Phase Space & Hamiltonians

Recall for a Lagrangian $L(q_n, \dot{q}_n, t)$, generalized momenta are $p_n = \frac{\partial L}{\partial \dot{q}_n} = p_n(q_k, \dot{q}_k, t)$. Then equation of motion is

$$\frac{d}{dt} p_n = \frac{\partial L}{\partial q_n}.$$

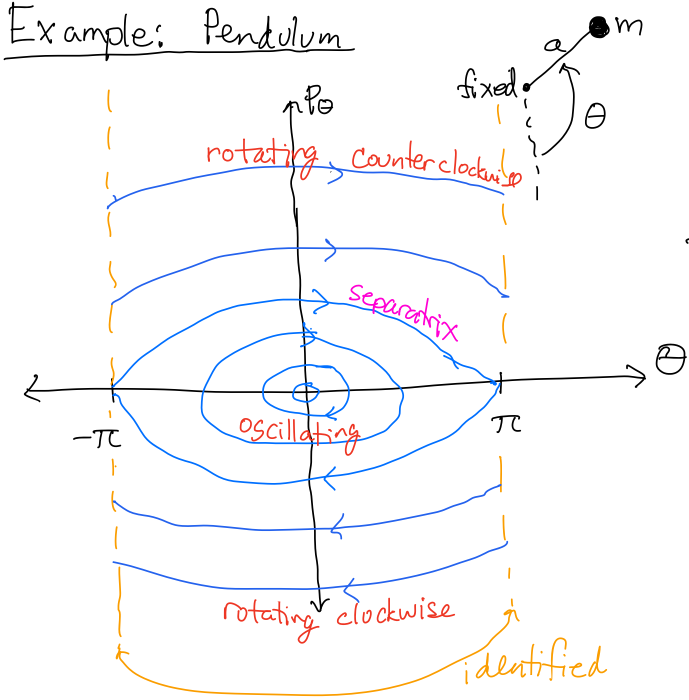
Usually, can invert: $\dot{q}_n(q_k, p_k, t)$.

State of system is:

$(q_n, \dot{q}_n, t)$ (configuration space) $\quad \dot{q}_n = \frac{dq_n}{dt}$ depends on $q_n(t)$, or

(q_n, p_n, t) (phase space) $\quad p_n, q_n$ independent, on equal footing.

Example: Pendulum



$$T = \tfrac{1}{2} m a^2 \dot{\theta}^2$$
$$V = mga(1 - \cos\theta)$$
$$p_\theta = m a^2 \dot{\theta}$$

phase space is infinite cylinder

$-\pi \le \theta \le \pi$, but $\pi, -\pi$ are same

$-\infty < p_\theta < \infty$

Time evolution = flow in phase space.

Define Hamiltonian $= H = \sum_n p_n \dot{q}_n - L(q_k, \dot{q}_k, t) =$ a function of q_k, p_k, t, not $\dot{q}_k$. Must eliminate all $\dot{q}_k$!!!

Example: Particle in a potential: $L = \frac{1}{2} m \sum_{i=1}^{3} \dot{r}_i^2 - V(r_i)$. $\quad p_i = \frac{\partial L}{\partial \dot{r}_i} = m\dot{r}_i$

So $H = p_i \dot{r}_i - L = p_i\left(\frac{p_i}{m}\right) - \left[\frac{1}{2} m \left(\frac{p_i}{m}\right)\left(\frac{p_i}{m}\right) - V(r_i)\right] \Rightarrow H = \sum_i \frac{p_i^2}{2m} + V(r_i) = E$

Consider the differential of H, from its definition:

$$dH = \sum_n \cancel{p_n\, d\dot{q}_n} + \sum_n \dot{q}_n\, dp_n - \sum_n \overset{\dot{p}_n}{\frac{\partial L}{\partial q_n}}\, dq_n - \sum_n \cancel{\frac{\partial L}{\partial \dot{q}_n}\, d\dot{q}_n} - \frac{\partial L}{\partial t}\, dt$$

cancel, because $p_n = \dfrac{\partial L}{\partial \dot{q}_n}$

So $dH = \sum_n \left(\dot{q}_n\, dp_n - \dot{p}_n\, dq_n \right) - \dfrac{\partial L}{\partial t}\, dt$. Therefore,

$H = H(q_n, p_n, t)$.

