Physics 661 **Homework 8** **Due Friday, April 12, 2024**

Reading assignment: sections 18.4 through 19.6 of the text.

Problem 1. Consider a particle of mass m bound in a simple harmonic oscillator of natural frequency ω in one dimension. Initially (for $t < 0$), it is in the ground state. At $t = 0$, a perturbation to the Hamiltonian is turned on, of the form:

$$W(X,t) = kX^2 e^{-t/\tau} \qquad (t \geq 0)$$

where k and τ are constant numbers.

(a) Using first-order time-dependent perturbation theory, calculate the probabilities that, after a very long time ($t \gg \tau$), the system will have made a transition to each of the excited states.

(b) Using your answers for part (a), infer the probability that after a long time the system will remain in the ground state.

(c) Now use second-order time-dependent perturbation theory to directly calculate the probability that the system will remain in the ground state. Check that your answer agrees with the result found in part (b).

Problem 2. A hydrogen atom in its ground state $[(n, \ell, m) = (1, 0, 0)]$ is placed between the plates of a capacitor. At times $t < 0$, there is no electric field due to the capacitor, but for later times the capacitor produces a field:

$$\vec{E} = E_0 \hat{z} e^{-t/\tau}.$$

Using first-order time-dependent perturbation theory, compute the probability for the atom to be found at $t \gg \tau$ in each of the 1st excited states $[(n, \ell, m) = (2, 0, 0)$ and $(2, 1, 1)$ and $(2, 1, 0)$ and $(2, 1, -1)]$.

Problem 3. In this problem you will consider a 2-state system subject to a harmonic perturbation, both exactly and in the approximation of time-dependent perturbation theory.

Consider an orthobasis of states $|1\rangle$ and $|2\rangle$ that are eigenstates of an unperturbed Hamiltonian H_0 with eigenvalues E_1 and E_2. This means that

$$H_0 \quad = \quad E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2|.$$

Assume, without loss of generality, that $E_1 < E_2$. There is also a time-dependent potential that connects the two levels, with matrix elements:

$$W_{11} = W_{22} = 0, \qquad\qquad W_{12} = W_{21}^* = \hbar\gamma e^{i\omega t},$$

where γ is a real number, so that

$$W(t) = \hbar\gamma e^{i\omega t}|1\rangle\langle 2| + \hbar\gamma e^{-i\omega t}|2\rangle\langle 1|.$$

Thus ω is the driving frequency of the perturbation, and the total Hamiltonian is

$$H = H_0 + W(t).$$

Let us write the normalized state at time t as

$$|\psi(t)\rangle \;=\; c_1(t)e^{-itE_1/\hbar}|1\rangle + c_2(t)e^{-itE_2/\hbar}|2\rangle.$$

Suppose that at time $t = 0$, the system is known to be in the lower level, so that $c_1(0) = 1$ and $c_2(0) = 0$. Our goal is to calculate the rate of transition to the higher level state $|2\rangle$.

(a) Consider the *exact* differential equations for the coefficients $c_1(t)$ and $c_2(t)$. Starting from the Schrodinger equation, show that they can be written as

$$\begin{aligned} i\hbar\frac{dc_1}{dt} &= W_{12}e^{-i\omega_{21}t}c_2, \\ i\hbar\frac{dc_2}{dt} &= W_{21}e^{i\omega_{21}t}c_1. \end{aligned}$$

where $\omega_{21} = (E_2 - E_1)/\hbar$.
(b) Solve these equations for $c_1(t)$ and $c_2(t)$. Do this by assuming a solution of the form:

$$\begin{aligned} c_1(t) &= e^{i(\omega-\omega_{21})t/2}[\cos(\Omega t) - b_1\sin(\Omega t)]; \\ c_2(t) &= e^{-i(\omega-\omega_{21})t/2}b_2\sin(\Omega t), \end{aligned}$$

where we define

$$\Omega^2 = \gamma^2 + (\omega - \omega_{21})^2/4,$$

and b_1 and b_2 are constant quantities that you will determine in terms of ω, γ, and ω_{21}. [Hint: plug $c_1(t)$ and $c_2(t)$ into the second differential equation first. Since this equation is supposed to be satisfied for all t, you can consider separately the parts of the equation that are proportional to $\cos(\Omega t)$ and $\sin(\Omega t)$.]

(c) From your results in the previous part, find the transition probability,

$$\mathcal{P}_{1\to 2}(t) = |c_2(t)|^2.$$

What is its maximum value if $\omega = \omega_{21}$? Sketch $\mathcal{P}_{1\to 2}(t)$ for this case, for $0 \leq t \leq 2\pi/\gamma$, with labeled axes.

(d) What is the maximum value of $\mathcal{P}_{1\to 2}(t)$ if $|\omega - \omega_{21}| = \gamma$? What is its maximum value if $|\omega - \omega_{21}| = 10\gamma$? What is its maximum value if $|\omega - \omega_{21}| = 100\gamma$?

(e) Now using the approximation of first order time-dependent perturbation theory, find the transition amplitude $a^{(1)}_{1\to 2}(t)$ and the corresponding probability $\mathcal{P}_{1\to 2}(t) = |a^{(1)}_{1\to 2}(t)|^2$. Do the two approaches agree for small values of γ?

(f) When ω is very close to ω_{21}, show that the perturbative answer you got in part (d) can be interpreted in terms of a rate:

$$\Gamma(1 \to 2) \quad \approx \quad N\gamma^2\delta(\omega - \omega_{21}),$$

where N is a certain number that you will find. [Hint: note the relationship between eqs. (18.3.9) and (18.3.20) of the text.] To apply this formula in practice, you would integrate it over a spectrum of driving frequencies ω.