Physics 751 **Homework 1** **Due Wednesday, January 23, 2019**

Reading assignment: sections 1.1 through 1.9 of Carroll.

Problem 1. Consider the parameterized path $x^\mu(\lambda)$ through spacetime, where $t = t_0\lambda$ and $x = v_0 t_0 \lambda(1-\lambda)$ and $y = z = 0$. The parameter λ runs from 0 to 1, and t_0 and v_0 are constants. In this problem, work in units in which $c = 1$. Suppose that this path is the worldline of a particle. (This problem is a smoothed out version of the twin "paradox". If you think of the particle as an observer carrying a tiny clock, then the proper time is the time read by the clock.)
(a) Draw a spacetime diagram showing the worldline of the particle for $0 \leq \lambda \leq 1$, in the x, t plane.
(b) What are the velocity and the acceleration of the particle as a function of the coordinate time t?
(c) Explain why the maximum possible value of v_0 is $c = 1$.
(d) Find the elapsed proper time $\Delta\tau$ along the path from $t = 0$ to $t = t_0$, in terms of t_0 and v_0.
(e) Using your answer to the previous part, make a graph of the numerical value of $\Delta\tau/t_0$, as a function of $0 \leq v_0 \leq 1$.
(f) What value of v_0 maximizes the proper time, and what is that proper time? What value of v_0 minimizes the proper time, and what is that proper time? (Hint: you should get an answer of the form $\Delta\tau = t_0\pi/n$, where n is a specific integer that you will find.)

Problem 2. Consider a Lorentz transformation that does a boost with velocity v along the z direction from one inertial frame with coordinates x^μ to another $x^{\mu'}$:

$$\Lambda^{\mu'}{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix}$$

with $\gamma = 1/\sqrt{1-v^2}$.
(a) Suppose that at a given point in spacetime, there is a 4-vector current density $J^\mu = (\rho, J_x, J_y, J_z)$ measured in the original (unprimed) frame, where ρ is the charge density, and $\vec{J}$ is the 3-vector current density. Find the 4-vector current density $J^{\mu'}$ in the primed frame at that point. What is the charge density ρ' in that frame? What is the current density $\vec{J}'$ in that frame?

(b) Find the inverse Lorentz transformation $(\Lambda^{-1})^{\mu}{}_{\nu'}$. [Hint: in Carroll's notation, as he explains on page 18, this is denoted $\Lambda^{\mu}{}_{\nu'}$. I keep the $^{-1}$ explicit, as I find it less confusing, even though it is more cumbersome.]

(c) Suppose that in the unprimed frame, there are electric and magnetic fields $\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$ and $\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$. Use the transformation properties of tensors and the form of the antisymmetric rank (0,2) electromagnetic field strength tensor $F_{\mu\nu}$ in eq. (1.69) to find $F_{\mu'\nu'}$ in the primed frame. Simplify your answer as much as possible (do not allow any γ^2 or v^2 to appear in the final result). What are the components of the electric and magnetic fields $E_{x'}, E_{y'}, E_{z'}, B_{x'}, B_{y'}, B_{z'}$ in the primed frame? (Note that these fields do *not* transform as components of 4-vectors, even though they are components of 3-vectors.)

(d) Now consider the symmetric rank $(2, 0)$ stress-energy tensor $T^{\mu\nu}$ for a perfect fluid with mass density ρ and pressure P in the unprimed frame, as in eq. (1.111).
What is the stress-energy tensor $T^{\mu'\nu'}$ in the primed frame?

(e) An "equation of state" is a relationship that holds between ρ and P for a given type of fluid. Referring to part (d), what equation of state must hold for the stress-energy tensor of a perfect fluid to be invariant under boosts? The answer may appear strange, but it is actually sensible; it is a possibility mentioned in section 1.9.

(f) Consider the constant tensors $\eta_{\mu\nu}$ (the Minkowski metric), and $\eta^{\mu\nu}$ and δ^{μ}_{ν}. Use the rules for tensor transformations to find, by explicit calculation (rather than just writing down the answer), each of the corresponding tensors in the primed frame (that is, find $\eta_{\mu'\nu'}$ and $\eta^{\mu'\nu'}$ and $\delta^{\mu'}_{\nu'}$) and verify that they are unchanged.