Can also write: $\quad dH = \sum_n \left(\dfrac{\partial H}{\partial p_n}\, dp_n + \dfrac{\partial H}{\partial q_n}\, dq_n \right) + \dfrac{\partial H}{\partial t}\, dt.$

Compare two expressions for dH, match coeffs of dp_n, dq_n, dt:

$$\underbrace{\dot{q}_n = \frac{\partial H}{\partial p_n} \qquad \dot{p}_n = -\frac{\partial H}{\partial q_n}}_{\text{Hamilton's equations}} \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

explicit dependence on time.

Construction of $H(q_n, p_n, t)$ from $L(q_n, \dot{q}_n, t)$ is an example of...

Legendre Transformation

Consider a function $F(x,y)$. Its differential is:

$dF = \dfrac{\partial F}{\partial x}\, dx + \dfrac{\partial F}{\partial y}\, dy.$ Now introduce a new variable u,

and a new function $G(x,y,u) = uy - F(x,y)$. Its differential is

$$dG = u\, dy + y\, du - \frac{\partial F}{\partial x}\, dx - \frac{\partial F}{\partial y}\, dy.$$

Something special happens if we assume $u = \dfrac{\partial F}{\partial y}$. Then the dy terms in dG cancel, and we get

$dG = y\, du - \dfrac{\partial F}{\partial x}\, dx.$ So in that case, $G = G(x,u)$.

no y!

The point; given a function $F(x,y)$, construct a function $G(x,u)$,
where $u = \dfrac{\partial F}{\partial y}$.

"same information"

We "traded" $y \leftrightarrow u$.

Can reverse it!
$$\begin{cases} G(x,u) = uy - F(x,y) & \text{(eliminate } y) \\ F(x,y) = uy - G(x,u) & \text{(eliminate } u) \end{cases}$$

with $y(x,u) = \dfrac{\partial G}{\partial u}$ and $u(x,y) = \dfrac{\partial F}{\partial y}$

Summary table:

general	Classical mech	Thermodynamics
x	q_n	V = volume
y	$\dot{q}_n$	S = entropy
$F(x,y)$	$L(q_n,\dot{q}_n)$	$U(V,S)$ = internal energy
u	p_n	$T = \dfrac{\partial U}{\partial S}$ = temperature
$G(x,u)$	$H(q_n,p_n)$	$-F(V,T)$ = Helmholtz free energy

Thermo has many other examples:

V = volume S = entropy P = pressure T = temperature $\}(x,y,u)$

Internal Energy $U(S,V)$ Helmholtz Free Energy $F(V,T)$

Enthalpy $H(S,P)$ Gibbs Free Energy $G(T,P)$. $\}$ F, G

Then
$$\begin{cases} G(T,P) = H(S,P) - TS & \text{with } T = \dfrac{\partial H}{\partial S} \quad \text{eliminate } S \\ H(S,P) = G(T,P) + TS & \text{with } S = -\dfrac{\partial G}{\partial T} \quad \text{eliminate } T \end{cases}$$

Conservation Laws

$$\frac{dH}{dt} = \underbrace{\sum_n \left(\underbrace{\frac{\partial H}{\partial q_n}}_{-\dot{p}_n} \underbrace{\frac{dq_n}{dt}}_{\dot{q}_n} + \underbrace{\frac{\partial H}{\partial p_n}}_{\dot{q}_n} \underbrace{\frac{dp_n}{dt}}_{\dot{p}_n} \right)}_{0} + \frac{\partial H}{\partial t}$$

So $\dfrac{dH}{dt} = \dfrac{\partial H}{\partial t}$ = explicit time dependence

So $\boxed{H = \text{energy is conserved if no } \underline{\text{explicit}} \text{ time dependence.}}$

Suppose q_n is cyclic (doesn't appear in L). Then recall

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} = 0 \implies \frac{d}{dt} p_n = 0 \implies p_n \text{ conserved.}$$

We also know $\frac{\partial H}{\partial q_n} = -\dot{p}_n = 0$. So H can't depend on q_n either.

