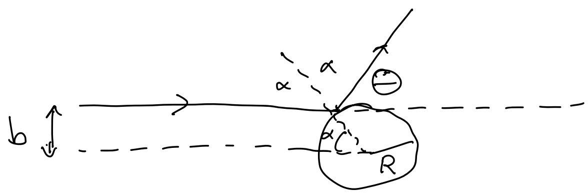
Scattering from a hard sphere



$\alpha = \arcsin\left(\frac{b}{R}\right)$

$\theta = \pi - 2\alpha$

So $b = R\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = R\cos\theta/2$.

Number of particles between b and $b+db$ is

$$\underbrace{n}_{\text{particles/(area)(time)}}\underbrace{(2\pi b\, db)}_{\text{area}} = n\,2\pi R\cos\left(\frac{\theta}{2}\right) R\, \left|d\left(\cos\left(\frac{\theta}{2}\right)\right)\right| = n\pi R^2 \underbrace{\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}_{\frac{1}{2}\sin\theta} d\theta$$

$$= \frac{1}{2} n\pi R^2\, d(\cos\theta)$$

Number of scattered particles in solid angle $d\Omega = d\phi\, d(\cos\theta)$

is $$dS = \frac{1}{2} n\pi R^2 \overbrace{d(\cos\theta)\,\frac{d\phi}{2\pi}}^{d\Omega} = \frac{1}{4} n R^2\, d\Omega$$

So $$\frac{d\sigma}{d\Omega} = \frac{\frac{1}{4} n R^2}{n} = \frac{R^2}{4}, \quad \text{so} \quad \frac{d\sigma}{d(\cos\theta)} = 2\pi\frac{d\sigma}{d\Omega} = \frac{\pi R^2}{2}$$

and $$\sigma = \int_{-1}^{1} d(\cos\theta)\,\frac{d\sigma}{d(\cos\theta)} = \pi R^2.$$ Note: units = area.

In this case, cross-section = geometric cross-section.

Alternative method: plug into formula.

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{R\cos\theta/2}{\sin\theta}\underbrace{\left|\frac{d\left(R\cos\theta/2\right)}{d\theta}\right|}_{-\frac{1}{2}\sin\theta/2} = \frac{R^2}{2}\underbrace{\frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin\theta}}_{1/2}$$

$$= \frac{R^2}{4}$$

Rutherford Scattering

$$V = -\frac{k}{r} \qquad k = \begin{cases} +GMm & \text{Kepler} \\ -Qq & \text{Coulomb} \end{cases}$$

Already found $b = \frac{|GMm|}{mv_\infty^2} \cot\frac{\theta}{2} = \frac{|k|}{2E}\cot\frac{\theta}{2}$

$$\text{So } \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{k^2}{4E^2}\underbrace{\frac{1}{\sin\theta}}_{\frac{1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}\underbrace{\cot\frac{\theta}{2}}_{\frac{\cos\frac{\theta}{2}}{\sin\theta/2}}\left|\underbrace{\frac{d}{d\theta}\cot\frac{\theta}{2}}_{-\frac{1}{2\sin^2\theta/2}}\right| = \left(\frac{k^2}{4E}\right)^2\frac{1}{\sin^4(\theta/2)}$$

blows up as $\theta \to 0$!

The $\frac{1}{r}$ potential has "infinite range". Also true of any potential that falls off slower than $\frac{1}{r}$ for large r.

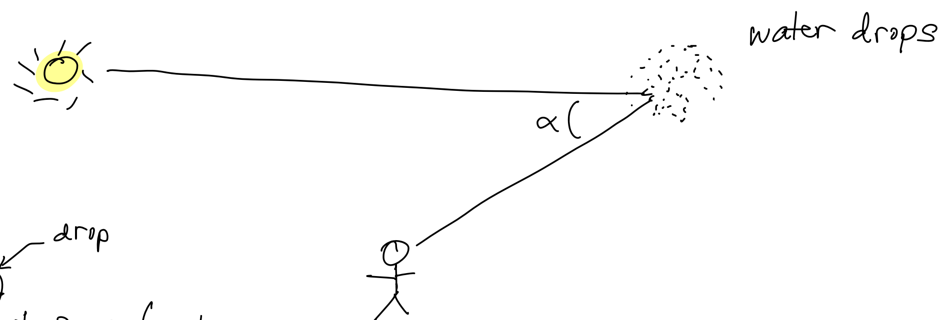
To get a finite cross-section, modify the question slightly. Count only scatters by more than θ_{min}. So:

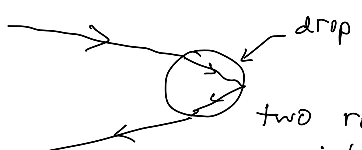
$$\sigma(\theta > \theta_{min}) = \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-1}^{\cos\theta_{min}} d(\cos\theta_{min}) \left(\frac{k^2}{4E}\right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}}_{\left(\frac{k^2}{4E}\right)^2 2\left(\frac{1+\cos\theta_{min}}{1-\cos\theta_{min}}\right)}$$

$$= \frac{k^4}{4E^2}\left(\frac{1+\cos\theta_{min}}{1-\cos\theta_{min}}\right)$$

Note: $\sigma \to \infty$ as $\theta_{min} \to 0$.

