PHYS 660 Homework 8 Due 10:00 AM on Wednesday, November 6, 2024

Reading assignment: sections 8.1-8.4 of the class notes (and review sections 7.1-7.5).

Problem 1. Consider a quantum particle of mass m in a 1-d harmonic oscillator potential with angular frequency ω, with minimum at the origin, as usual. For the observable $A = PX + XP$, find:

(a) The expression in simplest form for A in terms of the usual ladder operators.

(b) The expectation value for A for the nth excited energy eigenstate $|n\rangle$.

(c) The uncertainty for A in the state $|n\rangle$.

Problem 2. Two particles labeled 1 and 2 (they are distinguishable, but happen to have the same mass) are governed by a coupled harmonic oscillator Hamiltonian in one dimension:

$$H = \frac{1}{2m}(P_1^2 + P_2^2) + \frac{1}{2}m\omega^2 X_1^2 + \frac{1}{2}m\omega^2 X_2^2 + \frac{1}{2}m\Omega^2(X_1 - X_2)^2.$$

Here ω and Ω are constants with dimensions of angular frequency; note that ω parametrizes the restoring force for the particles to the origin, and Ω describes the attractive force between the particles. The operators X_1, P_1 and X_2, P_2 satisfy the canonical commutation relations $[X_1, P_1] = i\hbar$ and $[X_2, P_2] = i\hbar$ and $[X_1, P_2] = [X_2, P_1] = 0$ and $[X_j, X_k] = 0$ and $[P_j, P_k] = 0$ for all $j, k = 1, 2$. Consider the new coordinate operators:

$$U = \frac{1}{\sqrt{2}}(X_1 + X_2), \qquad V = \frac{1}{\sqrt{2}}(X_1 - X_2),$$

and momenta:

$$P_u = \frac{1}{\sqrt{2}}(P_1 + P_2), \qquad P_v = \frac{1}{\sqrt{2}}(P_1 - P_2)$$

(a) Derive all of the commutation relations of all pairs of operators from the set U, V, P_u, P_v.

(b) Write the Hamiltonian in terms of the operators U, V, P_u, P_v.

(c) Define appropriate creation and destruction operators for the U, V, P_u, P_v system, so that the Hamiltonian has a simple form in terms of them. From this, infer the eigenvalues of the Hamiltonian, and write down a suitable notation for the energy eigenbasis kets.

(d) Obtain the wavefunction of the ground state in the u, v representation $\psi(u, v)$, and use it to obtain the normalized ground state wavefunction in the x_1, x_2 representation, $\psi(x_1, x_2) = \langle x_1, x_2|\psi\rangle$.

(e) Is the Hamiltonian invariant under translations? What about parity?

Problem 3. Consider the isotropic harmonic oscillator for a single particle in 2-d:

$$H = \frac{1}{2m}(P_x^2 + P_y^2) + \frac{1}{2}m\omega^2(X^2 + Y^2).$$

(a) Write down the energy eigenvalue expression for the basis kets $|n_x, n_y\rangle$, where n_x and n_y are the energy level quantum numbers for the corresponding 1-dimensional harmonic oscillator problem. Find a general expression for the degeneracy of the nth excited energy state.

(b) Does the state $|n_x, n_y\rangle$ have a well-defined parity? If so, what is it? If not, why not?

(c) Consider the angular momentum operator $L_z = XP_y - YP_x$. Write it, as well as H, in simplest form in terms of the annihilation and creation operators a_x, $a_x^\dagger$, a_y, $a_y^\dagger$.

(d) Use your results from part (c) to compute the commutator $[L_z, H]$. Are they compatible operators?

(e) What is L_z acting on the ground state $|0, 0\rangle$?

(f) For the states with $n = n_x + n_y = 1$, compute L_z acting on each of the basis states from part (a). Use this to write a 2×2 matrix representation for the operator L_z. Find its eigenvalues and the corresponding eigenstates.

(g) For the states with $n = 2$, compute L_z acting on each of the basis states from part (a), and a matrix representation for the operator L_z. Find its eigenvalues and the corresponding eigenstates.

(h) For the states with $n = 3$, compute L_z acting on each of the basis states from part (a), and a matrix representation for the operator L_z. Find its eigenvalues; you do **not** need to find the eigenstates.

(i) Based on your experience from parts (e)-(h), do H, L_z form a C.S.C.O. (complete set of commuting observables)?

(j) Also based on your experience from parts (e)-(h), make a guess what the eigenvalues of L_z are for general n.

Problem 4. Consider a 1-d quantum harmonic oscillator with the potential minimum at $x = 0$ and with mass m and angular frequency ω. The system is prepared in a state $|\psi\rangle$ that has position wavefunction

$$\psi(x) \quad = \quad A\exp(-x^2/2L^2)$$

where L is an arbitrary length, and A is another constant. (Note that this wavefunction has a Gaussian form, like the ground state, but it is **not** the ground state.)

(a) If the energy is measured, what is the probability that the result is the lowest possible value? (Your answer should be a non-trivial function of m, ω, $\hbar$, and L.)

(b) What is the probability to find the system in the first excited energy state $|1\rangle$?