So $\boxed{\frac{\partial H}{\partial q_n} = 0 \iff \dot{p}_n = 0 \iff p_n = \text{conserved quantity}}$

Modified Hamilton's Principle

Recall equations of motion came from $\delta\left(\int_{t_1}^{t_2} dt \; L(q_n, \dot{q}_n, t)\right) = 0$, for trajectories with fixed $q_n(t_1)$ and $q_n(t_2)$, so that $\delta q_n(t_1) = \delta q_n(t_2)$.

Let's rewrite it in terms of H:

$$\underbrace{\delta\left(\int_{t_1}^{t_2} dt \left[\sum_n p_n \dot{q}_n - H(q_k, p_k, t)\right]\right)} = 0$$

$$\int_{t_1}^{t_2} dt \sum_n \left[\dot{q}_n \delta p_n + p_n \delta \dot{q}_n - \frac{\partial H}{\partial p_n}\delta p_n - \frac{\partial H}{\partial q_n}\delta q_n\right]$$

Now use $p_n \delta \dot{q}_n = \frac{d}{dt}(p_n \delta q_n) - \delta q_n \dot{p}_n$. So:

$$0 = \int_{t_1}^{t_2} dt \sum_n \left[\underbrace{\frac{d}{dt}(p_n \delta q_n)}_{p_n \delta q_n \big|_{t=t_2} - p_n \delta q_n \big|_{t=t_1}} + \delta p_n\left(\dot{q}_n - \frac{\partial H}{\partial p_n}\right) + \delta q_n\left(-\dot{p}_n - \frac{\partial H}{\partial q_n}\right)\right]$$

If we assume $\delta q_n(t_2) = \delta q_n(t_1) = 0$, then Hamilton's principle says:

$$\dot{q}_n = \frac{\partial H}{\partial p_n} \quad \text{and} \quad \dot{p}_n = -\frac{\partial H}{\partial q_n}$$

What boundary conditions do we need on $\delta p_n(t_1)$, $\delta p_n(t_2)$?

Don't <u>need</u> any. But usual to assume $\delta p_n(t_1) = \delta p_n(t_2) = 0$ also.

Two reasons:

1) Treat q_n, p_n on same footing.

2) Can add any total time derivative to $H(q_n, p_n, t)$.

So $H(q_n, p_n, t) \to H(q_n, p_n, t)' = H(q_n, p_n, t) + \frac{d}{dt} F(q_n, p_n, t)$

have same equations of motion from action principle.

$$\delta(\text{Action}) \to \delta(\text{Action}) + \underbrace{\int_{t_1}^{t_2} dt \, \frac{d}{dt}\left[\sum_n \left(\delta q_n \frac{\partial F}{\partial q_n} + \delta p_n \frac{\partial F}{\partial p_n}\right)\right]}_{\sum_n \left(\delta q_n \frac{\partial F}{\partial q_n} + \delta p_n \frac{\partial F}{\partial p_n}\right)\Big|_{t=t_1}^{t=t_2} = 0}$$

Example Charged particle in EM fields

In cgs units: $\begin{cases} \vec{B}(\vec{r}, t) = \vec{\nabla} \times \vec{A}(\vec{r}, t) \\ \vec{E}(\vec{r}, t) = -\vec{\nabla}\Phi(\vec{r}, t) - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}(\vec{r}, t) \end{cases}$

Then $$\boxed{L(\vec{r}, \dot{\vec{r}}, t) = \frac{1}{2} m \dot{\vec{r}}^2 - \underset{\text{charge}}{q}\,\Phi(\vec{r}, t) + \frac{q}{c}\,\dot{\vec{r}} \cdot \vec{A}(\vec{r}, t)}$$

EOM: $$\frac{\partial L}{\partial \dot{r}_i} = m\dot{r}_i + \frac{q}{c} A_i(\vec{r}, t) = p_i$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) = m\ddot{r}_i + \frac{q}{c}\frac{\partial A_i}{\partial r_j}\frac{dr_j}{dt} + \frac{q}{c}\frac{\partial A_i}{\partial t}$$

$$\frac{\partial L}{\partial r_i} = -q\frac{\partial \Phi}{\partial r_i} + \frac{q}{c}\,\dot{\vec{r}} \cdot \frac{\partial \vec{A}}{\partial r_j}$$

