Some inertia tensors

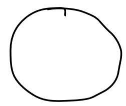
1) Any mass distribution constrained to lie on z-axis:  $\overleftrightarrow{I} = I \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

For a stick of mass M, length L, $I = \frac{ML^2}{12}$ about COM, $\frac{ML^2}{3}$ about end.

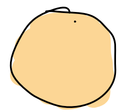
2) Any lamina in the xy plane:

$$\overleftrightarrow{I} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{xx}+I_{yy} \end{pmatrix} \leftarrow \text{principal axes}$$

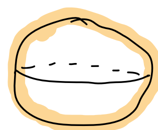
3) Hoop, radius R: $MR^2 \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$



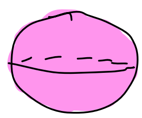
4) Disk, radius R: $MR^2 \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$



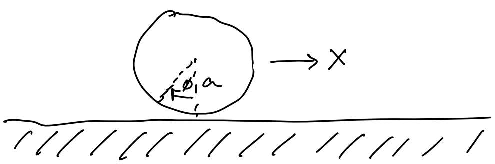
5) Spherical shell, $\frac{2}{3} MR^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



6) Solid sphere $\frac{2}{5} MR^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



Billiard ball slipping $\rightarrow$ rolling



If sliding, $\vec{F}_f = -\mu g M$

coefficient of friction

opposite motion of point of contact.

So $M\ddot{x} = -\mu g M \Rightarrow \boxed{\ddot{x} = -\mu g}$

Also, $\frac{d}{dt} L_z = \underbrace{I}_{\frac{2}{5}Ma^2} \ddot{\phi} = \text{Torque} = F_f a = \mu g M a$

about COM, out of page

So $\boxed{\ddot{\phi} = \frac{5}{2} \frac{\mu g}{a}}$

At $t=0$, strike ball in center: $\left.\begin{matrix} x=0, & \dot{x}=v_0 \\ \phi=0, & \dot{\phi}=0 \end{matrix}\right\}$ at $t=0$.

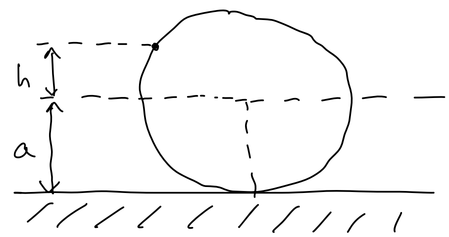
Then at later times: $\begin{cases} \dot{x} = v_0 - \mu g t \\ \dot{\phi} = \frac{5}{2}\frac{\mu g t}{a} \end{cases}$

<u>When does sliding stop and rolling begin?</u> Need $\dot{x} = a\dot{\phi}$, or

$$v_0 - \mu g t = \frac{5}{2}\frac{\mu g t}{a} \implies \boxed{t = \frac{2v_0}{7\mu g}.}$$

After ball starts rolling, sliding stops, $F_f = 0$.

<u>What if we strike ball above center?</u>



"Strike" means <u>impulse</u> $= \int_{-\Delta t}^{0} dt\, F_x(t) = M v_0$ after strike.

Torque $= \frac{dL}{dt} = h F_x$. So <u>torque impulse</u> $= \int_{-\Delta t}^{0} dt\; h\, F_x(t) = M v_0 h$

$= \Delta L = I\omega_0$. So, at time $t=0$,

$\begin{cases} x=0, & \dot{x} = v_0 \\ \phi = 0, & \dot{\phi} = \omega_0 \end{cases}$ at $t=0$, where $\omega_0 = \frac{M v_0 h}{I} = \frac{5}{2}\frac{h v_0}{a^2}$.

To make ball <u>immediately</u> roll, need $\dot{x} = a\dot{\phi}$, so $v_0 = \frac{5 h v_0}{2a}$,

or $h = \frac{2}{5}a$.

