Stability & Small Oscillations

Given $L(q_k, \dot{q}_k) = L(q_1, q_2, \cdots q_N, \dot{q}_1, \cdots \dot{q}_N)$, may find static equilibrium solutions:

$$q_k(t) = q_{k0} = \text{constants}$$
$$\dot{q}_k = 0, \quad \ddot{q}_k = 0.$$

Look for "nearby" solutions = perturbations, such that

$$q_k(t) = q_{k0} + \eta_k(t) \longleftarrow \text{small}$$
$$\dot{q}_k(t) = 0 + \dot{\eta}_k(t).$$

Suppose $L = T - V$, where $T = \frac{1}{2} M_{jk}(q_n)\, \dot{q}_j \dot{q}_k$ and $V = V(q_n)$. Expand in small η_n:

$$T = \frac{1}{2}\Big(\underbrace{M_{jk}(q_{n0})}_{\substack{\text{constant,}\\ \text{call this}\\ m_{jk}}} + \underbrace{\eta_n \frac{\partial M_{jk}}{\partial q_n}\Big|_{q_i = q_{i0}}}_{\text{neglect, 3 } \eta\text{'s}} + \underbrace{\frac{1}{2}\eta_n \eta_m \frac{\partial^2 M_{jk}}{\partial q_n \partial q_m}\Big|_{q = q_{i0}} + \ldots}_{\text{neglect, 4 } \eta\text{'s.}}\Big)\dot{\eta}_j \dot{\eta}_k$$

$$V = \underbrace{V(q_{n0})}_{\substack{\text{Constant,}\\ \text{doesn't contribute}\\ \text{to eqns of motion}}} + \underbrace{\eta_j \frac{\partial V}{\partial q_j}\Big|_{q_i = q_{i0}}}_{\substack{=0, \text{ for}\\ \text{equilibrium}}} + \frac{1}{2}\eta_j \eta_k \underbrace{\frac{\partial^2 V}{\partial q_j \partial q_k}\Big|_{q_i = q_{i0}}}_{\text{constant, call it } V_{jk}} + \mathcal{O}(\eta^3)$$

So $L = \frac{1}{2} m_{jk} \dot{\eta}_j \dot{\eta}_k - \frac{1}{2} V_{jk} \eta_j \eta_k + \mathcal{O}(\eta^3)$.

In matrix form,

$$L = \frac{1}{2}(\dot{\eta}_1\ \dot{\eta}_2 \cdots \dot{\eta}_N) \underbrace{\begin{pmatrix} \\ \underset{\sim}{m} \\ \\ \end{pmatrix}}_{\text{symmetric}} \begin{pmatrix} \dot{\eta}_1 \\ \vdots \\ \dot{\eta}_N \end{pmatrix} - \frac{1}{2}(\eta_1 \cdots \eta_N) \underbrace{\begin{pmatrix} \\ \underset{\sim}{V} \\ \\ \end{pmatrix}}_{\text{symmetric}} \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_N \end{pmatrix}$$

Equations of motion: $\frac{\partial L}{\partial \dot{\eta}_j} = m_{jk}\dot{\eta}_k$ $\qquad \frac{\partial L}{\partial \eta_j} = -V_{jk}\eta_k$

So $\frac{d}{dt}\left(m_{jk}\dot{\eta}_k\right) = -V_{jk}\eta_k \implies \boxed{m_{jk}\ddot{\eta}_k = -V_{jk}\eta_k .}$

