

Reading assignment: sections 7.4, 7.5, and 5.3 of the class text.

Problem 1. Consider a spin-less particle of mass m , moving in three dimensions, with the Hamiltonian given by:

$$H = \frac{1}{2m}(P_x^2 + P_y^2 + P_z^2) + \frac{1}{2}m\omega^2 \left(\frac{5}{2}X^2 + \frac{5}{2}Y^2 + 3XY + Z^2 \right).$$

(a) Use the change of coordinates:

$$\begin{aligned} x &= cx' + sy' \\ y &= -sx' + cy' \\ z &= z', \end{aligned}$$

where c and s are the cosine and sine of an arbitrary rotation angle, to rewrite the Hamiltonian in terms of the operators X' , Y' , Z' , $P_{x'}$, $P_{y'}$, and $P_{z'}$. (You may use the fact that this is an orthogonal rotation on the coordinate system, so $P_x^2 + P_y^2 + P_z^2 = P_{x'}^2 + P_{y'}^2 + P_{z'}^2$.)

(b) Choose specific values for c and s (remembering that $c^2 + s^2 = 1$), so that H will not contain a term proportional to $X'Y'$. Rewrite the Hamiltonian with this choice. (There is more than one valid choice here.)

(c) What are the three smallest energy eigenvalues of H , and what are their degeneracies?

For all of the remaining problems, consider the 1-dimensional harmonic oscillator with mass m and frequency ω .

Problem 2. At time $t = 0$, the oscillator is in the state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ in the usual energy basis.

(a) Find the state at later times, $|\psi(t)\rangle$, in the energy basis.

(b) By direct calculation using the results of part (a) (without appealing to Ehrenfest's Theorem), find the expectation values $\langle X \rangle$ and $\langle P \rangle$ as functions of time t .

(c) Check that your results from part (b) obey Ehrenfest's Theorem.

(d) At time t , compute the probability that a measurement of the position yields $x > 0$. Evaluate the minimum and the maximum probabilities numerically, and check that they are between 0 and 1.

Problem 3. At time $t = 0$, suppose the oscillator has wavefunction

$$\psi(x, 0) = Cx^2 \exp(-m\omega x^2/2\hbar)$$

where C is a positive real constant.

- (a) By requiring the wavefunction to be normalized, find the constant C .
- (b) If the energy is measured, what are the possible outcomes and their probabilities?
- (c) Find the state ket $|\psi(t)\rangle$ in the energy basis, as a function of time.
- (d) Compute the expectation values $\langle X \rangle$, $\langle P \rangle$, $\langle X^2 \rangle$, and $\langle P^2 \rangle$, and the uncertainties ΔX and ΔP , each as a function of time.
- (e) At time t , what is the probability that a measurement of the position yields $x > 0$?

Problem 4. The coherent states $|\alpha\rangle$ are the eigenstates of the annihilation operator a with complex eigenvalue α , as discussed in class.

- (a) For two different coherent states $|\alpha\rangle$ and $|\beta\rangle$, compute $\langle \beta | \alpha \rangle$ and $|\langle \beta | \alpha \rangle|^2$ in simplest form. (Your answers should be exponentials. Therefore, they cannot vanish, which shows that the set of states $|\alpha\rangle$ do *not* satisfy orthogonality.)
- (b) Suppose that the oscillator is in the state $|\alpha\rangle$ at time $t = 0$. Find the probability $P(t)$ to find it again in the state $|\alpha\rangle$ at a later time t . [Make use of eqs. (7.4.31)-(7.4.33) of the notes, and your result from part (a).] Put your answer in simplest form, and check that it obeys the sanity condition $P(0) = 1$ at time $t = 0$.
- (c) Suppose that $|\alpha| \gg 1$, as is true for a macroscopic oscillator. For what very small length of time after $t = 0$ does the probability $P(t)$ remain greater than 0.5? [Hint: expand $\cos(\omega t)$ to second order in t , for small t .]
- (d) Are there any other time or times at which $P(t) = 1$? If so, when? If not, why not?
- (e) What is the minimum value of $P(t)$, and at what time or times is it achieved?
- (f) Estimate your answers to parts (c) and (e) numerically for a macroscopic oscillator with $m = 0.2$ kg, $\omega = 10 \text{ sec}^{-1}$, and $x_0 = 0.1$ meters.