<u>If $h < \frac{2}{5}a$,</u> solve $\ddot{x} = -\mu g$ and $\ddot{\phi} = \frac{5}{2}\frac{\mu g}{a}$, with initial conditions $\left(\dot{x} = v_0 \quad \dot{\phi} = \frac{5 h v_0}{2a^2}\right)$ at $t=0$, so

$\begin{cases} \dot{x} = v_0 - \mu g t \\ \dot{\phi} = \frac{5}{2a}\left(\frac{h v_0}{a} + \mu g t\right) \end{cases}$ Rolling starts when $\dot{x} = a\dot{\phi}$, so

$$v_0 - \mu g t = \frac{5 h v_0}{2a} + \frac{5}{2}\mu g t \implies t = \frac{2v_0}{7\mu g}\left(1 - \frac{5h}{2a}\right)$$

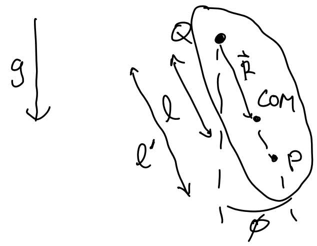
If $h > \frac{2}{5}a$, then $a\dot\phi > \dot x$, ball rolls faster than it slides, so force is in positive x direction (until rolling starts). So

$$\left.\begin{aligned}\dot x &= v_0 + \mu g t\\ \dot\phi &= \frac{5}{2a}\left(\frac{hv_0}{a} - \mu g t\right)\end{aligned}\right\}$$

continues until $v_0 + \mu g t = \frac{5}{2}\left(\frac{hv_0}{a} - \mu g t\right)$, so $t = \frac{2v_0}{\mu g}\left(\frac{5h}{2a} - 1\right)$ to start rolling.

Reversible (Kater's) Pendulum

Rigid body constrained to rotate about one axis $\hat y \leftarrow$ fixed in both inertial, body frames.



Torque about Q: $M\vec R \times g = -Mg\ell\sin\phi$

$$\frac{d}{dt}(I_Q\dot\phi) = I_Q\ddot\phi$$

$$\text{So}\quad \ddot\phi = -\frac{\overbrace{Mg\ell}^{\Omega^2}}{I_Q}\sin\phi$$

Parallel axis thm: $I_Q = \underbrace{I_{COM}}_{\text{call this } Ma^2\text{, defines } a} + M\ell^2 = M(a^2+\ell^2)$

So $\Omega_Q^2 = \frac{g\ell}{a^2+\ell^2} = \frac{g}{\ell'}$ where $\ell' = \ell + \frac{a^2}{\ell}$.

Define point P at distance ℓ' from Q, on line through COM.

What if we suspend body from P instead?

Then $I_P = I_{COM} + M(\ell'-\ell)^2 = M(a^2 + (\ell'-\ell)^2)$.

So $\ddot\phi = -\frac{Mg(\ell'-\ell)}{M(a^2+(\ell'-\ell)^2)}\sin\phi$, so $\Omega_P^2 = \frac{g(\ell'-\ell)}{a^2+(\ell'-\ell)^2}$

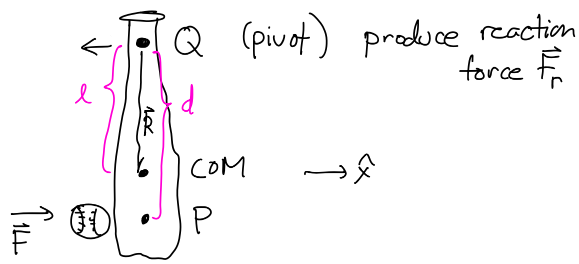
$$= \frac{g\,a^2/\ell}{a^2 + \left(\frac{a^2}{\ell}\right)^2} = \frac{g\,a^2/\ell}{a^2(\ell^2+a^2)/\ell^2} = \frac{g\ell}{\ell^2+a^2} = \Omega_Q^2$$

So, adjust ℓ' to make $\Omega_P = \Omega_Q$, then $\boxed{g = \ell' \Omega^2}$

Don't have to know where COM is!

Used for high accuracy maps of g in 19th century.

Baseball bat sweet spot



$$M\ddot{\vec{R}} = (F - F_R)\,\hat{x} \quad \Rightarrow$$

$$M\ell\ddot{\phi} = F - F_R$$

Also $\dfrac{dL}{dt} = \underbrace{I}_{M(\ell^2+a^2)}\ddot{\phi} = Fd$

So $\ddot{\phi} = \dfrac{F - F_R}{M\ell} = \dfrac{Fd}{M\underbrace{(\ell^2+a^2)}_{\ell'\ell}}$

So $1 - \dfrac{F_R}{F} = \dfrac{d}{\ell'} \quad \Rightarrow \quad \dfrac{F_R}{F} = 1 - \dfrac{d}{\ell'}$.