Rainbows



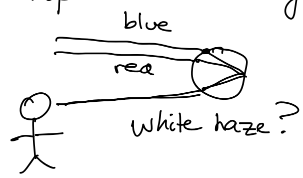


color	ω (sec^{-1}) x 10^{15}	n_{water}		
violet	4.71	1.344		
blue	4.39	1.340		
green	3.49	1.335		
yellow	3.19	1.333		
red	2.48	1.329		

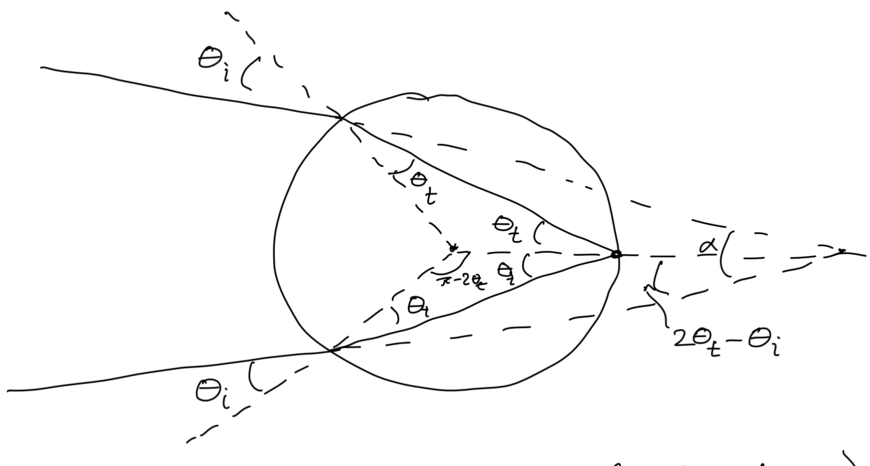
dispersion

Different colors $\rightarrow$ different n $\rightarrow$ different angles

BUT there is more to it. Sphere not like a prism, curved surface of drop means any color can reach our eye at any angle



Find angle α as a function of angle θ_i at which light hits drop.

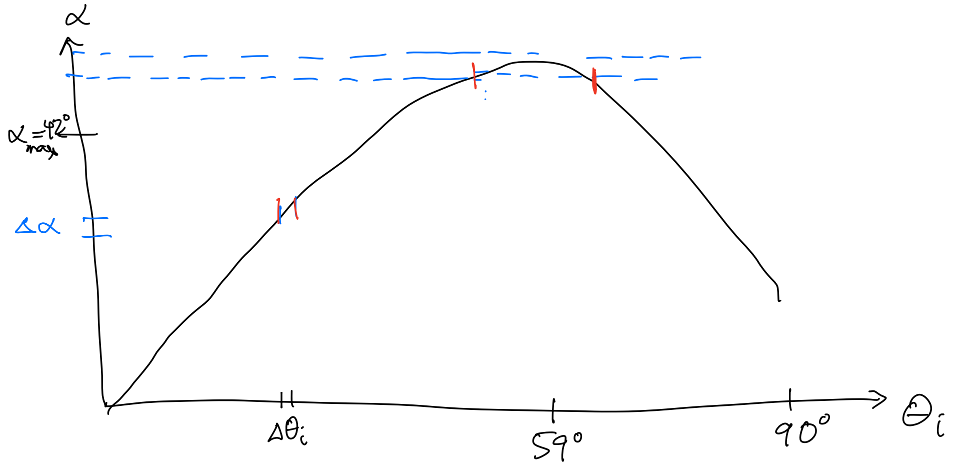


So $\alpha = 4\theta_t - 2\theta_i$ $\qquad \dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{1}{n}$ (Snell's Law)

$$\Rightarrow \quad \theta_t = \arcsin\left(\frac{\sin\theta_i}{n}\right), \quad \text{so}$$

$$\alpha = 4 \arcsin\left(\frac{\sin\theta_i}{n}\right) - 2\theta_i$$

For $n \simeq 1.335$, $\begin{cases} \alpha \approx \left(\frac{4}{n} - 2\right)\theta_i \approx 0.99625\theta_i & \text{for small } \theta_i \\ \text{maximum } \alpha = 42^\circ & (\text{at } \theta_i \approx 59^\circ) \\ \alpha \approx 14^\circ & \text{for } \theta_i = 90^\circ \end{cases}$



Look for light near $\alpha = 20^\circ$, only a narrow band of θ_i contributes.

