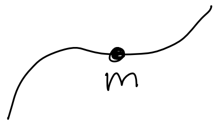
Constraints

What if not all N coordinates q_k are independent?

Example 1 Mass constrained to be on a curve:

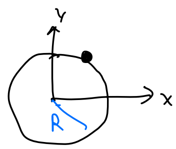


$$f(x,y,z)=0$$

Example 2 Mass constrained to be on a moving curve:

$$f(x,y,z,t)=0$$

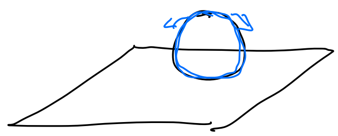
Example 3 Mass restricted to some region, say outside a sphere:



$$x^2+y^2+z^2 \geq R^2$$

(more generally, $f(x,y,z,t) \geq 0$, inequality).

Example 4 Coin rolling on a plane



Constraint involves velocities.

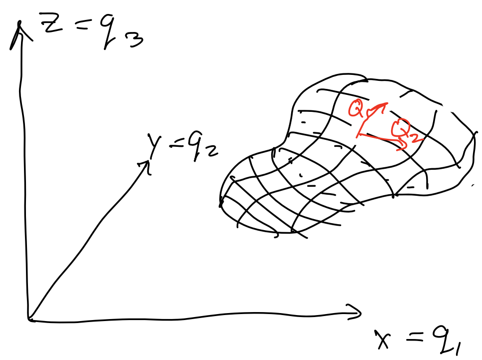
Holonomic constraints

$$f_\alpha(q_k, t) = 0 \qquad \alpha = 1, \dots, n$$

Note: no $\dot{q}_k$, no inequalities. Can depend on time.

If N coordinates q_k $(k=1,\dots,N)$ are related by n constraints $f_\alpha(q_k,t)=0$, there are $N-n$ degrees of freedom.

Think of configuration space as an $N-n$ dimensional abstract space. For $N=3$ and $n=1$, can picture it like this:



Many ways to choose q_k,
" " " " Q_J

On the surface of constraint,
$$L(q_k, \dot{q}_k, t) = L(Q_J, \dot{Q}_J, t)$$

If holonomic, can solve to get $Q_J = 1, \dots, N-n$.

Method 1: Use $L(Q_J, \dot{Q}_J, t)$, and eqns of motion are
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_J}\right) - \frac{\partial L}{\partial Q_J} = 0 \qquad J = 1, \dots, N-n.$$

(N−n equations, N−n unknowns.)

Method 2: Use $L' = L(q_k, \dot{q}_k, t) + \lambda_\alpha f_\alpha(q_k, t)$

new coordinates, no dynamics of their own. (Sum α)

Equations of motion:
$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_k}\right) - \dfrac{\partial L}{\partial q_k} - \lambda_\alpha \dfrac{\partial f_\alpha}{\partial q_k} = 0 & \text{(N equations)} \\[2ex] \dfrac{d}{dt}\cancelto{0}{\left(\dfrac{\partial L'}{\partial \dot{\lambda}_\alpha}\right)} - \dfrac{\partial L'}{\partial \lambda_\alpha} = 0 \Rightarrow f_\alpha(q_k, t) = 0 & \text{(n equations)} \end{cases}$$

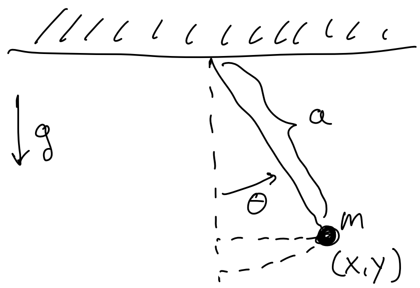
(N+n equations, N+n unknowns.)

Why is method 2 useful? Sometimes just easier. Also, if
$L(q_k, \dot{q}_k, t) = T(q_k, \dot{q}_k) - V(q_k)$, then get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \underbrace{-\frac{\partial V}{\partial q_k}}_{\text{force}} \underbrace{- \lambda_\alpha \frac{\partial f_\alpha}{\partial q_k}}_{\text{force of constraint}}$$

For example, if string produces constraint, this is tension in string.