For no reaction force, hit the ball at $d = \ell'$ = point P
= <u>center of percussion</u>.

Euler's Equations

Rigid body body, subject to a torque $\vec{\Gamma}^{ext}$.

$$\left(\frac{d\vec{L}}{dt}\right)_{inertial} = \vec{\Gamma}^{ext}.$$

Valid if: 1) origin is fixed point
2) origin if COM.

Also, $\left(\dfrac{d}{dt}\right)_{inertial} = \left(\dfrac{d}{dt}\right)_{body} + \vec{\omega}\times$ acting on vectors.

Apply this to $\vec{L}$: $\boxed{\left(\dfrac{d\vec{L}}{dt}\right)_{body} = \vec{\Gamma}^{ext} - \vec{\omega}\times\vec{L}}$

Choose body-fixed frame along <u>principal axes</u> $\hat{e}_i$ $i = 1, 2, 3$.

then $I_{ij} = I_i\,\delta_{ij}$ so $\overleftrightarrow{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$. Now write:

$\vec{\Gamma}^{ext} = \hat{e}_i \Gamma_i^{ext}$ and $\vec{L} = \hat{e}_i L_i$ and $\vec{\omega} = \hat{e}_i \omega_i$

(sums over $i = 1, 2, 3$ implicit).

Now $L_i = I_{ij}\omega_j = I_i \omega_i$ (no sum on i)

So $$\frac{d\vec{L}}{dt} = \hat{e}_i I_i \frac{d\omega_i}{dt} = \hat{e}_1\left(I_1 \frac{d\omega_1}{dt}\right) + \hat{e}_2\left(I_2\frac{d\omega_2}{dt}\right) + \hat{e}_3\left(I_3\frac{d\omega_3}{dt}\right)$$

Also $$\vec{\omega}\times\vec{L} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix} = \hat{e}_1\,\omega_2\omega_3(I_3 - I_2) + \hat{e}_2\,\omega_1\omega_3(I_1 - I_3) + \hat{e}_3\,\omega_1\omega_2(I_2 - I_1)$$

So:
$$\boxed{\begin{aligned} I_1\frac{d\omega_1}{dt} &= \omega_2\omega_3(I_2 - I_3) + \Gamma_1^{ext} \\ I_2\frac{d\omega_2}{dt} &= \omega_1\omega_3(I_3 - I_1) + \Gamma_2^{ext} \\ I_3\frac{d\omega_3}{dt} &= \omega_1\omega_2(I_1 - I_2) + \Gamma_3^{ext} \end{aligned}}$$

These are rates of change in <u>body frame</u>.

Solve for $\omega_i(t)$.

Torque-free Motion

$\Gamma_i^{ext} = 0$ for $i = 1, 2, 3$

Conserved quantities: $E = T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$

This tells us that $\omega_1, \omega_2, \omega_3$ are <u>bounded</u>.

$\vec{L}$ not conserved in body frame; rotates.

But $\vec{L}\cdot\vec{L} = \vec{L}_{inertial}\cdot\vec{L}_{inertial}$, so

$$L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

Again, $\omega_1, \omega_2, \omega_3$ <u>bounded</u>. (Not immediately obvious from differential eqns!)

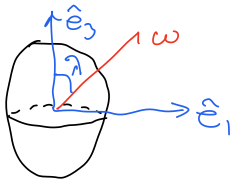
Symmetric top (no gravity) $\quad I_1 = I_2 \neq I_3$ (Earth, football)

Euler's eqns become:
$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_1 - I_3)$$
$$I_2 \dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I_1)$$
$$I_3 \dot{\omega}_3 = 0$$

so $\omega_3 = \text{constant}$!