So $$m\ddot{r}_i = q\underbrace{\left(-\frac{\partial \Phi}{\partial r_i} - \frac{1}{c}\frac{\partial A_i}{\partial t}\right)}_{E_i} + \frac{q}{c}\dot{r}_j \underbrace{\left(\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right)}_{[\dot{\vec{r}}\cdot(\underbrace{\vec{\nabla}\times\vec{A}}_{\vec{B}})]_i}$$

So $$\boxed{m\ddot{\vec{r}} = q\vec{E} + \frac{q}{c}\dot{\vec{r}}\cdot\vec{B} = \text{Lorentz Force Law.}}$$

Canonical momentum: $\vec{p} = \underbrace{m\dot{\vec{r}}}_{} + \frac{q}{c}\vec{A}$

mechanical momentum = kinematic momentum

$m\dot{\vec{r}} = \vec{p} - \frac{q}{c}\vec{A}$. So:

$$H = \vec{p}\cdot\dot{\vec{r}} - L = \vec{p}\cdot\frac{(\vec{p}-\frac{q}{c}\vec{A})}{m} - \frac{1}{2}m\left(\vec{p}-\frac{q}{c}\vec{A}\right)^2 + q\Phi - \frac{q}{c}\frac{(\vec{p}-\frac{q}{c}\vec{A})\cdot\vec{A}}{m}$$

$$\boxed{H = \frac{1}{2m}\left(\vec{p}-\frac{q}{c}\vec{A}\right)^2 + q\Phi.}$$

Is H conserved? $\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = -q\frac{\partial \Phi}{\partial t} + \frac{q}{c}\dot{\vec{r}}\cdot\frac{\partial \vec{A}}{\partial t}$

Not 0 in general, but will be if EM potentials don't depend on time.

Hamilton's EOM:

$\dot{r}_i = \frac{\partial H}{\partial p_i} = \frac{1}{m}\left(p_i - \frac{q}{c}A_i\right)$ (definition of $\vec{p}$)

$\dot{p}_i = -\frac{\partial H}{\partial q_i} = -q\frac{\partial \Phi}{\partial r_i} + \frac{1}{m}\left(\vec{p}-\frac{q}{c}\vec{A}\right)_j\left(\frac{-q}{c}\right)\frac{\partial A_j}{\partial r_i}$ or

$$\dot{p}_i = -q(\nabla\Phi)_i - \frac{q}{c}\dot{r}_j\frac{\partial A_j}{\partial r_i}$$
$$\parallel$$
$$\frac{d}{dt}\left(m\dot{r}_i + \frac{q}{c}A_i\right) = m\ddot{r}_i + \frac{q}{c}\dot{r}_j\frac{\partial A_i}{\partial r_j}$$

$\Rightarrow$ Lorentz force law again

Poisson brackets

Given any two functions on phase space, $F(q_1,\dots,q_N, p_1\cdots p_N, t)$ $G(q_1\cdots q_N, p_1\cdots p_N, t)$,

define

$$\boxed{\{F,G\}_{PB} = \sum_{i=1}^{N}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}\right)}$$

By construction: $\{F,G\}_{PB} = -\{G,F\}_{PB}$ (antisymmetric)

$\{aF + bG, K\}_{PB} = a\{F,K\}_{PB} + b\{G,K\}_{PB}$ (linear)

$\{FG, K\}_{PB} = F\{G,K\}_{PB} + G\{F,K\}_{PB}$ (Leibniz rule)

$\{F,\{G,K\}\} + \{G,\{K,F\}\} + \{K,\{F,G\}\} = 0$ (Jacobi rule)

Poisson brackets of phase-space coordinates:

$$\{q_i, p_j\}_{PB} = \underbrace{\frac{\partial q_i}{\partial q_k}}_{\delta_{ik}}\underbrace{\frac{\partial p_j}{\partial p_k}}_{\delta_{jk}} - \underbrace{\frac{\partial p_j}{\partial q_k}}_{0}\underbrace{\frac{\partial q_i}{\partial p_k}}_{0} = \delta_{ij}$$

