PHYS 661 **Homework Set 1** **Due Thursday, January 23, 2025**

Reading assignment: review the course textbook as needed to answer the following problems.

Problem 1. Consider a quantum system consisting of two independent spin-1/2 operators, $\vec{S}_1$ and $\vec{S}_2$, so that the state space is spanned by orthonormal S_{1z} and S_{2z} eigenstates $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$. Here the first label is $\uparrow$ for $S_{1z} = +\hbar/2$ and $\downarrow$ for $S_{1z} = -\hbar/2$, and the second label is $\uparrow$ for $S_{2z} = +\hbar/2$ and $\downarrow$ for $S_{2z} = -\hbar/2$. At time $t = 0$, the system happens to be in the state:

$$|\psi(0)\rangle = a\Big(|\uparrow\uparrow\rangle + 2\,|\uparrow\downarrow\rangle + 3\,|\downarrow\uparrow\rangle + 4\,|\downarrow\downarrow\rangle\Big),$$

where a is a constant that should not appear in your answers.

(a) At time $t = 0$, we measure the z-component of spin 1, S_{1z}. What is the probability of finding $+\hbar/2$? What is the state immediately after this measurement?

(b) Suppose the measurement mentioned in the previous part has happened, and the result was indeed $+\hbar/2$. If we then measure the x-component of spin 2, S_{2x}, what results can be found, and with what probabilities? (Hint: you will need to find the eigenvalues and corresponding eigenstates of S_{2x}.)

(c) Instead of performing the above measurements, we let the system evolve under the influence of the Hamiltonian $H = \omega S_{1z}$. Find the state of the system at time t. Use this to find the expectation value of the x-component of spin 1, $\langle S_{1x}\rangle$, as a function of time.

Problem 2. A spinless particle is in a stationary state of an unknown, but spherically symmetric, potential. Its wavefunction in spherical coordinates is known to be:

$$\psi(r,\theta,\phi) = c\,e^{-r/a}\sin(\theta)\cos(\phi).$$

where a and c are positive real constants.

(a) By requiring the wavefunction to have norm equal to 1, find c in terms of a.

(b) If L^2 and L_z are simultaneously measured, what are the possible pairs of results, and their probabilities?

(c) If the energy eigenvalue of the state is 0, what is the potential $V(r)$?

Problem 3.. Two non-identical particles have spins 1/2 and 3/2. Each of them is in the same spatial state with orbital angular momentum quantum number $l = 1$. If $\vec{J}$ is the total angular momentum, what are the allowed eigenvalues of J^2, and their degeneracies? (Hint: start by working out the grand total number of states, so that you will have something to check against. It should be a number greater than 50 and less than 100.)

Problem 4. Consider the effects of the hyperfine splitting of the ground state of the Hydrogen atom in the presence of a constant uniform external magnetic field $\vec{B} = B\hat{z}$. Since we will only be working with the ground state, you can ignore the orbital angular momentum $\vec{L}$. Let the electron spin operator be $\vec{S}$ and the proton spin operator be $\vec{I}$, and let their total be $\vec{J} = \vec{S} + \vec{I}$. (Recall that the electron and the proton both are spin 1/2 particles.) Then the Hamiltonian for the system is:

$$H = \frac{E_\gamma}{\hbar^2}\vec{S}\cdot\vec{I} + 2\frac{\mu}{\hbar}\vec{B}\cdot\vec{S},$$

where μ is a constant and E_γ is the energy of the famous 21.4 cm line. (Give your answers below in symbolic form in terms of μ and E_γ, not their numerical values.)

(a) Find the matrix representation of the Hamiltonian, in the product orthobasis for the two spins labeled by the eigenvalues of L_z and I_z. (Hint: start by evaluating each of the two operators $\vec{S}\cdot\vec{I}$ and $\vec{B}\cdot\vec{S}$ acting on each of the product orthobasis states.)

(b) In the limit that B is so large that E_γ can be neglected, find the energy eigenvalues, and the corresponding energy eigenstates in ket notation, again using the product orthobasis.

(c) In the limit that B is so small that it can be neglected, find the energy eigenvalues, and the corresponding energy eigenstates in ket notation, again using the product orthobasis.

(d) Redo part (c), but this time find the energy eigenstates in the total angular momentum orthobasis labeled by the eigenvalues of J^2 and J_z. (Hint: look up "dot product of angular momenta trick" in the index.)

(e) Find the energy eigenvalues for general B, and expand them to show that the special limits you obtained in parts (b) and (c) follow. (You do not need to find the energy eigenstates.)