PHYS 660 **Homework 3** **Due Wednesday, September 18, 2024**

Reading assignment: sections 2.7-2.8 and 3.1-3.2 of the class notes.

Problem 1. Consider Hermitian operators A_x, A_y, A_z with the following matrix representations on a state space with dimension 3:

$$A_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad A_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Note that these matrices are representations of the operators in an orthonormal basis $|1\rangle$, $|0\rangle$, $|-1\rangle$, which are labeled by the non-degenerate eigenvalues of A_z, and which we can therefore call "the A_z basis".

(a) Find the eigenvalues and the corresponding normalized eigenkets of A_x (in the A_z basis). Do the same for A_y.

(b) Consider the state in which $A_z = +1$. In this state, calculate the expectation values $\langle A_y \rangle$ and $\langle A_y^2 \rangle$ and the uncertainty ΔA_y.

(c) If the system is in the state with $A_z = -1$, and A_y is measured, what are the possible outcomes and their probabilities?

(d) If the system is in the state with $A_y = 0$, and A_z is measured, what are the possible outcomes and their probabilities?

(e) If the system is in the state with $A_x = -1$, and A_y is measured, what are the possible outcomes and their probabilities?

(f) Consider the state $|\psi\rangle = c\,(2|1\rangle + 2|0\rangle + |-1\rangle)$ where c is a real positive constant. If this ket has unit norm, what is c? If A_z^2 is then measured, and a result $+1$ is obtained, what is the normalized state ket immediately after the measurement? What was the probability of this result? If A_z is then immediately measured, what are the possible outcomes and their respective probabilities?

(g) The system is in a state for which the probabilities of measuring various outcomes for A_z are $P(A_z = 1) = 1/2$ and $P(A_z = 0) = 1/4$ and $P(A_z = -1) = 1/4$. Convince yourself that the most general normalized state ket with this property is

$$|\psi\rangle = \frac{e^{i\delta_1}}{\sqrt{2}}|1\rangle + \frac{e^{i\delta_2}}{2}|0\rangle + \frac{e^{i\delta_3}}{2}|-1\rangle$$

where δ_1, δ_2, and δ_3 are arbitrary real numbers. It has been claimed, correctly, that if $|\psi\rangle$ is a normalized ket, then $e^{i\theta}|\psi\rangle$ is a physically equivalent normalized ket. Does this mean that the factors $e^{i\delta_n}$ are physically irrelevant? To answer this, calculate the probability $P(A_x = 0)$.

Problem 2. Find expressions for the following matrix elements, where X and P are the position and momentum operators for a particle moving in 1 dimension, with eigenkets $|x\rangle$ and $|p\rangle$ respectively, with the Dirac normalizations $\langle x'|x\rangle = \delta(x - x')$ and $\langle p'|p\rangle = \delta(p - p')$.
(a) $\langle x|X|p\rangle$
(b) $\langle p|P|x\rangle$
(c) $\langle x|PX|p\rangle$
(d) $\langle p|PX|x\rangle$
(e) $\langle x_2|P^2|x_1\rangle$
(f) $\langle p_2|X^2|p_1\rangle$
(g) $\langle x_2|XP|x_1\rangle$

Problem 3. [This exercise is intended to help you to understand one reason why the states for quantum mechanics form a *complex* vector space.] Show that for a particle moving in one dimension, if the position-space wave function $\psi(x) = \langle x|\psi\rangle$ is real, then the expectation value of the momentum operator P must vanish: $\langle P\rangle = 0$. [Hint: show that for any p, the probabilities for the momenta $+p$ and $-p$ are equal.] This generalizes to the case that $\psi(x) = e^{i\theta} f(x)$, where θ is a constant and $f(x)$ is real; recall that the overall phase of the wavefunction is not physically relevant.

Problem 4. Suppose that the position-space wave function $\psi(x)$ for a particle moving in one dimension has momentum expectation value $\langle P\rangle = p_0$. Show that the state with position-space wave function $e^{ip_1 x/\hbar}\psi(x)$ must have momentum expectation value $p_0 + p_1$.