So
$$\left.\begin{aligned} \dot{\omega}_1 &= -\Omega \omega_2 \\ \dot{\omega}_2 &= \Omega \omega_1 \\ \dot{\omega}_3 &= 0 \end{aligned}\right\} \quad \text{where } \Omega = \omega_3 \frac{(I_3 - I_1)}{I_3} = \text{constant}$$

Similar to Foucault pendulum equations. Start with



$$\left.\begin{aligned} \omega_1 &= \omega \sin\lambda \\ \omega_2 &= 0 \\ \omega_3 &= \omega \cos\lambda \end{aligned}\right\} \text{ at } t = 0.$$

Then
$$\left\{\begin{aligned} \omega_1 &= \omega \sin\lambda \cos(\Omega t) \\ \omega_2 &= \omega \sin\lambda \sin(\Omega t) \\ \omega_3 &= \omega \cos\lambda \end{aligned}\right.$$

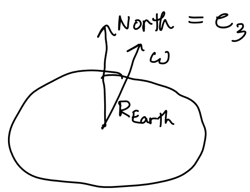
In the body frame, $\vec{\omega}$ precesses about $\hat{e}_3$ axis

Chandler wobble

Earth has $\frac{I_3 - I_1}{I_1} \approx \frac{1}{305}$. So

$$\Omega \approx \frac{\omega}{305} = \frac{2\pi/\text{day}}{305} \Rightarrow T_{\text{wobble}} \approx 305 \text{ days}$$

But λ is really small.



$$\lambda = \frac{(\text{few meters})}{R_{\text{Earth}}}$$

has varied by a factor of ~ 3 in last 100 years.

Earth not a rigid body.

Observed ≈ 435 days

Asymmetric Top

I_1, I_2, I_3 all different. Suppose $\vec{\omega}$ is very close to one of the axes (call it $\hat{e}_3$). Write

$\omega_1 = \eta_1$, $\omega_2 = \eta_2$, $\omega_3 = \omega_0 + \eta_3$, where η_i all small, $\omega_0 =$ constant

Expand to linear order in the η_i:

$$\left.\begin{aligned} I_1\dot{\eta}_1 &= (I_2 - I_3)\omega_0\eta_2 \\ I_1\dot{\eta}_2 &= (I_3 - I_1)\omega_0\eta_1 \end{aligned}\right\} \quad \ddot{\eta}_1 = -\Omega^2\eta_1 \quad \text{and} \quad \ddot{\eta}_2 = -\Omega^2\eta_2$$

$$\boxed{\Omega^2 = \omega_0^2\frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}}$$

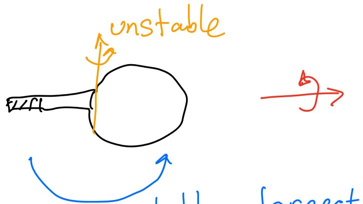
$$I_1\dot{\eta}_3 = (I_1 - I_2)\cancel{\eta_1\eta_2} = 0 \quad (\text{So } \eta_3 = \text{constant})$$

Small oscillations are stable is $\Omega^2 > 0$.

Runaway instability if $\Omega^2 < 0$, $\quad \eta_1, \eta_2 \approx e^{|\Omega| t}$.

Intermediate Axis Theorem = Tennis Racket Theorem:

Instabity occurs if the axis of rotation I_3 is between I_1 and I_2.



Youtube: Veritasium tennis racket

Is the Earth's rotation axis unstable? No. Rotation $\vec{\omega}$ is close to principal axis $\hat{e}_{\text{North}}$ with largest I.

Reason: tidal effects → heat (and other energy dissipation).

$$\text{Energy} = T + (\text{heat}) = \text{constant}$$

grows in time

minimize $\frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$ while conserving $L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$.

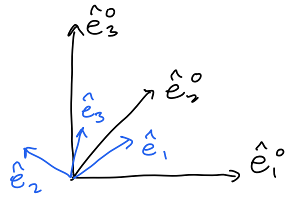
Solution: $\omega_i \neq 0$ only for largest I_i.

Astrophysical bodies (planets, stars, larger asteroids, galaxies) tend to rotate about largest I because:

(a) Centrifugal forces deform them
(b) Tidal effects and energy dissipation $\Big\}$ not rigid
(c) $\vec{\omega} \approx \hat{e}_{\text{largest } I}$ is stable even for rigid bodies

Euler angles for rigid body in inertial frame.

(Several different conventions.)



Specify rotation to get body-fixed $\hat{e}_1\ \hat{e}_2\ \hat{e}_3$ from inertial axes $\hat{e}_1^{\,o}\ \hat{e}_2^{\,o}\ \hat{e}_3^{\,o}$.