$\{q_i, q_j\}_{PB} = 0$ and $\{p_i, p_j\} = 0$

Try this: $$\{H, F\}_{PB} = \sum_i\left(\underbrace{\frac{\partial H}{\partial q_i}}_{-\dot{p}_i}\frac{\partial F}{\partial p_i} - \underbrace{\frac{\partial H}{\partial p_i}}_{-\dot{q}_i}\frac{\partial F}{\partial q_i}\right)$$

$$= -\left(\sum_i \frac{dq_i}{dt}\frac{\partial F}{\partial q_i} + \frac{dp_i}{dt}\frac{\partial F}{\partial p_i}\right) = -\frac{dF}{dt} + \frac{\partial F}{\partial t}.$$

So we learn:

$$\frac{dF}{dt} = -\{H, F\}_{PB} + \frac{\partial F}{\partial t}$$

$\frac{\partial F}{\partial t}$: explicit time dependence (built into definition)

$\frac{dF}{dt}$: total time dependence

Special cases:

1) $F = q_i$. Then $\frac{dq_i}{dt} = -\{H, q_i\} + 0 = -\frac{\partial H}{\partial q_j}\underbrace{\frac{\partial q_i}{\partial p_j}}_{0} + \frac{\partial H}{\partial p_j}\underbrace{\frac{\partial q_i}{\partial q_j}}_{\delta_{ij}}$

So $\dot{q}_i = \frac{\partial H}{\partial p_i}$ ✓

2) $F = p_i$. Then $\frac{dp_i}{dt} = -\{H, p_i\} + 0 = -\frac{\partial H}{\partial q_i}$ ✓

3) F has no explicit time dependence, and $\{H, F\}_{PB} = 0$.

Then F is conserved: $\frac{dF}{dt} = 0$. F = a symmetry generator

Canonical Transformations

Given coordinates on phase space $\vec{z} = (q_1 \cdots q_n, p_1 \cdots, p_n)$

$\vec{z}$: 2n-dimensional vector

with $H(q_i, p_i, t)$ we might like a different set:

$\vec{Z} = (Q_1, \ldots, Q_N, P_1, \ldots, P_N)$, with $\tilde{H}(Q_i, P_i, t)$. (Might mix up q's and p's.)

When will this preserve Hamilton's equations?

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \text{ and } \dot{q}_i = \frac{\partial H}{\partial p_i} \iff \dot{P}_i = -\frac{\partial \tilde{H}}{\partial Q_i} \text{ and } \dot{Q}_i = \frac{\partial \tilde{H}}{\partial P_i}.$$

Various ways of answering this.

Symplectic form = $(2n)\times(2n)$ matrix

$$\Omega = \left(\begin{array}{ccc|ccc} & & & 1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & 1 \\ \hline -1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & -1 & & & \end{array}\right) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Then $\dot{z}_i = \Omega_{ij} \dfrac{\partial H}{\partial z_j}$ are Hamilton's EOM.

Let $Z_i = Z_i(z_j)$. Then $\dot{Z}_i = \dfrac{d}{dt}(Z_i) = \dfrac{dz_j}{dt} \underbrace{\dfrac{\partial Z_i}{\partial z_j}}_{2n\times 2n \text{ matrix}}$

$J_{ij} = \dfrac{\partial Z_i}{\partial z_j}$

"Jacobian matrix"

So $\dot{Z}_i = J_{ij} \Omega_{jk} \underbrace{\dfrac{\partial H}{\partial z_k}}_{\frac{\partial H}{\partial Z_\ell} \underbrace{\frac{\partial Z_\ell}{\partial z_k}}_{J_{\ell k}}}$

So $\dot{Z}_i = \underbrace{J_{ij} \Omega_{jk} (J^T)_{k\ell}} \dfrac{\partial H}{\partial Z_\ell}$ So $\tilde{H} = H$ and

$J \Omega J^T = I$ ← $2n \times 2n$ identity matrix,

<u>Words:</u> J is a <u>symplectic Jacobian</u> $\Leftrightarrow$ Hamilton's EOM preserved.