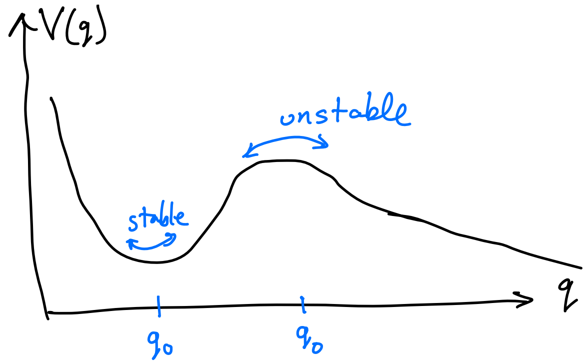
Matrix form:
$$\begin{pmatrix} & & \\ & \underset{\sim}{m} & \\ & & \end{pmatrix}\begin{pmatrix}\ddot{\eta}_1 \\ \ddot{\eta}_2 \\ \vdots \\ \ddot{\eta}_N\end{pmatrix} = -\begin{pmatrix} & & \\ & \underset{\sim}{V} & \\ & & \end{pmatrix}\begin{pmatrix}\eta_1 \\ \vdots \\ \eta_N\end{pmatrix}$$

If η_j remains small for all time $\implies$ stable

" " starts small and becomes large $\implies$ unstable.

For $N=1$, might have:

two equilibrium points, in this case one stable, one unstable.



$m\ddot{\eta} = -V\eta \implies \ddot{\eta} = -\frac{V}{m}\eta .$ Try $\eta = \eta_0 e^{\lambda t}$, take real part later.

So $\lambda^2 \cancel{\eta_0 e^{\lambda t}} = -\frac{k}{m}\cancel{\eta_0 e^{\lambda t}} \implies \lambda = \pm\sqrt{\frac{-k}{m}}$.

<u>Stable oscillations</u> if $\frac{V}{m} > 0$, $\quad \eta = \eta_{0+}e^{i\omega t} + \eta_{0-}e^{-i\omega t}$,

where $\omega = \sqrt{\frac{V}{m}}$.

<u>Unstable mode</u> if $\frac{V}{m} < 0$, $\quad \eta = \eta_{0+}e^{\lambda t} + \eta_{0-}e^{-\lambda t}$

grows, violates assumption of small η

where $\lambda = \sqrt{\frac{-V}{m}}$.

Now look at general $N > 1$. Try for <u>normal modes</u>.

<u>Normal mode:</u> all η_i's oscillate at same ω.

$$\eta_j(t) = \underset{\text{complex!}}{\eta_{j0}} e^{i\omega t}. \qquad \text{Vector notation} \quad \begin{pmatrix} \eta_{10} \\ \eta_{20} \\ \vdots \\ \eta_{N0} \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}$$

So
$$\begin{pmatrix} \underset{\sim}{m} \end{pmatrix} \underbrace{(i\omega)^2}_{-\omega^2} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix} \cancel{e^{i\omega t}} = \begin{pmatrix} \underset{\sim}{V} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix} \cancel{e^{i\omega t}}, \quad \text{or}$$

$$\left[\begin{pmatrix} \underset{\sim}{V} \end{pmatrix} - \omega^2 \begin{pmatrix} \underset{\sim}{m} \end{pmatrix}\right] \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix} = 0. \quad \text{Algebraic equation.}$$

Matrix form $\underbrace{(\underset{\sim}{V} - \omega^2 \underset{\sim}{m})}_{\text{symmetric matrix}} \vec{z} = 0.$ Look for solution

with $\vec{z} \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, with some ω^2, given real matrices V, m.

<u>Claim:</u> The solutions for ω^2 must have $\mathrm{Det}(\underset{\sim}{V} - \omega^2 \underset{\sim}{m}) = 0$.

<u>Proof:</u> if $\mathrm{Det}\,\underset{\sim}{A} \neq 0$, then the inverse matrix $\underset{\sim}{A}^{-1}$ exist, such that $\underset{\sim}{A}^{-1}\underset{\sim}{A} = I$. So, if $\mathrm{Det}(\underset{\sim}{V} - \omega^2 \underset{\sim}{m}) \neq 0$, then

$$(\underset{\sim}{V} - \omega^2 \underset{\sim}{m})^{-1} (\underset{\sim}{V} - \omega^2 \underset{\sim}{m}) \vec{z} = 0 \implies \vec{z} = 0.$$

So, look for solutions to $\mathrm{Det}(\underset{\sim}{V} - \omega^2 \underset{\sim}{m}) = 0$. This is a polynomial in ω^2 of degree N.