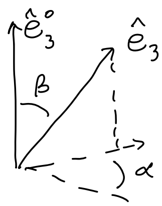
First, rotate by angle α about $\hat{e}_3^{\,o}$ (counterclockwise).

Next, rotate by angle β about new $\hat{e}_2$ = "line of nodes".

Finally, rotate by angle γ about new $\hat{e}_3$.

Now α, β, γ = functions of time, independent

Facts:



Spherical coordinates of $\hat{e}_3$ in the inertial frame:

$$(\theta, \phi) = (\beta, \alpha).$$

γ = additional rotation about new $\hat{e}_3$

$0 \leq \alpha < 2\pi \qquad 0 \leq \beta \leq \pi \qquad 0 \leq \gamma \leq 2\pi.$

$$T = \tfrac{1}{2}\left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2\right) \quad \text{where} \quad (\text{F+W p } 156)$$

$$\omega_1 = -\dot{\alpha} \sin\beta \cos\gamma + \dot{\beta} \sin\gamma$$

$$\omega_2 = \dot{\alpha} \sin\beta \sin\gamma + \dot{\beta} \cos\gamma$$

$$\omega_3 = \dot{\alpha} \cos\beta + \dot{\gamma}$$

Note: if $V = 0$, then α = cyclic, $p_\alpha = \dfrac{\partial L}{\partial \dot{\alpha}}$ = constant

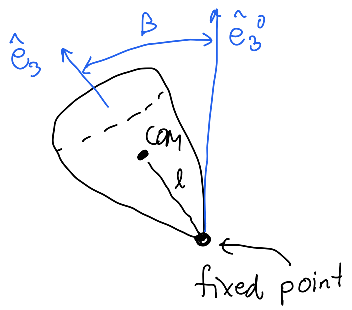
Symmetric Top $I_1 = I_2$

Plug ω's into T, use $\sin^2 + \cos^2 = 1$,

$$T = \tfrac{1}{2} I_1 (\dot\alpha^2 \sin^2\beta + \dot\beta^2) + \tfrac{1}{2} I_3 (\dot\alpha \cos\beta + \dot\gamma)^2$$

Both α, γ are cyclic, $p_\alpha = \frac{\partial L}{\partial \dot\alpha} = \text{constant}$ $\quad p_\gamma = \frac{\partial L}{\partial \dot\gamma} = \text{constant}$

Put into uniform gravitational field:



$$V = Mgl\cos\beta$$ ← α, γ still cyclic.

$$p_\gamma = I_3(\dot\alpha\cos\beta + \dot\gamma) = \text{constant}$$
$$= I_3 \omega_3$$

$$p_\alpha = I_1 \dot\alpha \sin^2\beta + \cos\beta\, I_3 (\dot\alpha\cos\beta + \dot\gamma) = I_1 \dot\alpha \sin^2\beta + \cos\beta\, p_\gamma = \text{constant}$$

So $\displaystyle \dot\alpha = \frac{p_\alpha - p_\gamma \cos\beta}{I_1 \sin^2\beta}$ $\qquad \displaystyle \dot\gamma = p_\gamma \left(\frac{1}{\tan^2\beta\, I_1} + \frac{1}{I_3} \right) - \frac{\cos\beta}{I_1 \sin^2\beta} p_\alpha$

If we know $\alpha(0)$, $\gamma(0)$, and $p_\alpha, p_\gamma = \text{constants}$, then $\dot\alpha$, $\dot\gamma$ depend only on $\beta(t)$.

EOM for β: $\displaystyle \frac{d}{dt}\frac{\partial L}{\partial \dot\beta} = I_1 \ddot\beta$

$$\frac{\partial L}{\partial \beta} = I_1 \sin\beta\cos\beta\, \dot\alpha^2 + I_3(\dot\alpha\cos\beta + \dot\gamma)\,\dot\alpha(-\sin\beta) + Mgl\sin\beta$$

Plug in for $\dot\alpha$, $\dot\gamma$ in terms of p_α, p_γ, simplify:

$$I_1 \ddot\beta = \frac{\cos\beta}{I_1 \sin^3\beta}\left(p_\alpha^2 - 2 p_\alpha p_\gamma \cos\beta + p_\gamma^2\right) - \frac{p_\alpha p_\gamma}{I_1 \sin\beta} + Mgl\sin\beta.$$

