PHYS 660 **Homework 2** **Due Wednesday, September 11, 2024**

Reading assignment: Read through section 2.6 of the text.

Problem 1. Suppose that U is a unitary operator.
(a) Show, using results given in class and the notes, that if A is a Hermitian operator, then so is $U^\dagger A U$.
(b) Show, using results given in class and the notes, that if V is another unitary operator, then so is $U^\dagger V U$.

Problem 2. Define the trace of an operator A to be

$$\mathrm{Tr}[A] = \sum_j \langle j|A|j\rangle,$$

where the kets $|j\rangle$ form a complete orthonormal basis.
(a) Show that $\mathrm{Tr}[A]$ does not depend on the choice of orthonormal basis. To do this, assume that some kets $|q\rangle$ are *another* orthonormal basis, and show that $\mathrm{Tr}[A]$ as given above is also equal to $\sum_q \langle q|A|q\rangle$.
(b) Show that $\mathrm{Tr}[AB] = \mathrm{Tr}[BA]$.

Problem 3. Suppose that A and B are operators, such that $[A, B]$ commutes with B. Show that $[A, B^n] = n[A, B]B^{n-1}$. [Hint: you can do this by mathematical induction. In particular, show that if it is true for some n, then it is true also for $n + 1$.]

Problem 4. Consider a vector space spanned by three orthobasis kets $|1\rangle$, $|2\rangle$, and $|3\rangle$, and a Hermitian operator Ω defined by

$$\begin{aligned} \Omega\,|1\rangle &= 2\,|1\rangle\,, \\ \Omega\,|2\rangle &= 3\,|2\rangle - i\,|3\rangle \\ \Omega\,|3\rangle &= i\,|2\rangle + 3\,|3\rangle\,. \end{aligned}$$

The orthobasis kets can be represented by vectors $v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, and $v_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$, respectively.

(a) Write down the 3×3 matrix representation of Ω in this basis.
(b) Find the eigenvalues of Ω. Two of them should be equal (degenerate).
(c) For the non-degenerate eigenvalue, show that the most general form of the eigenvector is $\begin{pmatrix}0\\a\\-ia\end{pmatrix}$, where a is an arbitrary complex number. Also, re-write this answer in ket form.

(d) Find the most general element of the subspace of eigenvectors corresponding to the degenerate eigenvalue that you found in part (b). In other words, for the degenerate eigenvalue, find the most general form for a single eigenvector; you should write it in terms of two arbitrary complex numbers. Write your answer both in three-component vector and ket forms. Check that this vector is orthogonal to the result in part (c).

Problem 5. Again consider a three-dimensional vector space spanned by an orthobasis consisting of three kets $|1\rangle$, $|2\rangle$, and $|3\rangle$, and two Hermitian operators A and B defined by

$$A\,|1\rangle = 2a\,|1\rangle + a\,|3\rangle\,, \tag{1}$$

$$A\,|2\rangle = a\,|2\rangle\,, \tag{2}$$

$$A\,|3\rangle = a\,|1\rangle + 2a\,|3\rangle\,, \tag{3}$$

and

$$B\,|1\rangle = b\,|1\rangle + 2b\,|3\rangle\,, \tag{4}$$

$$B\,|2\rangle = 0, \tag{5}$$

$$B\,|3\rangle = 2b\,|1\rangle + b\,|3\rangle \tag{6}$$

where a and b are constant real numbers. Again, the orthobasis kets can be represented by vectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, respectively.

(a) What are the 3×3 matrix representations of the operators A and B in this basis?

(b) Show that $[A, B] = 0$, which implies that A and B can be simultaneously diagonalized.

(c) Find the eigenvalues of A and the eigenvalues of B. [One of them has non-degenerate eigenvalues; you would be wise to use this to plan your strategy for the next part.]

(d) Find an orthonormal basis v_1', v_2', and v_3' of vectors that are eigenvectors of both A and B. For each of them, give their eigenvalues with respect to each of A and B.

(e) Write the matrix U that transforms the orthonormal basis you found in part (d) to the original orthonormal basis. This means that $v_1' = Uv_1$ and $v_2' = Uv_2$ and $v_3' = Uv_3$. Check by direct computation that U is unitary.

(f) Check that U diagonalizes the matrices A and B, by computing $U^\dagger AU$ and $U^\dagger BU$.

[Note that this illustrates a special case of Theorem 2.6.7 of the text.]