PHYS 660 **Homework 4** **Due Friday, September 27, 2024**

Reading assignment: sections 2.9, 3.3, and 6.1-6.3 of the class notes.

Problem 1. Consider a particle free to move in 1-d everywhere along the line $-\infty < x < \infty$. Suppose that its wavefunction is:

$$\psi(x) = \begin{cases} C\cos(\pi x/2L) & \text{(for } |x| \leq L\text{)}, \\ 0 & \text{(for } |x| \geq L\text{)}. \end{cases}$$

(a) If the wavefunction is normalized to unity, what is the magnitude of the constant C? Sketch the wavefunction, as a function of x.
(b) If the position of the particle is measured, what is the probability that it is found to have $|x| > L/6$? (Sanity check: is your answer unitless, and between 0 and 1?)
(c) What are the expectation value and the uncertainty of X in this state?
(d) Find the normalized momentum wavefunction $\widetilde{\psi}(p)$. Sketch it as as function of p.
(e) If the momentum of the particle is measured, what is the probability that it is found to have $|p| > p_0$? You may leave your answer in terms of a definite integral. Is there a maximum value that the result of the momentum measurement can be?
(f) Is the state an eigenstate of the Hamiltonian, $H = P^2/2m$? If so, what is its energy eigenvalue? If not, why not?

Problem 2. Consider a particle free to move throughout all space in 3-d. Suppose that its wavefunction in spherical coordinates is $\psi(\vec{r}) = Ce^{-r^2/a^2}$, where C and a are constants.
(a) If the wavefunction is normalized to unity, what is the magnitude of the constant C?
(b) If the position of the particle is measured, what is the probability that it will be found to be further from the origin than a? You may leave your answer as a definite integral, although it is possible to evaluate it numerically.
(c) What is the expectation value of the radial coordinate operator R?

Problem 3. Consider a Hilbert space with dimension 2, which has an orthobasis of eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$, which are defined to be eigenstates of an observable S_z, with eigenvalues $\hbar/2$ and $-\hbar/2$, respectively. There are also observables S_x and S_y, with matrix representations:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(This describes a spin-1/2 system.) Suppose the system is in the state described by the ket

$$a\,|\uparrow\rangle + b\,|\downarrow\rangle\,,$$

where a and b are non-zero complex numbers. For each of the following observable operators:

$$S_x, \quad S_y, \quad S_z, \quad S_z^2, \quad \text{and} \quad (S_x^2 + S_y^2 + S_z^2),$$

compute the expectation value, the uncertainty, the allowed results of a measurement, and the probabilities of each of those allowed measurement results. Make sure that your answers are in simplified form, and satisfy any sanity checks that you can think of.

Problem 4. Consider a particle of mass m moving in a 1-d potential, so that its Hamiltonian is

$$H = \frac{P^2}{2m} + V(X).$$

Here $V(X)$ is an arbitrary function of X, and the position and momentum operators satisfy $[X, P] = i\hbar$. Suppose that the eigenvectors of H are denoted by $|\psi_n\rangle$, where n is a discrete label, so that

$$H\,|\psi_n\rangle = E_n\,|\psi_n\rangle\,,$$

where E_n are the energy eigenvalues.

(a) Show that

$$\langle\psi_n|P|\psi_k\rangle = \alpha\,\langle\psi_n|X|\psi_k\rangle\,,$$

where α is a quantity that you will determine, which depends on the difference between E_n and E_k. (Hint: consider the commutator $[X, H]$.)

(b) From the result in part (a), show that the following rule is true:

$$\langle\psi_n|P^2|\psi_n\rangle = \beta\sum_k (E_n - E_k)^2 |\,\langle\psi_n|X|\psi_k\rangle\,|^2$$

where β is a constant quantity that you will find. (Hint: completeness is your friend.)