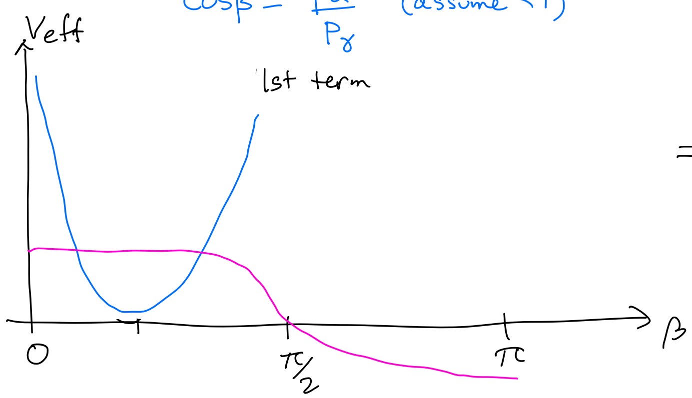
Solve for $\beta(t)$, plug in to get $\dot\alpha$, $\dot\gamma$, solve for $\alpha(t)$, $\gamma(t)$.

Effective potential version: $\quad E = T + V = \text{constant}$

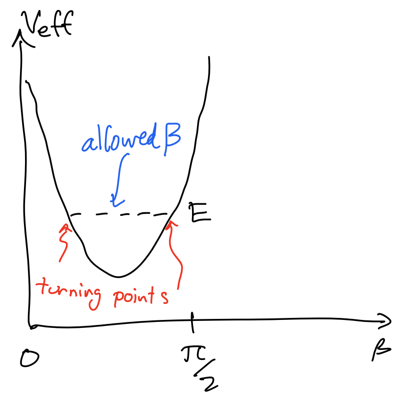
$$E = \tfrac{1}{2} I_1 \dot{\beta}^2 + V_{eff}(\beta)$$ ← everything without $\dot{\beta}$. Part from V, part from T,

$$V_{eff}(\beta) = \underbrace{\frac{(p_\alpha - p_\gamma \cos\beta)^2}{2 I_1 \sin^2\beta}}_{\text{local minimum at}} + Mgl\cos\beta + \frac{p_\gamma^2}{2I_3}$$ ← constant

$\cos\beta = \dfrac{p_\alpha}{p_\gamma}$ (assume <1)



$\Rightarrow$



Sleeping top $\quad \beta = 0$. Then $p_\alpha = p_\gamma$ is forced. Is it stable?

$$\text{So } V_{eff}(\beta) = \frac{p_\gamma^2}{2I_1}\frac{(1-\cos\beta)^2}{\sin^2\beta} + Mgl\cos\beta + \text{constant}$$

$$1 - \cos\beta = 1 - \left(1 - \frac{\beta^2}{2} + \frac{\beta^4}{24} + \dots\right) \quad \text{So } (1-\cos\beta)^2 = \frac{\beta^4}{4} - \frac{\beta^6}{24} + \dots$$

$$\sin^2\beta = \left(\beta - \frac{\beta^3}{6} + \dots\right)^2 = \beta^2 - \frac{\beta^4}{6} + \dots$$

$$\text{So } V_{eff}(\beta) = \frac{p_\gamma^2}{2I_1}\left(\frac{\beta^2}{4}\right) + Mgl\left(1 - \frac{\beta^2}{2}\right) + \text{constant}$$

$$= \left(\frac{p_\gamma^2}{8I_1} - \frac{Mgl}{2}\right)\beta^2 + \text{constant}$$

So, for small β, $\quad I_1 \ddot{\beta} = \left(-\dfrac{p_\gamma^2}{4I_1} + Mgl\right)\beta$

If $p_\gamma = 0$, falls over. $\quad$ Stability, $\boxed{\dfrac{p_\gamma^2}{4I_1} > Mgl}$