$\Leftrightarrow$ Canonical transformation

Can write $J_{ij} = \left(\begin{array}{c|c} \dfrac{\partial Q_i}{\partial q_j} & \dfrac{\partial Q_i}{\partial p_j} \\ \hline \dfrac{\partial P_i}{\partial q_j} & \dfrac{\partial P_i}{\partial p_j} \end{array}\right)$. Now compute:

$$(J \Omega J^T)_{ij} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}_{ik} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}_{k\ell} \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{pmatrix}_{\ell j}$$

$$= \left(\begin{array}{c|c} \{Q_i, Q_j\}_{PB} & \{Q_i, P_j\}_{PB} \\ \hline \{P_i, Q_j\}_{PB} & \{P_i, P_j\}_{PB} \end{array}\right) \stackrel{?}{=} \left(\begin{array}{c|c} 0 & \delta_{ij} \\ \hline -\delta_{ij} & 0 \end{array}\right)$$

Works if, and only if, Poisson brackets are preserved.

Example: No mixing of coordinates with momenta. $Q_i(q_j)$. Let $\Theta_{ij} = \frac{\partial Q_i}{\partial q_j}$.

Then Jacobian is $J = \begin{pmatrix} \Theta_{ij} & 0 \\ \frac{\partial P_i}{\partial q_j} & \frac{\partial P_i}{\partial p_j} \end{pmatrix}$.

Works if $P_i = (\Theta^{-1})_{ij}\, p_j$. <u>But:</u> P_i <u>can</u> depend on q_j.

Can construct P_i directly from Lagrangian $L(q_i, \dot{q}_i, t) = L(Q_i, \dot{Q}_i, t)$.

Example: Infinitesimal Canonical Transformation.

$$\begin{aligned} Q_i &= q_i + \epsilon\, A_i(q_k, p_k) \\ P_i &= p_i + \epsilon\, B_i(q_k, p_k) \end{aligned} \Rightarrow J = \begin{pmatrix} I + \epsilon\, \partial A_i/\partial q_j & \epsilon\, \partial A_i/\partial p_j \\ \epsilon\, \partial B_i/\partial q_j & I + \epsilon\, \partial B_i/\partial p_j \end{pmatrix}$$

Plug into $J \Omega J^T = \Omega$, <u>drop</u> ϵ^2. Find:

$$\frac{\partial A_i}{\partial q_j} = -\frac{\partial B_j}{\partial p_i} \quad \text{and} \quad \frac{\partial A_i}{\partial p_j} = \frac{\partial A_j}{\partial p_i} \quad \text{and} \quad \frac{\partial B_i}{\partial q_j} = \frac{\partial B_j}{\partial q_i}$$

are all needed. To satisfy these, choose:

$$A_i = \frac{\partial G}{\partial p_i} \quad \text{and} \quad B_i = -\frac{\partial G}{\partial q_i} \quad \text{for } \underline{\text{same}}\ G(q_k, p_k).$$

Reason: partial derivatives commute: $\frac{\partial^2 G}{\partial p_i \partial q_j} = \frac{\partial^2 G}{\partial q_j p_i}$ etc.

The function $G(q_k, p_k)$ is called the generator of the infinitesimal canonical transformation.

Can build big ones out of repeated infinitesimal ones.

Sub-example: Time evolution is a canonical transformation.

Let $\left\{ \begin{array}{l} G(q_k, p_k) = H(q_k, p_k). \\ \epsilon = dt \end{array} \right\}$ Then $A_i = \dfrac{\partial H}{\partial p_i} = \dot{q}_i$

$B_i = -\dfrac{\partial H}{\partial q_i} = \dot{p}_i$

So $$Q_i = q_i + dt\left(\frac{dq_i}{dt}\right) = q_i(t+dt)$$

$$P_i = p_i + dt\left(\frac{dp_i}{dt}\right) = p_i(t+dt)$$

More generally, any choice of G gives a "flow" on phase space.