Example: Pendulum of length a, mass m



Could use x, y (q_k's), but let's use just θ (Q)

Kinetic energy: $T = \frac{1}{2} m (a\dot{\theta})^2$

Potential energy: $V = -mga\cos\theta + \text{const}$

So $L(\theta, \dot{\theta}) = \frac{1}{2} m a^2 \dot{\theta}^2 + mga\cos\theta$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(ma^2\dot{\theta}\right) = ma^2\ddot{\theta} \qquad \frac{\partial L}{\partial \theta} = -mga\sin\theta$$

So $\boxed{\ddot{\theta} = -\frac{g}{a}\sin\theta.}$ $\quad \theta = 0 =$ stable equilibrium.

Small oscillations: $\theta = \theta_0 + \epsilon \Rightarrow \ddot{\epsilon} = -\frac{g}{a}\epsilon$.

Angular frequency $\omega^2 = \frac{g}{a} \Rightarrow \omega = \sqrt{\frac{g}{a}}$.

Method 2: use coordinates $(r, \theta) = q_k$, and constraint $r - a = 0$

$$L' = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta + \lambda(r-a)$$

Eqns of motion: $\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{r}}\right) - \frac{\partial L'}{\partial r} = 0 \Rightarrow$

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0} \quad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\theta}}\right) - \frac{\partial L'}{\partial \theta} = 0 \Rightarrow \boxed{\frac{d}{dt}(mr^2\dot{\theta}) + mgr\sin\theta = 0} \quad (2)$$

$$\frac{\partial L'}{\partial \lambda} = 0 \Rightarrow \boxed{r = a} \quad (3)$$

So (2) $\Rightarrow ma^2\ddot{\theta} = -ga\sin\theta \Rightarrow \ddot{\theta} = -\frac{g}{a}\sin\theta$ ✓

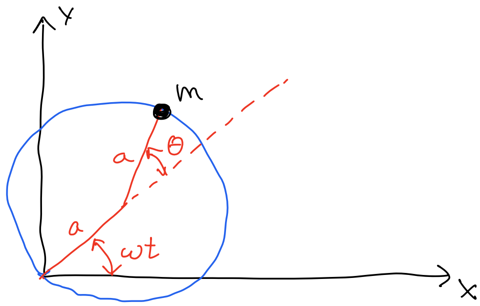
(3) $\Rightarrow \lambda = -mg\cos\theta - ma\dot{\theta}^2 =$ tension force in string

Example: bead on a rotating wire hoop

All motion in plane.

Hoop = circle, radius a, one point fixed at origin

No potential, just kinetic (neglect gravity)



Center of hoop: $x_c = a\cos(\omega t)$ $\quad y_c = a\sin(\omega t)$

Angle of m with respect to x-axis: $\omega t + \theta$.

So position of m is $\begin{cases} x = a\cos(\omega t) + a\cos(\omega t+\theta) \\ y = a\sin(\omega t) + a\sin(\omega t+\theta) \end{cases}$

Use θ = configuration variable.

$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$, write in terms of θ, $\dot{\theta}$.

$$\dot{x} = -a\omega\sin(\omega t) - a(\omega+\dot{\theta})\sin(\omega t+\theta)$$
$$\dot{y} = a\omega\cos(\omega t) + a(\omega+\dot{\theta})\cos(\omega t+\theta)$$

So
$$\dot{x}^2+\dot{y}^2 = a^2\omega^2\overset{1}{\cancel{\left[\sin^2(\omega t)+\cos^2(\omega t)\right]}} + a^2(\omega+\dot{\theta})^2\overset{1}{\cancel{\left[\sin^2(\omega t+\theta)+\cos^2(\omega t+\theta)\right]}}$$
$$+2a^2\omega(\omega+\dot{\theta})\left[\underbrace{\sin(\omega t)\sin(\omega t+\theta)+\cos(\omega t)\cos(\omega t+\theta)}_{\cos(\omega t+\theta-\omega t)\,=\,\cos\theta}\right]$$

So $L = T = \frac{ma^2}{2}\left[\omega^2 + (\omega+\dot{\theta})^2 + 2\omega(\omega+\dot{\theta})\cos\theta\right]$

No explicit time dependence!