Fundamental Theorem of Algebra: there are N solutions, but some may be repeated.

Claim: all solutions must have real ω^2 (but possibly $\omega^2<0$).

Proof: Start with $\left(V_{jk} - \omega^2 m_{jk}\right) z_k = 0$,

multiply by z_j^*, sum over j:

$$\omega^2 \underbrace{\left(z_j^* m_{jk} z_k\right)}_{\text{real}} = \underbrace{\left(z_j^* V_{jk} z_k\right)}_{\text{real}},$$

because $\left(z_j^* V_{jk} z_k\right)^* = \left(z_j V_{jk}^* z_k^*\right) = z_k^* V_{kj} z_j$

V is real symmetric

relabel indices $j \leftrightarrow k$

$= z_j^* V_{jk} z_k$

So $\omega^2 = \dfrac{\text{real}}{\text{real}} = \text{real}.$ ✓

Claim: general solution can be rewritten as

$$\eta_k(t) = \sum_{I=1}^{N} e^{i\omega_I t}\, C_I\, \eta_{Ik}$$

C_I: complex constants

η_{Ik}: real constants, forming N orthogonal vectors.

Proof: text, pages 92-97.

Notes: (1) some of the ω_I might be the same.

(2) some of the ω_I might be imaginary $(\omega_I^2<0)$
$\Leftrightarrow$ instability

(3) For stability of equilibrium, need <u>all</u> ω_I $(I=1,\dots,N)$ real

(4) Orthogonality: $\displaystyle\sum_k \eta_{Ik}\eta_{Jk} = \delta_{IJ} = \begin{cases} 1 & \text{if } I=J \\ 0 & \text{if } I\neq J. \end{cases}$

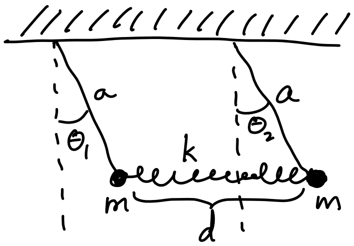
(5) To find normal modes:

(a) Solve $\mathrm{Det}\left(\underset{\sim}{V} - \omega^2 \underset{\sim}{m}\right) = 0$ for ω^2.

(b) For each $\omega^2 = \omega_I^2$, solve $\left(\underset{\sim}{V} - \omega_I^2 \underset{\sim}{m}\right)\vec{\eta} = 0$ for $\vec{\eta}$.

⑥ If $\omega_I^2 = 0$ is a solution, ("zero mode"), then stability is not known. (Keep higher order in η.)

Example: Coupled Pendulums



$$T = \tfrac{1}{2} m (a\dot{\theta}_1)^2 + \tfrac{1}{2} m (a\dot{\theta})^2$$

$$V_1 = -mga\cos\theta_1 \qquad V_2 = -mga\cos\theta_2$$

Potential energy of spring with constant is $\frac{1}{2}k(\Delta x)^2$, so

$V_{12} = \frac{1}{2}k\left[a\sin\theta_2 - a\sin\theta_1\right]^2$. Use small angle approximations:

$\cos\theta_1 \approx 1 - \frac{1}{2}\theta_1^2 + \dots \qquad \cos\theta_2 = 1 - \frac{1}{2}\theta_2^2 + \dots$

$\sin\theta_2 - \sin\theta_1 = \theta_2 - \theta_1$. So the total potential energy, up to an additive constant, is $V = -\frac{1}{2}mga(\theta_1^2 + \theta_2^2) - \frac{1}{2}ka^2(\theta_1 - \theta_2)^2$

