Physics 751 **Homework 1** **Due Wednesday, January 21, 2026**

Reading assignment: sections 1.1 through 1.9 of Carroll.

Problem 0. Read carefully, and acknowledge that you have read and understood, the statement in the syllabus for this course regarding the prohibition on use of AI and other outside sources to do homework problems.

Problem 1. Consider the parameterized path $x^\mu(\lambda)$ through spacetime, where $t = t_0\lambda$ and $x = v_0 t_0 \lambda(1-\lambda)$ and $y = z = 0$. The parameter λ runs from 0 to 1, and t_0 and v_0 are constants. In this problem, work in units in which $c = 1$. Suppose that this path is the worldline of a particle. (This problem is a smoothed out version of the twin "paradox". If you think of the particle as an observer carrying a tiny clock, then the proper time is the time read by the clock.)
(a) Draw a spacetime diagram showing the worldline of the particle for $0 \leq \lambda \leq 1$, in the x, t plane.
(b) What are the velocity and the acceleration of the particle as functions of the coordinate time t?
(c) Explain why the maximum possible value of v_0 is $c = 1$.
(d) Find the elapsed proper time $\Delta\tau$ along the path from $t = 0$ to $t = t_0$, in terms of t_0 and v_0.
(e) Using your answer to the previous part, make a graph of the numerical value of $\Delta\tau/t_0$, as a function of $0 \leq v_0 \leq 1$.
(f) What value of v_0 maximizes the proper time, and what is that proper time? What value of v_0 minimizes the proper time, and what is that proper time? (Hint: you should get an answer of the form $\Delta\tau = t_0\pi/n$, where n is a specific integer that you will find.)

Problem 2. Consider a Lorentz transformation that does a boost with velocity v along the z direction from one inertial frame with coordinates x^μ to another $x^{\mu'}$:

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix}$$

with $\gamma = 1/\sqrt{1-v^2}$.
(a) Suppose that at a given point in spacetime, there is a 4-vector current density $J^\mu = (\rho, J_x, J_y, J_z)$ measured in the original (unprimed) frame, where ρ is the charge density, and $\vec{J}$ is the 3-vector current density. Find the 4-vector current density $J^{\mu'}$ in the primed frame at that point. What is the charge density ρ' in that frame? What is the current

density $\vec{J}'$ in that frame?

(b) Find the inverse Lorentz transformation $(\Lambda^{-1})^{\mu}{}_{\nu'}$. [Hint: in Carroll's notation, as he explains on page 18, this is denoted $\Lambda^{\mu}{}_{\nu'}$. I keep the $^{-1}$ explicit, as I find it less confusing, even though it is more cumbersome.]

(c) Suppose that in the unprimed frame, there are electric and magnetic fields $\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$ and $\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$. Use the transformation properties of tensors and the form of the antisymmetric rank (0,2) electromagnetic field strength tensor $F_{\mu\nu}$ in eq. (1.69) to find $F_{\mu'\nu'}$ in the primed frame. Simplify your answer as much as possible (do not allow any γ^2 or v^2 to appear in the final result). What are the components of the electric and magnetic fields $E_{x'}, E_{y'}, E_{z'}, B_{x'}, B_{y'}, B_{z'}$ in the primed frame? (Note that these fields do *not* transform as components of 4-vectors, even though they are components of 3-vectors.)

(d) Now consider the symmetric rank $(2,0)$ stress-energy tensor $T^{\mu\nu}$ for a perfect fluid with mass density ρ and pressure P in the unprimed frame, as in eq. (1.111).
What is the stress-energy tensor $T^{\mu'\nu'}$ in the primed frame?

(e) An "equation of state" is a relationship that holds between ρ and P for a given type of fluid. Referring to part (d), what equation of state must hold for the stress-energy tensor of a perfect fluid to be invariant under boosts? The answer may appear strange, but it is actually sensible; it is a possibility mentioned in section 1.9.

(f) Consider the constant tensors $\eta_{\mu\nu}$ (the Minkowski metric), and $\eta^{\mu\nu}$ and δ^{μ}_{ν}. Use the rules for tensor transformations to find, by explicit calculation (rather than just writing down the answer), each of the corresponding tensors in the primed frame (that is, find $\eta_{\mu'\nu'}$ and $\eta^{\mu'\nu'}$ and $\delta^{\mu'}_{\nu'}$) and verify that they are unchanged.