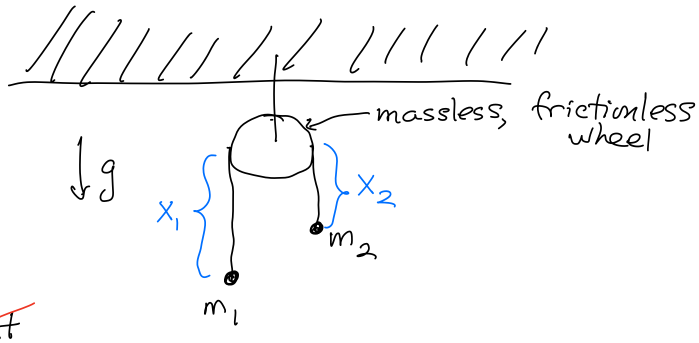
$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = ma^2\frac{d}{dt}\left[(\omega+\dot{\theta}) + \omega\cos\theta\right] = ma^2\left[\ddot{\theta} - \omega\sin\theta\,\dot{\theta}\right]$$

$$\frac{\partial L}{\partial\theta} = -ma^2\omega(\omega+\dot{\theta})\sin\theta$$

So $\ddot{\theta} - \cancel{\omega\sin\theta}\,\dot{\theta} + \omega^2\sin\theta + \cancel{\omega\sin\theta\,\dot{\theta}} = 0 \quad\Longrightarrow\quad \boxed{\ddot{\theta} = -\omega^2\sin\theta}$

Just like pendulum! Rotating system $\Longleftrightarrow$ constant centrifugal force.

Atwood's machine



$$T = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2$$

$$V = m_1 g x_1 + m_2 g x_2 + \cancel{\text{const}}$$

Constraint: $x_1 + x_2 = a = \text{constant}.$

So take $L = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 - g m_1 x_1 - g m_2 x_2 + \lambda(x_1 + x_2 - a)$

Equations of motion:

$$\left.\begin{aligned} m_1\ddot{x}_1 - g m_1 + \lambda &= 0 \\ m_2\ddot{x}_2 - g m_2 + \lambda &= 0 \end{aligned}\right\} \quad m_1\ddot{x}_1 - m_2\ddot{x}_2 = g(m_1 - m_2)$$

$$x_1 + x_2 = a \qquad \Longrightarrow \quad \ddot{x}_1 = -\ddot{x}_2$$

So $\ddot{x}_1 = g\left(\dfrac{m_1 - m_2}{m_1 + m_2}\right) \qquad \ddot{x}_2 = g\left(\dfrac{m_2 - m_1}{m_1 + m_2}\right) \qquad \lambda = \dfrac{2m_1m_2}{m_1 + m_2}\,g$

constant accelerations, positive for larger mass

constant tension in string (force)

Generalized momenta

$p_k = \dfrac{\partial L}{\partial \dot{q}_k}$ Eqns of motion say $\dfrac{dp_k}{dt} = \dfrac{\partial L}{\partial q_k}$

Cyclic coordinates are any q_k such that $\dfrac{\partial L}{\partial q_k} = 0$. In that case,

p_k = constant (conserved). To find conserved quantities, find cyclic coordinates.

Example: $L(x,y,z,\dot{x},\dot{y},\dot{z}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + V(x,y)$

So $\frac{\partial L}{\partial z} = 0$, z is a cyclic coordinate, $p_z = m\dot{z} = \text{constant}$

Example: motion in central potential, in a plane.