BUT, look for light near $\alpha = \alpha_{max} = 42^\circ$, a large range of θ_i contributes. Intensity enhancement near $\alpha_{max} = 42^\circ$

Sharp intensity peak, Different colors $\rightarrow$ different n $\rightarrow$ different α_{max}

Max occurs when $\frac{d\alpha}{d\theta_i} = 0$. (stationary point)

$$\frac{d\alpha}{d\theta_i} = 4 \underbrace{\frac{d}{d\theta_i}\left[\arcsin\left(\frac{\sin\theta_i}{n}\right)\right]}_{\frac{1}{\sqrt{1 - \frac{\sin^2\theta_i}{n^2}}} \frac{\cos\theta_i}{n}} - 2$$

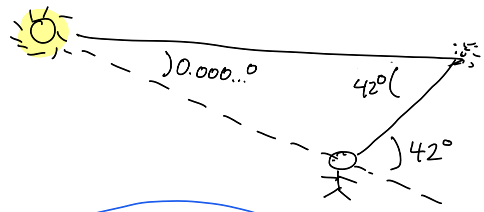
(Chain rule with $\frac{d}{dx} \arcsin u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$)

So $\frac{d\alpha}{d\theta_i} = 0 \Rightarrow 2\cos\theta_i = n\sqrt{1 - \frac{\sin^2\theta_i}{n^2}} \Rightarrow 4\cos^2\theta_i = n^2 - \sin^2\theta_i$

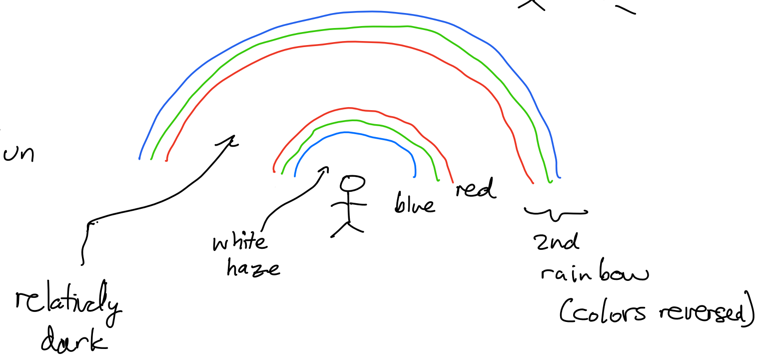
$$= n^2 - 1 + \cos^2\theta_i$$

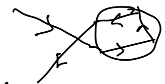
$\Rightarrow \cos^2\theta_{i,max} = \frac{n^2 - 1}{3}$ $\longleftarrow$ plug in to get α_{max}

color	$\theta_{i,max}$	α_{max}
violet	58.77°	40.51°
blue	59.00°	41.07°
green	59.29°	41.79°
yellow	59.41°	42.08°
red	59.64°	42.66°



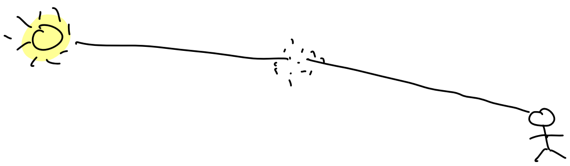
Look directly away from Sun

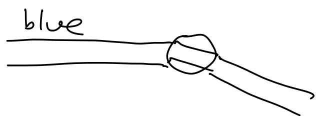




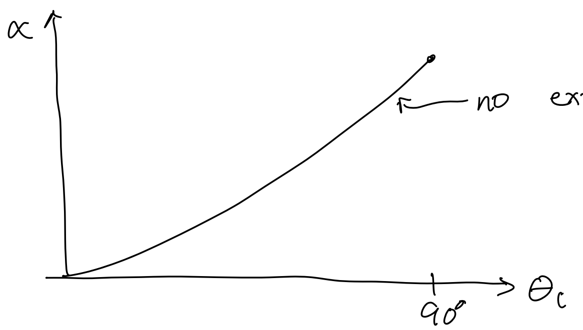
2 refractions, 2 reflections

Non-rainbow

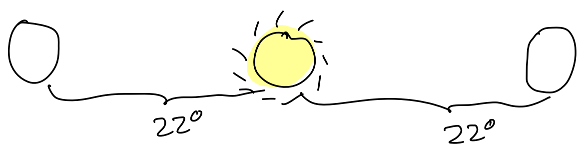




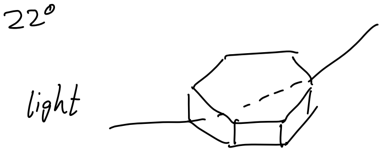
colors separate, but no enhancement, no rainbow