General Canonical Transformation, Type 1

Recall that we could add $\frac{d}{dt}(\text{anything})$ to $L(q_n, \dot{q}_n, t)$ in Hamilton's principle. So, write:

$$p_n \dot{q}_n - H(q_n, p_n, t) = P_n \dot{Q}_n - \tilde{H}(Q_n, P_n, t) + \underbrace{\frac{d}{dt} \overbrace{F(q_n, Q_n, t)}^{\text{generator}}}_{\frac{\partial F}{\partial q_n}\dot{q}_n + \frac{\partial F}{\partial Q_n}\dot{Q}_n + \frac{\partial F}{\partial t}}$$

Collect terms:

$$\dot{q}_n\left(p_n - \frac{\partial F}{\partial q_n}\right) + \dot{Q}_n\left(-P_n - \frac{\partial F}{\partial Q_n}\right) - H(q_n, p_n, t) + \tilde{H}(Q_n, P_n, t) - \frac{\partial F}{\partial t} = 0$$

So, try to define a canonical transformation by

$$\boxed{p_n = \frac{\partial}{\partial q_n} F(q_k, Q_k, t) \qquad P_n = -\frac{\partial}{\partial Q_n} F(q_k, Q_k, t),}$$

and the new Hamiltonian is:

$$\boxed{\tilde{H}(Q_n, P_n, t) = H(q_n, p_n, t) + \frac{\partial}{\partial t} F(q_n, Q_n, t)}$$

Example: Start with $H(q,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$, so that

$$\dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q \quad \text{and} \quad \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}.$$

Now choose $F(q,Q,t) = qQ$.

then $p = \frac{\partial}{\partial q}(qQ) = Q$ (new coordinate = old momentum)

$P = -\frac{\partial}{\partial Q}(qQ) = -q$ (new momentum = $-$(old coordinate))

$$\tilde{H} = H = \frac{Q^2}{2m} + \frac{1}{2}m\omega^2 P^2.$$

Check Poisson brackets: $\{Q,P\}_{PB} = \underbrace{\frac{\partial Q}{\partial q}}_{\frac{\partial p}{\partial q}=0}\underbrace{\frac{\partial P}{\partial p}}_{\frac{\partial(-q)}{\partial p}=0} - \underbrace{\frac{\partial P}{\partial q}}_{\frac{\partial(-q)}{\partial q}=-1}\underbrace{\frac{\partial Q}{\partial p}}_{\frac{\partial p}{\partial p}=1} = 1$ ✓

Recall: this is the key check of a canonical transformation

Automatic: $\{Q,Q\}_{PB} = 0$ $\{P,P\}_{PB} = 0$

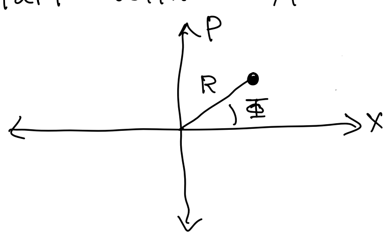
antisymmetric

Check EOM: $\dot{P} = -\frac{\partial \tilde{H}}{\partial Q} = -\frac{Q}{m} \Rightarrow -\dot{q} = -\frac{p}{m} \Rightarrow \dot{q} = \frac{p}{m}$ ✓

$\dot{Q} = \frac{\partial \tilde{H}}{\partial P} = m\omega^2 P \Rightarrow \dot{p} = -m\omega^2 q$ ✓

Example: Polar coordinates on phase space

Start with (x,p).



Try (R,Φ) defined by $\begin{cases} x = R\cos\Phi \\ p = R\sin\Phi \end{cases}$

Is this a canonical transformation?

$$\left.\begin{aligned} R &= \sqrt{x^2+p^2} \\ \Phi &= \arctan p/x \end{aligned}\right\} \Rightarrow \quad \frac{\partial R}{\partial x} = \cos\Phi \qquad \frac{\partial R}{\partial p} = \sin\Phi$$

$$\frac{\partial \Phi}{\partial x} = \frac{-\sin\Phi}{R} \qquad \frac{\partial \Phi}{\partial p} = \frac{\cos\Phi}{R}$$

So $\{R, \Phi\}_{PB} = \frac{\partial R}{\partial x}\frac{\partial \Phi}{\partial p} - \frac{\partial \Phi}{\partial x}\frac{\partial R}{\partial p} = (\cos\Phi)\left(\frac{\cos\Phi}{R}\right) - \left(\frac{-\sin\Phi}{R}\right)(\sin\Phi)$

$= \frac{\cos^2\Phi + \sin^2\Phi}{R} = \frac{1}{R} \neq 1$ Not canonical.