$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$.

The only cyclic coordinate is ϕ. $p_\phi = mr^2\dot{\phi} = \ell = \text{conserved}$

Consider the time derivative of $L(q_k, \dot{q}_k, t)$

$$\frac{dL}{dt} = \underbrace{\frac{dq_k}{dt}}_{\dot{q}_k}\underbrace{\frac{\partial L}{\partial q_k}}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)} + \frac{d\dot{q}_k}{dt}\frac{\partial L}{\partial \dot{q}_k} + \frac{\partial L}{\partial t} = \frac{d}{dt}\left(\dot{q}_k\frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial L}{\partial t}$$

So $\frac{d}{dt}\left(\dot{q}_k\frac{\partial L}{\partial \dot{q}_k} - L\right) = -\frac{\partial L}{\partial t}$. Now if $\frac{\partial L}{\partial t} = 0$,

(no explicit dependence of L on t), then

$$\boxed{\dot{q}_k\frac{\partial L}{\partial \dot{q}_k} - L = \text{constant} = E}$$

if L only depends on t through q_k, $\dot{q}_k$

Example: $L = \frac{1}{2}m\dot{x}^2 - V(x)$, so $E = \dot{x}(m\dot{x}) - \left(\frac{1}{2}m\dot{x}^2 - V(x)\right)$

or $E = \frac{1}{2}m\dot{x}^2 + V(x)$. (In 3-d, $L = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)^2 - V \Rightarrow E = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)^2 + V$)

Example: $L = \frac{1}{2}m\dot{x}^2 - V(x) - \underbrace{f\,x\cos(\omega t)}_{\text{explicit time dependence!}}$

Here $\frac{\partial L}{\partial t} = fx\omega\sin(\omega t)$, so $\frac{dE}{dt} = -fx\omega\sin(\omega t)$

Energy not conserved.

Noether's Theorem For each continuous symmetry of a system, there is a conserved quantity (charge, constant of motion).

<u>Transformations</u> are changes in the coordinates:

$$q_k(t) \to q_k(t) + s\,\delta q_k(t).$$

<u>Symmetries</u> are transformations that leave eqns of motion unchanged.

In terms of the Lagrangian, the transformation is a symmetry if:

$$\delta L = \frac{d}{dt} K(q_k(t), t)$$

(often assume $K=0$, but let's be general)

Also,

$$\delta L = \delta q_k \underbrace{\frac{\partial L}{\partial q_k}}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_k}\right)} + \underbrace{\delta \dot q_k}_{\frac{d}{dt}(\delta q_k)} \frac{\partial L}{\partial \dot q_k}$$

(chain rule)

use eqns of motion.

So

$$\delta L = \frac{d}{dt}\left(\delta q_k \frac{\partial L}{\partial \dot q_k}\right) = \frac{dK}{dt}.$$

Define

$$\boxed{Q = \delta q_k \frac{\partial L}{\partial \dot q_k} - K} = \text{conserved}, \quad \frac{dQ}{dt} = 0.$$

<u>Examples 1</u> $L = \frac{1}{2}m(\dot x^2 + \dot y^2) - V(x)$

Symmetry: $\delta x = 0$, $\delta y = 1$ $\quad \delta L = 0$ (so $K=0$)

Then $Q = 1\,\frac{\partial L}{\partial \dot y} = m\dot y = p_y$.

Cyclic coordinates $\longleftrightarrow$ symmetry.

<u>Example 2</u> $L = \frac{1}{2}m(\dot x^2 + \dot y^2) - V(x+y)$

Symmetry: $\delta x = 1$, $\delta y = -1$ $\quad \delta L = 0 \quad$ (so $K=0$)

Then $Q = \dfrac{\partial L}{\partial \dot x} - \dfrac{\partial L}{\partial \dot y} = m(\dot x - \dot y)$ is conserved.