Eqns of motion: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}_1} = ma^2\ddot{\theta}_1$; $\dfrac{\partial L}{\partial \theta_1} = -mga\theta_1 - ka^2(\theta_1 - \theta_2)$

$$\Rightarrow \boxed{\ddot{\theta}_1 = -\frac{g}{a}\theta_1 + \frac{k}{m}(\theta_2 - \theta_1)}, \qquad \text{and}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = ma^2\ddot{\theta}_2, \qquad \frac{\partial L}{\partial \theta_2} = -mga\theta_2 - ka^2(\theta_1 - \theta_2) \Rightarrow$$

$$\boxed{\ddot{\theta}_2 = -\frac{g}{a}\theta_2 + \frac{k}{m}(\theta_1 - \theta_2)} \qquad \text{So,}$$

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} -\frac{g}{a} - \frac{k}{m} & +\frac{k}{m} \\ +\frac{k}{m} & -\frac{g}{a} - \frac{k}{m} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

Try $\theta_k = e^{i\omega t}\theta_{k0}$,

$\ddot{} \rightarrow -\omega^2$

So
$$\begin{pmatrix} \frac{g}{a}+\frac{k}{m}-\omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{a}+\frac{k}{m}-\omega^2 \end{pmatrix} \begin{pmatrix} \theta_{10} \\ \theta_{20} \end{pmatrix} = 0$$

Require Det( ) = 0, or $\left(\omega^2 - \frac{g}{a} - \frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = 0 \Rightarrow$

This is a quadratic equation for ω^2. Instead of using quadratic formula,
$$\omega^2 - \frac{g}{a} - \frac{k}{m} = \pm\frac{k}{m} \Rightarrow \omega^2 = \frac{g}{a} \text{ or } \frac{g}{a} + \frac{2k}{m}$$

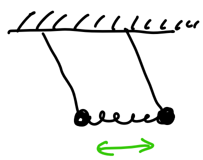
So $\omega_1 = \sqrt{\frac{g}{a}}$ and $\omega_2 = \sqrt{\frac{g}{a}+\frac{2k}{m}}$.

Mode 1 Mode 2

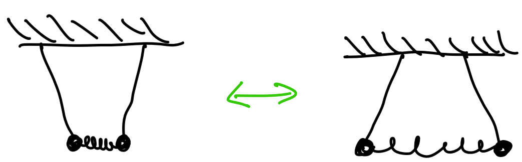
For Mode 1,
$$\begin{pmatrix} \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{pmatrix} \begin{pmatrix} \theta_{10} \\ \theta_{20} \end{pmatrix} = 0 \Rightarrow \boxed{\theta_{10} = \theta_{20}}$$

For Mode 2
$$\begin{pmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{pmatrix} \begin{pmatrix} \theta_{10} \\ \theta_{20} \end{pmatrix} = 0 \Rightarrow \boxed{\theta_{10} = -\theta_{20}}$$

Sketch the modes:



Mode 1: Spring does nothing. Pendula swing together, could have guessed $\omega = \sqrt{\frac{g}{a}}$, same as each separately



Mode 2: Spring compresses and expands, higher frequency of oscillation.

What is the general solution?

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \mathrm{Re}\left[\underbrace{C_1 e^{i\omega_1 t}}_{N_1 e^{i(\omega_1 t+\phi_1)}}\begin{pmatrix}1\\1\end{pmatrix} + \underbrace{C_2 e^{-i\omega_2 t}}_{N_2 e^{i(\omega_2 t+\phi_2)}}\begin{pmatrix}1\\-1\end{pmatrix}\right]$$

N_1 N_2 real.

So $$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}N_1\cos(\omega_1 t+\phi_1) + N_2\cos(\omega_2 t+\phi_2)\\ N_1\cos(\omega_1 t+\phi_1) - N_2\cos(\omega_2 t+\phi_2)\end{pmatrix}.$$

Solve for N_1, ϕ_1, N_2, ϕ_2 from initial conditions.

