PHYS 660 **Homework 10** **Due Friday, December 6, 2024**

Reading assignment: sections 10.1 and 11.1-11.4 of the text.

Problem 1. Consider a particle of mass μ trapped inside a ball of radius b that has a hard core of radius a, so that the potential in spherical coordinates is:

$$V(r) = \begin{cases} \infty & (\text{for } r < a), \\ 0 & (\text{for } a < r < b), \\ \infty & (\text{for } r > b). \end{cases}$$

This means that the eigenstates of H, L^2, and L_z have wavefunctions of the form $\Psi_{E,\ell,m}(r,\theta,\phi) = [Aj_\ell(kr) + Bn_\ell(kr)]Y_\ell^m(\theta,\phi)$ for the region $a < r < b$.

(a) Find all of the allowed energy eigenstates and eigenvalues for $\ell = 0$. [Hint: use boundary conditions to solve for the ratio B/A twice, and require the two expressions to be equal; then find the solutions for k by inspection.]

(b) For the case $\ell = 1$, find a transcendental equation whose solutions will yield the energy eigenvalues. Put your equation into the form $\tan[k(b-a)] = \{$an expression not involving sines or cosines$\}$. [Hint: one way to do this is to first put the transcendental equation into a form that is polynomial in ka, kb, and their sines and cosines; then use trigonometric identities for $\sin(kb - ka)$ and $\cos(kb - ka)$.]

(c) For the special case $\ell = 1$ and $b = 2a$, show that your transcendental equation can be written in the form

$$\tan X = \frac{X}{1 + NX^2},$$

where $X = ka$ and N is a certain fixed integer that you will discover. Solve the transcendental equation for X numerically to at least 3 digits of accuracy, and obtain the lowest energy for $\ell = 1$. How does it compare to the lowest energy for $\ell = 0$ that you found in part (a)?

Problem 2. Consider the ground state and the first excited states of the hydrogen atom (ignoring the electron's spin), in the eigenstate orthobasis of the CSCO H, L^2, L_z, denoted $|n,\ell,m\rangle$ for $n = 1, 2$.

(a) Consider the matrix elements of Z (the rectangular coordinate operator) for every pair of such states, $\langle n',\ell',m'|Z|n,l,m\rangle$ with $n = 1,2$ and $n' = 1,2$. Identify **one** of these matrix elements that is **non-zero**, and calculate it. (Hint: try not to waste time on any that are destined to be zero.)

(b) Find the expectation value and uncertainty of Z in the ground state.

(c) Find the expectation value and uncertainty of P_z in the ground state.

(d) Check that the uncertainty principle agrees with your answers to parts (b) and (c).

Problem 3. A particle is in an eigenstate of a Hamiltonian that has a spherically symmetric potential $V(r)$, with energy E, and is described by the normalized wavefunction in spherical coordinates

$$\langle \vec{r}|\psi_E\rangle = \psi_E(r,\theta,\phi) = Ae^{-r/b},$$

where A and b are constants and $\langle\psi_E|\psi_E\rangle = 1$.

(a) What are the units of A and b?

(b) What is the angular momentum of the state?

(c) Assume that $V(r)$ vanishes as $r \to \infty$. Use this to find the energy eigenvalue E, by matching leading terms in Schrödinger's equation as $r \to \infty$.

(d) Now that you have found E, consider finite r and find the potential $V(r)$.

(e) Find A by requiring that the wavefunction is normalized.

Problem 4. Consider a quantum system with two independent spin-1/2 operators, $\overrightarrow{S_1}$ and $\overrightarrow{S_2}$, so that the state space is spanned by an orthobasis of S_{1z} and S_{2z} eigenstates $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$. In each ket, the first entry labels states with S_{1z} eigenvalue $\pm\hbar/2$, and the second entry labels states with S_{2z} eigenvalue $\pm\hbar/2$. At time $t = 0$, the system happens to be in the state

$$|\psi(0)\rangle = \frac{1}{2}|\uparrow\uparrow\rangle + \frac{1}{2}|\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\downarrow\rangle\,.$$

(a) At time $t = 0$, we simultaneously measure S^2 and S_z, where $\vec{S} = \vec{S}_1 + \vec{S}_2$ is the total spin operator. What are the possible outcomes, and their probabilities?

(b) Suppose that instead of the above measurements, we let the system evolve until time t, with the Hamiltonian $H = \omega_1 S_{1z} + \omega_2 S_{2z}$, where ω_1 and ω_2 are constants. What is the state at time t? If we measure S_{1z} at time t, what are the possible outcomes, and their probabilities? Use these results to find the expectation value for S_{1z} as a function of time.

(c) Now suppose that instead the Hamiltonian of the system is $H = a\vec{S}_1 \cdot \vec{S}_2$, where a is a constant. What is the state at time t, and what are the possible outcomes and probabilities for a measurement of S_{1z}? Use these results to find the expectation value for S_{1z} as a function of time.