ tensor} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

Levi-Civita tensor

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma = \text{even permutation of } 0,1,2,3 \\ -1 & \text{if } \quad " \quad " \quad \text{odd} \quad " \quad " \quad " \\ 0 & \text{otherwise} \end{cases}$$

$= (0,4)$ tensor in SR (= "tensor density" in GR)

These examples all happen to have same components in <u>every</u> frame, and at <u>every</u> point.

EM field strength

$$F_{\mu\nu} = (0,2) \text{ tensor} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

This is antisymmetric: $F_{\nu\mu} = -F_{\mu\nu}$

Rules for tensors 1) Product of tensors with ranks (k_1, ℓ_1), (k_2, ℓ_2) is a tensor of rank $(k_1 + k_2, \ell_1 + \ell_2)$. (Just count raised/lowered indices)

2) If rank (k, ℓ) where $k, \ell \geq 1$, can <u>contract</u> indices to make $(k-1, \ell-1)$ tensor. Examples...

Start with $S^{\mu\nu\rho}{}_\sigma$ = rank $(3,1)$ tensor

$$X^{\mu\nu} = S^{\mu\nu\rho}{}_{\rho} \qquad Y^{\mu\nu} = S^{\mu\rho\nu}{}_{\rho} \qquad Z^{\mu\nu} = S^{\rho\mu\nu}{}_{\rho}$$

different in general.

$$A\cdot B = A^\mu B_\mu = A^\nu B_\nu$$ Repeated index = "contracted"

Can raise, lower indices using metric. Examples:

$$\eta^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}\eta_{\rho\sigma} \qquad \eta^{\mu\nu} = \eta^{\mu\rho}\delta_\rho^\nu \qquad \epsilon^{\mu\nu\rho\sigma} = \eta^{\mu\mu'}\eta^{\nu\nu'}\eta^{\rho\rho'}\eta^{\sigma\sigma'}\epsilon_{\mu'\nu'\rho'\sigma'}$$

Also:

$$\underbrace{S^{\mu}{}_{\nu}{}^{\kappa} = S^{\mu\rho\kappa}\eta_{\rho\nu} \qquad S_\rho{}^{\mu\nu} = \eta^{\mu\sigma}\eta^{\nu\kappa}S_{\rho\sigma\kappa} \qquad S_\mu{}^{\nu\kappa} = \eta_{\mu\sigma}\eta^{\nu\rho}S^{\sigma}{}_{\rho}{}^{\kappa}}_{\text{Traditional to use same letter for tensor related by } \eta\text{'s.}}$$

Can turn vectors $\longleftrightarrow$ dual vectors:

$$V_\mu = \eta_{\mu\nu}V^\nu \qquad V^\mu = \eta^{\mu\nu}V_\nu .$$

So

$$A\cdot B = A^\mu B_\mu = A^\mu \eta_{\mu\nu}B^\nu = A_\nu B^\nu = A_\mu B^\mu$$

Partial derivative of a (k,ℓ) tensor is a $(k,\ell+1)$ tensor (but ONLY in special relativity). So $\partial_\mu S^{\nu\rho} = T_\mu{}^{\nu\rho}$

Maxwell's Equations

$$J^\mu = (\underset{\text{charge}}{\rho}, \underset{\text{3-current}}{\vec{J}}) = \text{4-vector density}$$

Charge conservation: $\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{J} \implies \boxed{\partial_\mu J^\mu = 0}$

Potentials: $A^\mu = (\underset{\text{scalar}}{V}, \underset{\text{vector}}{\vec{A}}) = $ 4-vector

EM fields: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Gauge invariance: $\begin{cases} V \to V + \frac{\partial\lambda}{\partial t} \\ \vec{A} \to \vec{A} - \vec{\nabla}\lambda \end{cases} \iff A_\mu \to A_\mu + \partial_\mu\lambda$ (Note $A_\mu = \eta_{\mu\nu}A^\nu$

Maxwell's: $\partial_\mu F^{\mu\nu} = e A^\nu$ Gauss, Ampere

$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0$ (Faraday, no magnetic monopoles)

automatic if defined in terms of A_μ.