PHYS 660 **Homework 9** **Due Thursday, November 21, 2024**

Reading assignment: sections 8.5-8.7 and 9.1-9.5 of the text.

Problem 1. Consider a particle in a state with wavefunction $\psi = N(2x - z)e^{-\alpha r}$, where x, y, z are rectangular coordinates, r is the spherical radial coordinate, α is a positive constant, and N is a normalization constant which you need not compute.
(a) Write the wavefunction in terms of spherical harmonics $Y_l^m(\theta, \phi)$.
(b) If L_z is measured, what are the possible results, and their probabilities?

Problem 2. Consider a particle with spin 1/2, and let $\hat{n} = \hat{x}\sin\beta + \hat{z}\cos\beta$ be a fixed unit vector, where β is a fixed angle.
(a) Consider the basis of eigenstates of S_z, denoted $|\uparrow\rangle$ and $|\downarrow\rangle$ for eigenvalues $+\hbar/2$ and $-\hbar/2$ respectively. In that basis, construct the matrix representations of S_x, S_y, S_z, S^2, and $\hat{n} \cdot \vec{S}$.
(b) Suppose that we start with the spin in the state $|\uparrow\rangle$. What is the probability that the measurement of $\hat{n} \cdot \vec{S}$ yields the result $+\hbar/2$?
(c) Suppose that the measurement in part (b) has been carried out and the result for $\hat{n} \cdot \vec{S}$ was indeed $+\hbar/2$. Immediately afterwards, S_z is measured. What is the probability that the measurement yields $-\hbar/2$?
(d) Check that your results for parts (b) and (c) make sense when $\beta = 0$ and π. What happens when $\beta = \pi/2$?

Problem 3. Consider a spinless particle in a state with L^2 eigenvalue $2\hbar^2$ and L_z eigenvalue $\hbar$. As in the previous problem, let $\hat{n} = \hat{x}\sin\beta + \hat{z}\cos\beta$.
(a) Suppose that the angular momentum along the direction $\hat{n}$ is measured. What are the possible results, and their probabilities?
(b) For each of the possible results in part (a), suppose that L_z is then measured. What are the possible results, and their probabilities?
(c) Check that your results make sense when $\beta = 0$ and π and $\pi/2$.

Problem 4. Consider the isotropic 3-d harmonic oscillator problem, with potential $V(x, y, z) = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2)$. As discussed in class, the Hamiltonian H can be written as the sum of $H_x = \hbar\omega(a_x^\dagger a_x + 1/2)$, $H_y = \hbar\omega(a_y^\dagger a_y + 1/2)$, and $H_z = \hbar\omega(a_z^\dagger a_z + 1/2)$, which form a C.S.C.O. with corresponding orthonormal eigenbasis $|n_x, n_y, n_z\rangle$. Another choice of C.S.C.O. is H, L^2, and L_z, with corresponding eigenbasis $|n, \ell, m\rangle'$, where $n = n_x + n_y + n_z$. (The $'$ is just to distinguish the two types of orthobasis elements, since they both have three integer labels and therefore could be confused if we aren't careful.)
(a) Construct the operators L^2 and L_z in terms of the creation and annihilation operators.

You should find:

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\begin{aligned}
L^2 &= \hbar^2\big[N_1(a_x^{\dagger 2}a_y^2 + a_x^{\dagger 2}a_z^2 + a_y^{\dagger 2}a_x^2 + a_y^{\dagger 2}a_z^2 + a_z^{\dagger 2}a_x^2 + a_z^{\dagger 2}a_y^2) \\
&\quad +N_2(a_x^\dagger a_y^\dagger a_x a_y + a_x^\dagger a_z^\dagger a_x a_z + a_y^\dagger a_z^\dagger a_y a_z) \\
&\quad +N_3(a_x^\dagger a_x + a_y^\dagger a_y + a_z^\dagger a_z)\big]
\end{aligned}
$$

where N_1, N_2, and N_3 are certain integers that you will discover. Note that this result is in "normal-ordered" form, which means that the commutation relations have been used to ensure that no creation operator appears to the right of an annihilation operator.

(b) For the subspace of states with $n = 2$, find the action of L^2 on each of the $|n_x, n_y, n_z\rangle$ basis. Using these results, and using the ordering $|2,0,0\rangle$, $|0,2,0\rangle$, $|0,0,2\rangle$, $|1,1,0\rangle$, $|1,0,1\rangle$, $|0,1,1\rangle$, find the corresponding 6×6 matrix representation for L^2. Find the eigenvalues and normalized eigenvectors of L^2 for the $n = 2$ subspace in that basis.

(c) Compute the action of L_z on each of the simultaneous eigenvectors of H, L^2 found in the previous part. Within each sub-subspace of fixed $n = 2$ and fixed ℓ, find the eigenvalues and eigenvectors of L_z, and so conclude by writing the six $|2, \ell, m\rangle'$ orthobasis states as linear combinations of the six $|n_x, n_y, n_z\rangle$ eigenstates.