PHYS 660 Homework 7 Due 10:00 AM Friday, October 27, 2023

Reading assignment: sections 7.4, 7.5, and 5.3 of the class notes.

Problem 1. Consider a spin-less particle of mass m, moving in three dimensions, with the Hamiltonian given by:

$$H = \frac{1}{2m}(P_x^2 + P_y^2 + P_z^2) + \frac{1}{2}m\omega^2 \left(\frac{5}{2}X^2 + \frac{5}{2}Y^2 + 3XY + Z^2\right).$$

(a) Use the change of coordinates:

$$\begin{aligned} x &= cx' + sy' \\ y &= -sx' + cy' \\ z &= z', \end{aligned}$$

where c and s are the cosine and sine of an arbitrary rotation angle, to rewrite the Hamiltonian in terms of the operators X', Y', Z', $P_{x'}$, $P_{y'}$, and $P_{z'}$. (You may use the fact that this is an orthogonal rotation on the coordinate system, so $P_x^2 + P_y^2 + P_z^2 = P_{x'}^2 + P_{y'}^2 + P_{z'}^2$.)

(b) Choose specific values for c and s (remembering that $c^2 + s^2 = 1$), so that H will not contain a term proportional to $X'Y'$. Rewrite the Hamiltonian with this choice. (There is more than one valid choice here.)

(c) What are the three smallest energy eigenvalues of H, and what are their degeneracies?

For all of the remaining problems, consider the 1-dimensional harmonic oscillator with mass m and frequency ω.

Problem 2. At time $t = 0$, the oscillator is in the state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ in the usual energy basis.

(a) Find the state at later times, $|\psi(t)\rangle$, in the energy basis.

(b) By direct calculation using the results of part (a) (without appealing to Ehrenfest's Theorem), find the expectation values $\langle X\rangle$ and $\langle P\rangle$ as functions of time t.

(c) Check that your results from part (b) obey Ehrenfest's Theorem.

(d) At time t, compute the probability that a measurement of the position yields $x > 0$. Evaluate the minimum and the maximum probabilities numerically, and check that they are between 0 and 1.

Problem 3. At time $t = 0$, suppose the oscillator has wavefunction

$$\psi(x,0) = Cx^2 \exp(-m\omega x^2/2\hbar)$$

where C is a positive real constant.

(a) By requiring the wavefunction to be normalized, find the constant C.

(b) If the energy is measured, what are the possible outcomes and their probabilities?

(c) Find the state ket $|\psi(t)\rangle$ in the energy basis, as a function of time.

(d) Compute the expectation values $\langle X\rangle$, $\langle P\rangle$, $\langle X^2\rangle$, and $\langle P^2\rangle$, and the uncertainties ΔX and ΔP, each as a function of time.

(e) At time t, what is the probability that a measurement of the position yields $x > 0$?

Problem 4. The coherent states $|\alpha\rangle$ are the eigenstates of the annihilation operator a with complex eigenvalue α, as discussed in class.

(a) For two different coherent states $|\alpha\rangle$ and $|\beta\rangle$, compute $\langle\beta|\alpha\rangle$ and $|\langle\beta|\alpha\rangle|^2$ in simplest form. (Your answers should be exponentials. Therefore, they cannot vanish, which shows that the set of states $|\alpha\rangle$ do *not* satisfy orthogonality.)

(b) Suppose that the oscillator is in the state $|\alpha\rangle$ at time $t = 0$. Find the probability $P(t)$ to find it again in the state $|\alpha\rangle$ at a later time t. [Make use of the results at the bottom of page 148 of the text.] Put your answer in simplest form, and check that it obeys the sanity condition $P(0) = 1$ at time $t = 0$.

(c) Suppose that $|\alpha| \gg 1$, as is true for a macroscopic oscillator. For what very small length of time after $t = 0$ does the probability $P(t)$ remain greater than 0.5? [Hint: expand $\cos(\omega t)$ to second order in t, for small t.]

(d) Are there any other times at which $P(t) = 1$? If so, when? If not, why not?

(e) What is the minimum value of $P(t)$, and at what time or times is it achieved?

(f) Estimate your answers to parts (c) and (e) numerically for a macroscopic oscillator with $m = 0.2$ kg, $\omega = 10\,\text{sec}^{-1}$, and $x_0 = 0.1$ meters.