<u>Example 3</u> $\quad L = \frac{1}{2} m(\dot x^2 + \dot y^2 + \dot z^2) - V(z)$

Symmetry: $\delta x = -y$, $\delta y = +x$ (rotation) $\qquad \delta \dot x = -\dot y$, $\delta \dot y = \dot x$

$$L \to \frac{1}{2} m \underbrace{\left((\dot x - s\dot y)^2 + (\dot y + s\dot x)^2 + \dot z^2\right)}_{\dot x^2 \cancel{- 2s\dot x\dot y} + O(s^2) + \dot y^2 \cancel{+ 2s\dot x\dot y} + O(s^2) + \dot z^2} - V(z)$$

So $\delta L = 0 \implies K = 0$.

So $Q = -y \dfrac{\partial L}{\partial \dot x} + x \dfrac{\partial L}{\partial \dot y} = x p_y - y p_x = L_z$

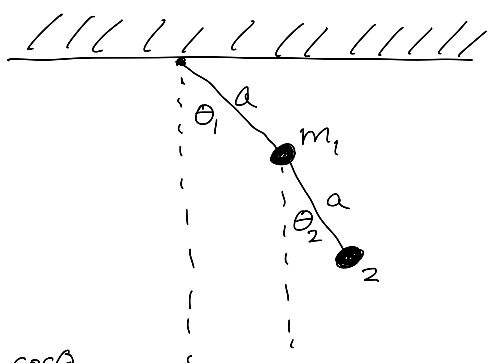
Time translation symmetry $\longrightarrow$ energy conserved

Space " " $\longrightarrow$ momentum conserved

Rotation symmetry $\longrightarrow$ angular momentum conserved

Gauge invariance $\longrightarrow$ charge conservation

<u>Double planar pendulum</u>



For m_1: $\quad T_1 = \frac{1}{2} m a^2 \dot\theta_1^2$

$\qquad V_1 = -m_1 g a \cos\theta_1$

For m_2: first find coordinates:

$x_2 = a\sin\theta_1 + a\sin\theta_2, \quad y_2 = a\cos\theta_1 + a\cos\theta_2$

So $T_2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) = \frac{1}{2} m_2 a^2 \left[\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos(\theta_1 - \theta_2)\, \dot{\theta}_1 \dot{\theta}_2 \right]$

(This used $\sin^2 + \cos^2 = 1$, and $\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 = \cos(\theta_1 - \theta_2)$.)

Also $V_2 = -m_2 g y_2 = -m_2 g (a\cos\theta_1 + a\cos\theta_2)$.

So
$$L = T_1 + T_2 - V_1 - V_2$$
$$= \frac{1}{2}(m_1 + m_2) a^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 a^2 \dot{\theta}_2^2 + m_2 a^2 \cos(\theta_1 - \theta_2)\, \dot{\theta}_1 \dot{\theta}_2$$
$$+ (m_1 + m_2) g a \cos\theta_1 + m_2 g a \cos\theta_2$$

$\Rightarrow$ Two equations of motion:
$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}_1}\right) - \dfrac{\partial L}{\partial \theta_1} = 0 \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}_2}\right) - \dfrac{\partial L}{\partial \theta_2} = 0 \end{cases}$$

Solutions very complicated in general. Chaotic motion above a certain energy.

Small change in initial conditions $\Longrightarrow$ large change in motion.

To get reliable solutions, look at limit of small oscillations. Keep only quadratic terms in L, linear terms in equations of motion. So:

$$\underbrace{\dot{\theta}_1 \dot{\theta}_2}_{\text{already quadratic}} \underbrace{\cos(\theta_1 - \theta_2)}_{\approx 1} \qquad \cos\theta_1 \approx 1 - \tfrac{1}{2}\theta_1^2, \quad \cos\theta_2 \approx 1 - \tfrac{1}{2}\theta_2^2$$

For simplicity, let's also take $m_1 = m_2 = m$. So

$$L \approx m a^2 \left(\dot{\theta}_1^2 + \tfrac{1}{2}\dot{\theta}_2^2 + \dot{\theta}_1 \dot{\theta}_2\right) - m g a \left(\theta_1^2 + \tfrac{1}{2}\theta_2^2\right)$$