Small oscillations about equilibrium β_0

$$\beta = \beta_0 + \eta(t)$$

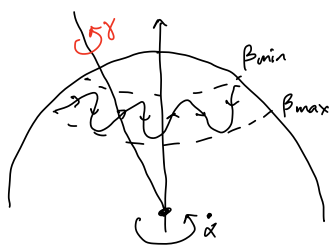
$$V_{eff} = V_{eff}(\beta_0) + \eta \underbrace{\left.\frac{\partial V_{eff}}{\partial \beta}\right|_{\beta=\beta_0}}_{=0,\ \text{solve for } \beta_0} + \frac{1}{2}\eta^2 \underbrace{\left.\frac{\partial^2 V_{eff}}{\partial \beta^2}\right|_{\beta=\beta_0}}_{\text{stable if } > 0}$$

$$\left.\frac{\partial V_{eff}}{\partial \beta}\right|_{\beta=\beta_0} = 0 \rightarrow \boxed{\left(p_\gamma^2 - 2p_\alpha p_\gamma \cos\beta_0 + p_\alpha^2\right)\cos\beta_0 - \left(p_\alpha p_\gamma - I_1 Mg\ell \sin^2\beta_0\right)\sin^2\beta_0 = 0}$$

Using $\sin^2\beta_0 = 1-\cos^2\beta_0$, this is a quartic equation.

Then
$$\left.\frac{\partial^2 V_{eff}}{\partial \beta^2}\right|_{\beta=\beta_0} = \frac{p_\alpha p_\gamma - I_1 Mg\ell\left(4-3\sin^2\beta_0\right)}{I_1 \cos\beta_0}.$$

$$I_1 \ddot{\eta} = -\left.\left(\frac{\partial^2 V_{eff}}{\partial \beta^2}\right)\right|_{\beta=\beta_0} \eta \Rightarrow \Omega^2 = \frac{p_\alpha p_\gamma - I_1 Mg\ell\left(4-3\sin^2\beta_0\right)}{I_1^2 \cos\beta_0} \underset{\text{stability}}{>} 0$$



Nutation: $\beta(t)$ oscillates between turning points

Precession: $\alpha(t)$ increases but not monotonically

Recall $\dot\alpha = \dfrac{p_\alpha - p_\gamma\cos\beta}{I_1\sin^2\beta}$, expand in η:

$$\dot\alpha(t) = \underbrace{\left(\frac{p_\alpha - p_\gamma\cos\beta_0}{I_1\sin^2\beta_0}\right)}_{\text{main precession}} + \underbrace{\eta(t)\left.\left[\frac{\partial}{\partial\beta}\left(\frac{p_\alpha - p_\gamma\cos\beta}{I_1\sin^2\beta}\right)\right]\right|_{\beta=\beta_0}}_{\text{variation in precession } \propto \cos(\Omega t+\delta) \text{ from } \eta.}$$

Fast top Go back to equation of motion for β_0, take limit $\dfrac{p_\gamma}{p_\alpha} \to \infty$.

$p_\gamma^2 \cos\beta_0 \approx 0$, so $\beta_0 \approx \frac{\pi}{2}$, $\cos\beta_0$ small.

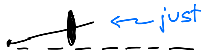
To next order: $p_\gamma^2\cos\beta_0 - p_\gamma p_\alpha \underbrace{\sin^2\beta_0}_{1} \approx 0$, so $\cos\beta_0 \approx \dfrac{p_\alpha}{p_\gamma} \ll 1$.

Try $\cos\beta_0 = \frac{p_\alpha}{p_\gamma} + \epsilon$, $\sin^2\beta_0 = 1 - \left(\frac{p_\alpha}{p_\gamma}+\epsilon\right)^2$. Expand to linear order in ϵ:

$\epsilon = {-I_1 Mg\ell}/{p_\gamma^2}$

Then mean precession rate is $\dot{\alpha}\Big|_{\beta=\beta_0} = \dfrac{p_\alpha - p_\gamma \cos\beta_0}{I_1 \sin^2\beta_0} \approx \dfrac{p_\alpha - p_\gamma\left(\frac{p_\alpha}{p_\gamma} - \frac{I_1 M g l}{p_\gamma^2}\right)}{I_1}$, or

$\dot{\alpha} \approx \dfrac{I_1 M g l}{p_\gamma}$ very small.



⇐ just above horizontal, precesses slowly

Frequency of nutation: $\Omega = \dfrac{p_\alpha p_\gamma \cancel{- I_1 M g l (4 - 3\sin^2\beta_0)}}{I_1^2 \underbrace{\cos\beta_0}_{p_\alpha/p_\gamma}} \approx \dfrac{p_\gamma^2}{I_1^2}$

Very rapid nutation, slow precession, nearly horizontal.