For example, suppose at $t=0$, $\begin{cases}\theta_1=\alpha, & \dot\theta_1 = 0\\ \theta_2 = 0, & \dot\theta_2 = 0.\end{cases}$

Then $$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \frac{\alpha}{2}\begin{pmatrix}\cos(\omega_1 t)+\cos(\omega_2 t)\\ \cos(\omega_1 t)-\cos(\omega_2 t)\end{pmatrix}.$$ Rewrite using

$\cos(u)+\cos(v) = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$ and

$\cos(u)-\cos(v) = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$. So:

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_2-\omega_1}{2}t\right)\\ \sin\left(\frac{\omega_2+\omega_1}{2}t\right)\sin\left(\frac{\omega_2-\omega_1}{2}t\right)\end{pmatrix}.$$

Now suppose $\frac{k}{m} \ll \frac{g}{a}$ (weak coupling). Then $\omega_2 = \sqrt{\frac{g}{a}+\frac{2k}{m}}$

$= \sqrt{\frac{g}{a}}\sqrt{1+\frac{2ka}{gm}} = \sqrt{\frac{g}{a}}\left(1+\frac{ka}{gm}+\dots\right) = \omega_1 + \Delta\omega$, where

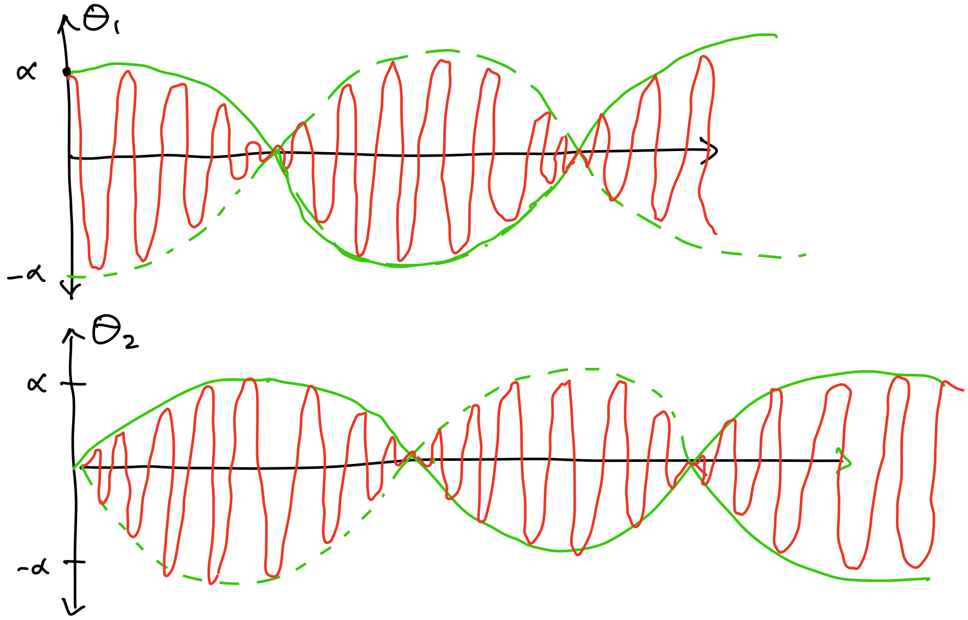
$\Delta\omega \approx \sqrt{\frac{g}{a}}\frac{ka}{gm} \ll \omega_1$. So

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\cos\left(\frac{\Delta\omega t}{2}\right)\cos(\omega_1 t)\\ \sin\left(\frac{\Delta\omega t}{2}\right)\sin(\omega_1 t)\end{pmatrix}$$

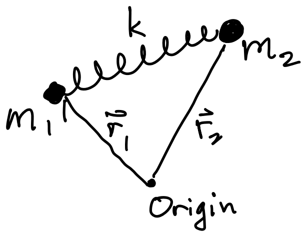
fast

slow

Sketch of motions:



Diatomic molecule Model with two masses m_1, m_2 connected by a spring with constant k and unstretched length d.