Equations of motion are simpler, but still couple θ_1, θ_2:

$$ma^2(2\ddot{\theta}_1 + \ddot{\theta}_2) = -2mga\,\theta_1 \qquad (\text{from } \theta_1)$$

$$ma^2(\ddot{\theta}_1 + \ddot{\theta}_2) = -mga\,\theta_2 \qquad (\text{from } \theta_2),$$

or $$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\frac{g}{a} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

Multiply both sides by inverse of matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, which is $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. The result is:

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\frac{g}{a} \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

Look for normal modes with angular frequency ω.

$$\theta_1 = \mathrm{Re}[\theta_{10} e^{i\omega t}]$$

$$\theta_2 = \mathrm{Re}[\theta_{20} e^{i\omega t}]$$

Here, θ_{10} and θ_{20} may be complex, and ω = same for both.

Key point: eqns of motion are linear, so can take Re[ ] at end, after solving. So $\frac{d^2}{dt^2} \to -\omega^2$, and

$$-\omega^2 \begin{pmatrix} \theta_{10} \\ \theta_{20} \end{pmatrix} = -\frac{g}{a} \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} \theta_{10} \\ \theta_{20} \end{pmatrix}, \quad \text{or}$$

$$\boxed{\begin{pmatrix} \omega^2 a/g - 2 & +1 \\ 2 & \frac{\omega^2 a}{g} - 2 \end{pmatrix} \begin{pmatrix} \theta_{10} \\ \theta_{20} \end{pmatrix} = 0}$$

This is an eigenvalue equation; we want to solve for the eigenvalue $\frac{\omega^2 a}{g}$ and the eigenvector $\begin{pmatrix} \theta_{10} \\ \theta_{20} \end{pmatrix}$ at same time.

First, the eigenvalues. For a solution, need

$$\mathrm{Det}\begin{vmatrix} \omega^2 a/g - 2 & 1 \\ 2 & \frac{\omega^2 a}{g} - 2 \end{vmatrix} = 0 \qquad \text{Let } \lambda = \frac{\omega^2 a}{g}. \text{ Then}$$

$$\underbrace{(\lambda-2)^2 - 2}_{\mathrm{Det}} = 0 \implies (\lambda - 2) = \pm\sqrt{2} \implies \lambda = 2 \pm \sqrt{2}$$

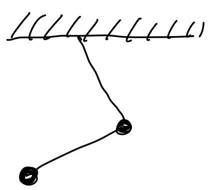
So $\omega^2 = \frac{g}{a}(2 \pm \sqrt{2})$ (Two modes). Now can plug back into boxed equation, solve for θ_{20} in terms of θ_{10}.

For $\omega_1^2 = \frac{g}{a}(2+\sqrt{2})$, get $\theta_{20} = \sqrt{2}\,\theta_{10}$ Mode 1

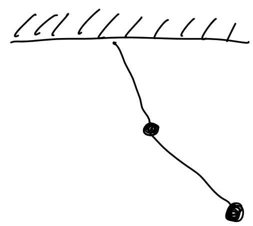
$\omega_2^2 = \frac{g}{a}(2-\sqrt{2})$, get $\theta_{20} = -\sqrt{2}\,\theta_{10}$ Mode 2

Note both $\omega^2 > 0$ (oscillatory motion, not damped)

Mode 1 (fast)



Mode 2 (slow)



General solution for small oscillations: linear combination

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \mathrm{Re}\left[A_1 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} e^{i(\omega_1 t + \phi_1)} + A_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{i(\omega_2 t + \phi_2)} \right]$$

Solve for A_1, A_2, ϕ_1, ϕ_2 in terms of initial conditions for θ_1, θ_2, $\dot{\theta}_1$, $\dot{\theta}_2$ at time $t = 0$, say.