Reading assignment: sections 2.1 through 2.8 of Carroll.

Problem 1. Which of the following are possibly valid tensor equations? For the rest, state why they are not.

(a) \( T^\nu_{\mu\rho} = R^\rho_{\mu\nu} \)
(b) \( A_\mu B_{\nu\rho} = X^\mu_{\nu\rho} \)
(c) \( C^\mu_{\nu\rho} = Y^\mu_{\rho} \)
(d) \( T^\mu_{\nu\rho} + R^\rho_{\mu\nu} = Z^\nu_{\rho\mu} \)
(e) \( A_\mu (B^\nu_{\nu} + C^\nu_{\nu}) = P^\nu_{\mu} \)
(f) \( A^\mu_{\mu} B^\nu_{\nu} = A^\nu_{\nu} B^\mu_{\mu} \)

(I’m not telling you anything about the tensors involved, so it is impossible for you to say whether any of these equations are really true. I’m asking if they could be valid.)

Problem 2. Let \( A \) be a vector field on \( \mathbb{R}^2 \) whose components in the standard Cartesian basis are \( A^x = xy \) and \( A^y = x^2 + y^2 \). What are the components of \( A \) in the basis defined by polar coordinates?

Problem 3. Consider a spacetime with the line element:

\[
 ds^2 = -dt^2 + dx^2 + dy^2 + (1 + ax^2)dz^2 + bx(dxdz + dxdz),
\]

where \( a \) and \( b \) are some constants. Assume that the coordinates \((t, x, y, z)\) are good for the whole spacetime.

(a) Write down a list of the 16 components \( g_{\mu\nu} \) of the metric.
(b) What are the 16 components \( g^{\mu\nu} \) of the inverse metric?
(c) What is the determinant of the metric? Use it to find a condition on \( a, b \) that is necessary for the metric to have Lorentzian signature for all \( t, x, y, z \).
(d) Consider a new coordinate system \((t', x', y', z')\), where \( t' = t \) and \( x' = x \) and \( y' = y \) and \( z' = z + kx \), for some constant \( k \). Find the line element and the metric components in the primed coordinate system.