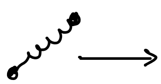
$$T = \frac{1}{2} m_1 \left(\frac{d\vec{r}_1}{dt}\right)^2 + \frac{1}{2} m_2 \left(\frac{d\vec{r}_2}{dt}\right)^2$$

$$V = \frac{1}{2} k \left(|\vec{r}_1 - \vec{r}_2| - d\right)^2$$

It's best to think ahead: count expected normal modes, and identify zero modes.

How many normal modes? Coordinates $x_1, y_1, z_1, x_2, y_2, z_2$, so $N=6$ normal modes.

Zero modes = no restoring force.

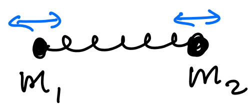
Three translation modes:  in orthogonal x, y, z directions

Two rotational modes:

Rotation about axis out of page, or about line in page perpendicular to spring.

(No rotational mode about axis between masses!)

<u>One</u> vibrational mode



Check: $6 = 3+2+\underline{\underline{1}}$ ✓

To find frequency of small oscillations, restrict to a line:

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k (x_2 - x_1 - d)^2$$

Equations of motion:
$$\begin{cases} m_1 \ddot{x}_1 = -k(x_1 - x_2 + d) \\ m_2 \ddot{x}_2 = -k(x_2 - x_1 - d) \end{cases}$$

Equilibrium solution: $x_1 = -d/2 \qquad x_2 = d/2$

So write $x_1 = -\frac{d}{2} + \eta_1 \qquad x_2 = +\frac{d}{2} + \eta_2$. Then

$$\begin{cases} m_1 \ddot{\eta}_1 = -k(\eta_1 - \eta_2) \\ m_2 \ddot{\eta}_2 = -k(\eta_2 - \eta_1) \end{cases} \Longrightarrow \begin{cases} -\omega^2 m_1 \eta_1 + k\eta_1 - k\eta_2 = 0 \\ -\omega^2 m_2 \eta_2 + k\eta_2 - k\eta_1 = 0 \end{cases}$$

or
$$\underbrace{\begin{pmatrix} k - \omega^2 m_1 & -k \\ -k & k - \omega^2 m_2 \end{pmatrix}}_{\text{Set Det} = 0} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0$$

$$\cancel{k^2} - \omega^2 (m_1 + m_2) k + (\omega^2)^2 m_1 m_2 - \cancel{k^2} = 0$$

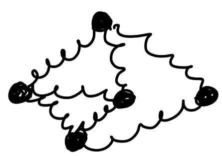
So
$$\omega^2 = \begin{cases} 0 & \longleftarrow \text{Translation in } x \text{ direction} \\ \dfrac{m_1 + m_2}{m_1 m_2} k & \longleftarrow \text{vibrational mode} \end{cases}$$

For vibrational mode: $k \begin{pmatrix} 1 - \frac{(m_1+m_2)m_1}{m_1 m_2} & -1 \\ -1 & 1 - \frac{(m_1+m_2)}{m_1 m_2} m_2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0,$

or $\begin{pmatrix} -\frac{m_1}{m_2} & -1 \\ -1 & -\frac{m_2}{m_1} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0 \Rightarrow m_1 \eta_1 = -m_2 \eta_2,$

or $\frac{\eta_1}{\eta_2} = -\frac{m_2}{m_1}$. The lighter mass moves more, this is motion about center-of-mass.

Vibrations in molecules with N atoms



Degrees of freedom: $3N$

Translations: 3

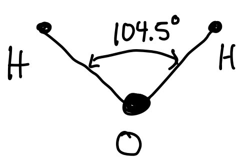
Rotations: 3 (unless collinear, then 2)

(No rotation about collinear axis.)