But $\left\{\frac{R^2}{2}, \Phi\right\}_{PB} = 1$, so $\left(Q = \frac{R^2}{2}, \; P = \Phi\right)$ are canonical.

(Not very useful, because R doesn't even have well-defined units!)

Action-angle variables for the harmonic oscillator

$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$ Define $(q,p) \to (I, \theta)$ where

$x = \sqrt{\frac{2I}{m\omega}}\sin\theta \qquad p = \sqrt{2Im\omega}\cos\theta.$

Now $\left(\sqrt{\frac{m\omega}{2I}}\,x\right)^2 + \left(\frac{p}{\sqrt{2Im\omega}}\right)^2 = \sin^2\theta + \cos^2\theta = 1$, so

$I = \frac{m\omega}{2}x^2 + \frac{1}{2m\omega}p^2$ and $\theta = \arctan\left(\frac{m\omega x}{p}\right)$

Check Poisson brackets!

$\{\theta, I\}_{PB} = \frac{\partial\theta}{\partial x}\frac{\partial I}{\partial p} - \frac{\partial I}{\partial x}\frac{\partial\theta}{\partial p}$

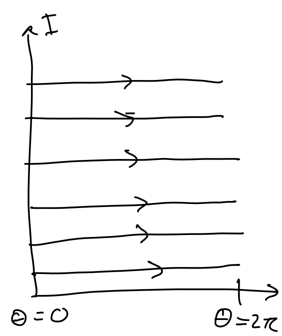
$= \left(\frac{m\omega p}{p^2 + m^2\omega^2x^2}\right)\left(\frac{p}{m\omega}\right) - (m\omega x)\left(\frac{-m\omega x}{p^2+m^2\omega^2x^2}\right) = 1 \;\checkmark$

The new Hamiltonian is:

$H(\theta, I) = \frac{1}{2m}\left(\sqrt{2Im\omega}\cos\theta\right)^2 + \frac{1}{2}m\omega^2\left(\sqrt{\frac{2I}{m\omega}}\sin\theta\right)^2 = \omega I$

Very simple! EOM:

$\dot\theta = \frac{\partial H}{\partial I} = \omega$ and $\dot I = -\frac{\partial H}{\partial\theta} = 0$



Finding the right phase-space coordinates makes everything better.

Canonical Transformations, Types 2, 3, 4

Recall $F_1(q_n, Q_n, t) \Rightarrow p_n = \frac{\partial F_1}{\partial q_n}, \quad P_n = -\frac{\partial F_1}{\partial Q_n}, \quad \tilde{H} = H + \frac{\partial F_1}{\partial t}$

That's "type 1."

Type 2: $F_2(q_n, P_n, t) \Rightarrow p_n = \frac{\partial F_2}{\partial q_n}, \quad Q_n = \frac{\partial F_2}{\partial P_n}, \quad \tilde{H} = H + \frac{\partial F_2}{\partial t}$

Type 3: $F_3(p_n, Q_n, t) \Rightarrow q_n = -\frac{\partial F_3}{\partial p_n}, \quad P_n = -\frac{\partial F_3}{\partial Q_n}, \quad \tilde{H} = H + \frac{\partial F_3}{\partial t}$

Type 4: $F_4(p_n, P_n, t) \Rightarrow q_n = -\frac{\partial F_4}{\partial p_n}, \quad Q_n = \frac{\partial F_4}{\partial P_n}, \quad \tilde{H} = H + \frac{\partial F_4}{\partial t}$

All of these preserve Poisson brackets:

$$\{Q_i, Q_j\} = 0 \qquad \{P_i, P_j\} = 0 \qquad \{Q_i, P_j\} = \delta_{ij}$$

and obey EOM: $\dot{Q}_n = \frac{\partial \tilde{H}}{\partial P_n}$ and $\dot{P}_n = -\frac{\partial \tilde{H}}{\partial Q_n}$.

There are other more complicated types, too.

All can be found by adding $\frac{d}{dt}F$ to Hamilton's principle.