Physics 686 **Homework 2** **Due Wednesday, September 15, 2021**

Reading assignment: up through section 3.2 of the course notes.

Problem 1. Prove that any Lorentz transformation matrix $L^\mu{}_\nu$ satisfies $\det(L) = \pm 1$. [Hint: Recall that $\det(AB) = \det(A)\det(B)$ for any matrices A, B.]

Problem 2. Prove that the following statements are true, where the N_i are certain integers that you will determine. Show your work; don't just look up the answers and quote them. Put the results for parts (e), (f), (g) into simplest form. [Hint: you do not need the explicit forms of the gamma matrices at all, just eqs. (3.1.35)-(3.1.37) in the lecture notes and the fact that $\mathrm{Tr}(\gamma_\mu) = 0$.]

(a) $\gamma^\mu \gamma_\nu \gamma_\mu = N_1 \gamma_\nu$.

(b) $\gamma^\mu \gamma_\nu \gamma_\rho \gamma_\mu = N_2 g_{\nu\rho}$.

(c) $\gamma^\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu = N_3 \gamma_\sigma \gamma_\rho \gamma_\nu$.

(d) $\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = N_4 (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho})$.
[Hint: use the cyclic property of the trace: $\mathrm{Tr}(ABCD) = \mathrm{Tr}(BCDA)$.]

(e) $\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho) = ?$

(f) $\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\kappa) = ?$

(g) $\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\kappa \gamma_\lambda) = ?$

(h) $[\gamma_\rho, [\gamma_\mu, \gamma_\nu]] = N_5 (g_{\rho\mu} \gamma_\nu - g_{\rho\nu} \gamma_\mu)$.
[Hint: Write out the left side as four explicit terms, and then use eq. (3.1.37) in the lecture notes several times. You may see this identity again sometime soon.]

Problem 3. In this problem, we will check the Lorentz invariance of the Dirac equation, and in the process determine the Lorentz transformation rule for Dirac spinors. Suppose that two coordinate systems are related by a Lorentz transformation

$$x'^\mu = L^\mu{}_\nu x^\nu.$$

The wavefunction $\Psi'(x')$ as reported by an observer in the primed frame should be related to that in the unprimed frame by

$$\Psi'(x') = \Lambda\Psi(x)$$

where Λ is a 4×4 matrix. Now, the Dirac equation in the unprimed frame is

$$(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m)\Psi(x) = 0,$$

and in the primed frame it is

$$(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m)\Psi'(x') = 0.$$

(a) Show that these equations are consistent provided that

$$\Lambda^{-1}\gamma^\rho L_\rho{}^\mu \Lambda = \gamma^\mu.$$

(b) Now suppose that $L^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$ with $\omega^\mu{}_\nu$ infinitesimal, as in eq. (2.3.11) of the lecture notes. Prove that the equation found in part (a) is satisfied if

$$\Lambda = 1 + \frac{1}{8}\omega^{\mu\nu}[\gamma_\mu, \gamma_\nu] + \ldots$$

[Hints: Use a result you found in Problem 2. Note that if ϵ is an infinitesimal matrix, then $(1+\epsilon)^{-1} = 1 - \epsilon$. Also use the fact that $\omega_{\mu\nu}$ is antisymmetric.]

Problem 4. Consider a Lorentz transformation for which $x^\mu \to x'^\mu = L^\mu{}_\nu x^\nu$, where $L^\mu{}_\nu$ is constant.

(a) Suppose that T^μ_ν is a tensor. Use eq. (2.3.20) in the lecture notes to write down how it transforms under a Lorentz transformation. Then, from this, prove that its trace, T^μ_μ, transforms as a scalar.

(b) Let A^μ be the 4-vector potential for electromagnetism, and $\partial_\mu = \partial/\partial x^\mu$ and $\partial^\mu = \partial/\partial x_\mu$ and let J^μ be the corresponding current density, as in section 2.4. Write down, using eq. (2.3.20), how each of these objects transforms under a Lorentz transformation.

(c) Use the results of part (b) to derive how the field-strength tensor $F^{\mu\nu}$ transforms under the Lorentz transformation, but without further appealing to eq. (2.3.20). Write the result in the form

$$F^{\mu\nu} \quad\to\quad X^{\mu\nu}{}_{\rho\sigma}\, F^{\rho\sigma},$$

and then check that it does indeed agree with eq. (2.3.20).

(d) Suppose that Maxwell's equations (2.4.13) and (2.4.15) are satisfied in the original coordinate system. Prove, using the above, that they are still satisfied in the primed coordinate system resulting from the Lorentz transformation.