Problem 1. [10 points] Consider all of the $2 \to 2$ partonic processes in QCD:

- $qq \to qq$
- $qq' \to qq'$
- $\overline{q}q \to \overline{q}q$
- $\overline{q}q \to \overline{q}q'$
- $q\overline{q} \to gg$
- $q\overline{q} \to q\overline{g}$
- $\overline{q}g \to \overline{q}g$
- $qg \to qg$

where $q$ represents any fixed quark flavor, and $q'$ represents a quark flavor that is definitely different from $q$. For the purposes of this problem, we consider only massless partons (those very light compared to the partonic $\sqrt{s}$), and so do not consider processes involving top quarks or antiquarks.

Several of the processes in this list actually have the same tree-level differential cross-sections:

\[ \frac{d\hat{\sigma}(qq' \to qq')}{dt} = \frac{d\hat{\sigma}(\overline{q}q' \to \overline{q}q)}{dt} = \frac{d\hat{\sigma}(qq' \to q\overline{q})}{dt}, \]
\[ \frac{d\hat{\sigma}(qq \to qq)}{dt} = \frac{d\hat{\sigma}(\overline{q}q \to \overline{q}q)}{dt}, \]
\[ \frac{d\hat{\sigma}(gg \to gg)}{dt} = \frac{d\hat{\sigma}(g\overline{g} \to g\overline{g})}{dt}. \]

So, there are really only 8 independent parton-level cross-sections in terms of which one can express the leading-order two-jet production cross-section for hadron colliders including the Tevatron and the LHC. Assume they are all known.

(a) Find an integral expression for the total Tevatron cross section for dijet production in $p\overline{p}$ collisions at leading order in QCD in terms of the 8 independent parton-level total cross sections and the PDFs $g(x), u(x), d(x), \overline{u}(x), \overline{d}(x), s(x)$. (Neglect charm and bottom PDFs; they are small.) I’ll start by including the contributions for three of the 8 independent parton-level cross-sections, and you fill in the contributions for the other 5:

\[
\sigma(p\overline{p} \to jj + X) = \int_0^{\sqrt{s}} d\hat{s} \int_0^1 dx \frac{1}{xs} \left\{ g(x)g(\hat{s}/sx) \left[ \hat{\sigma}(gg \to gg) + \sum_q \hat{\sigma}(gg \to q\overline{q}) \right] \\
+ [u(x)\overline{u}(\hat{s}/sx) + d(x)\overline{d}(\hat{s}/sx) + s(x)s(\hat{s}/sx)] 2\hat{\sigma}(qq \to qq) \\
+ ? \right\}
\]

Here $\sqrt{s}$ is the proton-antiproton collision energy in their center-of-momentum frame, and $\sqrt{\hat{s}}$ is the parton-parton collision energy in their center-of-momentum frame.

(b) Do the same for $pp$ collisions relevant for the LHC. (The answer is different.)
Problem 2. [20 points] Consider the parton-level process involving scattering of a quark with its antiquark: $q\bar{q} \rightarrow q\bar{q}$. (Note that this is one of the partonic processes that appeared in the previous problem.) We'll be calculating this to leading order in QCD, which means that you should only include diagrams with gluon exchange; photon exchange is relatively negligible. You will probably find section 5.2.4 and section 9.2.1 of the notes to be useful references. In fact, the “non-color part” of the calculation in this problem is essentially identical to the calculation of Bhabha scattering in QED, so large chunks of the calculation in the notes can be simply adopted to the present problem. Therefore, the hard part is getting the color part correct; watch out for the fact that the color factor for the interference term is different from the color factor for the non-interference terms in part (b). Treat the quark $q$ as massless.

(a) Draw the two Feynman diagrams, label them appropriately, and make a corresponding list of the initial and final state partons with their momenta, spin, spinor, and color index. Write down the reduced matrix element, using the QCD Feynman rules on page 199 and working in Feynman gauge ($\xi = 1$).

(b) Compute the reduced matrix element squared, summed (averaged) over final (initial) spins and colors. You should find:

$$|M|^2 = g_3^4 \left( \frac{4 s^2 + \hat{u}^2}{9 \frac{t^2}{s^2}} + \frac{4 \hat{t}^2 + \hat{u}^2}{9 s^2} - n^3 \frac{\hat{u}^2}{s^2} \right)$$

where $n$ is a certain positive rational number that you will find, and $\hat{s}$, $\hat{t}$, $\hat{u}$ are the partonic Mandelstam variables.

[Hint: You will need to compute Tr$[T^a T^b T^a T^b]$, with the adjoint indices $a$ and $b$ implicitly summed over. This can be done by using equations (8.1.16), (8.1.59), (8.1.61), and (8.1.67) in the notes.]

(c) Find the parton-level differential cross-section $\frac{d\hat{\sigma}(q\bar{q} \rightarrow q\bar{q})}{d\hat{t}}$.

Problem 3. [10 points] Use crossing symmetry to find some others of the $2 \rightarrow 2$ partonic differential cross-sections mentioned in Problem 1:

(a) Using the results of part (b) of the previous problem, find $\frac{d\hat{\sigma}(qq \rightarrow qq)}{d\hat{t}}$.

(b) Using the result for $qq' \rightarrow qq'$ in equation (9.2.16) of the notes, find both $\frac{d\hat{\sigma}(q\bar{q} \rightarrow q\bar{q})}{d\hat{t}}$ and $\frac{d\hat{\sigma}(q\bar{q} \rightarrow q'\bar{q}')}{{d\hat{t}}}$. 

[Hint: don’t worry, nothing weird happens with the color factors in this problem.]