So: $3N-6$ vibrational modes if not linear

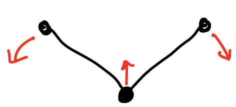
$3N-5$ " " if linear in equilibrium.

Example: water H_2O $N=3$, not linear in equilibrium.



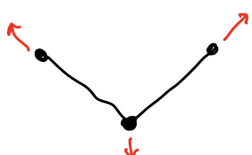
$\underbrace{\text{3 translational, 3 rotational modes}}_{\omega = 0}$

Bending



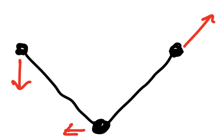
$\hbar\omega = 0.2046\,\text{eV}$

Stretching

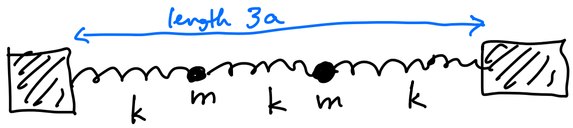


$\hbar\omega = 0.4754\,\text{eV}$

Asymmetric



$\hbar\omega = 0.4892\,\text{eV}$



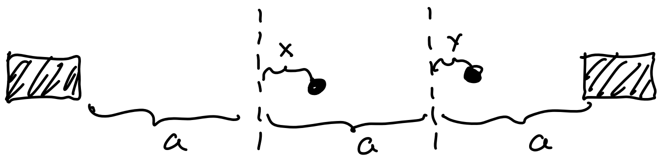
Example

3 equal springs, 2 equal masses, motion only in horizontal line.

spring constant k,
unstretched length d

Find normal mode frequencies and eigenvectors.

Choose coordinates x and y for the masses



$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2$$

$$V = \tfrac{1}{2}k(x+a-d)^2 + \tfrac{1}{2}(y+a-x-d)^2 + \tfrac{1}{2}(a-y-d)^2$$

$$= \tfrac{1}{2}k\big(x^2+a^2+d^2 + \cancel{2xa} - \cancel{2xd} - 2ad$$
$$+ y^2 + x^2 + a^2 + d^2 - 2yx + \cancel{2ya} - \cancel{2yd} - \cancel{2xa} + \cancel{2xd} - 2ad$$
$$+ a^2 + y^2 + d^2 - \cancel{2ya} + \cancel{2yd} - 2ad\big)$$

$$= \tfrac{1}{2}k\left(2x^2 + 2y^2 - 2xy + \text{constants}\right)$$

$$L = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 + k\left(x^2+y^2-xy\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \qquad \frac{\partial L}{\partial x} = k(2x-y) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y} \qquad \frac{\partial L}{\partial y} = k(2y-x).$$

So
$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \frac{k}{m}\begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \Rightarrow \underbrace{\begin{pmatrix}\frac{2k}{m}-\omega^2 & -\frac{k}{m}\\ -\frac{k}{m} & \frac{2k}{m}-\omega^2\end{pmatrix}}_{\text{Set Det}=0}\begin{pmatrix}x_0\\ y_0\end{pmatrix} = 0$$

$$\left(\frac{2k}{m}-\omega^2\right)^2 - \frac{k^2}{m^2} = 0 \Rightarrow$$

$$\omega^2 - \frac{2k}{m} = \pm\frac{k}{m}$$

So $\boxed{\omega^2 = \frac{k}{m}, \; \frac{3k}{m}}$ For $\omega^2 = \frac{k}{m}$, $\begin{pmatrix} \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$

$\Rightarrow x_0 = y_0$

For $\omega^2 = \frac{3k}{m}$, $\begin{pmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0 \Rightarrow x_0 = -y_0$.

So:

<u>Mode 1</u>

$\omega = \sqrt{\frac{k}{m}}$

<u>Mode 2</u>

$\omega = \sqrt{\frac{3k}{m}}$

Important lesson: could have guessed the normal mode vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ based on the symmetry of the problem. (Useful tip for a future homework...)