PHYS 660 **Homework 10** **Due 10:00am Friday, December 8, 2023**

Reading assignment: sections 9.4, 9.5, 10.1, and 11.1-11.3 of the class text.

Problem 1. Consider the isotropic 3-d harmonic oscillator problem, with potential $V(x, y, z) = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2)$. As discussed in class, the Hamiltonian H can be written as the sum of $H_x = \hbar\omega(a_x^\dagger a_x + 1/2)$, $H_y = \hbar\omega(a_y^\dagger a_y + 1/2)$, and $H_z = \hbar\omega(a_z^\dagger a_z + 1/2)$, which form a C.S.C.O. with corresponding orthonormal eigenbasis $|n_x, n_y, n_z\rangle$. Another choice of C.S.C.O. is H, L^2, and L_z, with corresponding eigenbasis $|n, \ell, m\rangle'$, where $n = n_x + n_y + n_z$. (The $'$ is just to distinguish the two types of orthobasis elements, since they both have three integer labels and therefore could be confused if we aren't careful.)

(a) Construct the operators L^2 and L_z in terms of the creation and annihilation operators. You should find:

$$\begin{aligned} L^2 &= \hbar^2\big[N_1(a_x^{\dagger 2}a_y^2 + a_x^{\dagger 2}a_z^2 + a_y^{\dagger 2}a_x^2 + a_y^{\dagger 2}a_z^2 + a_z^{\dagger 2}a_x^2 + a_z^{\dagger 2}a_y^2) \\ &\quad + N_2(a_x^\dagger a_y^\dagger a_x a_y + a_x^\dagger a_z^\dagger a_x a_z + a_y^\dagger a_z^\dagger a_y a_z) \\ &\quad + N_3(a_x^\dagger a_x + a_y^\dagger a_y + a_z^\dagger a_z)\big], \end{aligned}$$

where N_1, N_2, and N_3 are certain fixed integers that you will discover. Note that this result is in "normal-ordered" form, which means that the commutation relations have been used to ensure that no creation operator appears to the right of an annihilation operator.

(b) For the subspace of states with $n = 2$, find the action of L^2 in the $|n_x, n_y, n_z\rangle$ basis and its corresponding 6×6 matrix representation. Find the eigenvalues and normalized eigenvectors of L^2 for the $n = 2$ subspace in that basis.

(c) Compute the action of L_z on each of the simultaneous eigenvectors of H, L^2 found in the previous part. Within each sub-subspace of fixed $n = 2$ and fixed ℓ, find the eigenvalues and eigenvectors of L_z, and so conclude by writing the $|2, \ell, m\rangle'$ orthobasis states in terms of the $|n_x, n_y, n_z\rangle$ eigenstates.

Problem 2. Consider the ground state and the first excited states of the Hydrogen atom (ignoring the electron's spin), in the eigenstate orthobasis of the CSCO H, L^2, L_z, denoted $|n, \ell, m\rangle$ for $n = 1, 2$.

(a) Consider the matrix elements of Z (the rectangular coordinate operator) for every pair of such states, $\langle n', \ell', m'|Z|n, l, m\rangle$ with $n = 1, 2$ and $n' = 1, 2$. Identify **one** of these matrix elements that is **non-zero**, and calculate it. (Hint: try not to waste time on any that are destined to be zero.)

(b) Find the expectation value and uncertainty of Z in the ground state.

(c) Find the expectation value and uncertainty of P_z in the ground state.

(d) Check whether the uncertainty principle agrees with your answers to parts (b) and (c).

Problem 3. Consider a particle of mass μ trapped inside a ball of radius b that has a hard core of radius a, so that the potential in spherical coordinates is:

$$V(r) = \begin{cases} \infty & \text{(for } r < a\text{)}, \\ 0 & \text{(for } a < r < b\text{)}, \\ \infty & \text{(for } r > b\text{)}. \end{cases}$$

This means that the eigenstates of H, L^2, and L_z have wavefunctions of the form $\Psi_{E,\ell,m}(r,\theta,\phi) = [Aj_\ell(kr) + Bn_\ell(kr)]Y_\ell^m(\theta,\phi)$ for the region $a < r < b$.

(a) Find all of the allowed energy eigenstates and eigenvalues for $\ell = 0$. [Hint: use boundary conditions to solve for the ratio B/A twice, and require the two expressions to be equal; then find the solutions for k by inspection.]

(b) For the case $\ell = 1$, find a transcendental equation whose solutions will yield the energy eigenvalues. Put your equation into the form $\tan[k(b-a)] = \{$an expression not involving sines or cosines$\}$. [Hint: one way to do this is to first put the transcendental equation into a form that is polynomial in ka, kb, and their sines and cosines; then use trigonometric identities for $\sin(kb - ka)$ and $\cos(kb - ka)$.]

(c) For the special case $\ell = 1$ and $b = 2a$, show that your transcendental equation can be written in the form

$$\tan X = \frac{X}{1 + NX^2},$$

where $X = ka$ and N is a certain fixed integer that you will discover. Solve the transcendental equation for X to at least 3 digits of accuracy, and obtain the lowest energy for $\ell = 1$. How does it compare to the lowest energy for $\ell = 0$ that you found in part (a)?

Problem 4. Consider a quantum system with two independent spin-1/2 operators, $\vec{S_1}$ and $\vec{S_2}$, so that the state space is spanned by an orthobasis of S_{1z} and S_{2z} eigenstates $|\uparrow,\uparrow\rangle$, $|\uparrow,\downarrow\rangle$, $|\downarrow,\uparrow\rangle$, and $|\downarrow,\downarrow\rangle$. In each ket, the first entry labels states with S_{1z} eigenvalue $\pm\hbar/2$, and the second entry labels states with S_{2z} eigenvalue $\pm\hbar/2$. At time $t = 0$, the system happens to be in the state

$$|\psi(0)\rangle = |\uparrow,\downarrow\rangle\,.$$

The Hamiltonian of the system is $H = a\vec{S}_1 \cdot \vec{S}_2$, where a is a constant.

(a) At time $t = 0$, we simultaneously measure S^2 and S_z, where $\vec{S} = \vec{S}_1 + \vec{S}_2$ is the total spin operator. What are the possible outcomes, and their probabilities?

(b) Suppose that instead of the above measurements, we let the system evolve until time t, and then measure S_{1z}. What are the possible outcomes, and their probabilities as a function of t?