Sun dogs



intensity enhancement from oriented falling hexagonal ice crystals



Classical Mechanics as a Limit of Quantum Mechanics

Will use special case: particle moving in 1-d:

$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2, \qquad V = V(x).$

$E = T + V.$ Define $L(\dot{x}, x) = T - V$ = Lagrangian

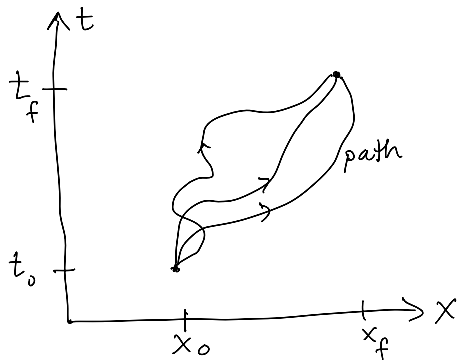
In QM, there are two ways to give time evolution of the wavefunction $\psi(x,t)$.

① Schrodinger eqn: $\boxed{i\hbar \frac{\partial \psi}{\partial t} = H\psi}$

② Propagator: $\psi(x,t) = \int_{-\infty}^{\infty} dx_0 \langle x,t | x_0, t_0 \rangle \psi(x_0, t_0)$

Want to show this is equivalent.

Assumption: $\langle x_f t_f | x_0 t_0 \rangle$ = sum of infinite number of complex amplitudes, each corresponding to a possible path $x(t)$ from (x_0, t_0) to (x_f, t_f):



$$\langle x_f t_f | x_0 t_0 \rangle \Big|_{path} = N \exp\left(\frac{i}{\hbar} S_{path}\right)$$

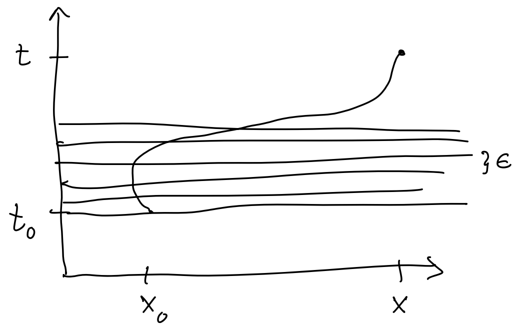
where

$$S_{path} = \int_{t_0\ path}^{t_f} dt \; L(x, \dot{x}, t) = \text{action}$$

So $\langle x_f t_f | x_0 t_0 \rangle = \sum_{\substack{paths \\ x(t)}} \exp\left(i \underbrace{S[x(t)]}_{\text{functional}} / \hbar\right)$

What is a sum over paths?

Discretize the continuous function $x(t)$



$$S[x(t)] = S[x_0, x_1, x_2, \dots x_N]$$

$$= \sum_{k=1}^{N-1} \underbrace{\epsilon}_{= \frac{t-t_0}{N} = dt} \left[\frac{m}{2} \Big(\underbrace{\frac{x_k - x_{k-1}}{\epsilon}}_{\frac{dx}{dt}} \Big)^2 - V(x_k) \right]$$

Now $\langle x_f t_f | x_0, t_0 \rangle = \lim_{N\to\infty} \int \frac{dx_1}{C} \int \frac{dx_2}{C} \int \frac{dx_3}{C} \dots \int \frac{dx_{N-1}}{C} \exp\left(\frac{i\, S[x_0 \dots, x_N]}{\hbar} \right)$

C = normalization factor, to be found.

Take one infinitesimal time step:

$$\psi(x, t+\epsilon) = \int \frac{dx_0}{C} e^{iS(x,x_0)/\hbar}\, \psi(x_0, t)$$

Expand in small ϵ:

$$LHS = \psi(x,t) + \epsilon \frac{\partial}{\partial t}\psi(x,t) + \frac{\epsilon^2}{2} \frac{\partial^2}{\partial t^2} \psi(x,t)$$

$$RHS = \int \frac{dx_0}{C} \exp\left(\frac{i}{\hbar} \left[\frac{m}{2} \left(\frac{x-x_0}{\epsilon} \right)^2 - V(x) \right] \epsilon \right) \psi(x_0, t)$$

$$= \Big(1 - \frac{i\epsilon}{\hbar} \underbrace{V(x)}_{\text{drop } \epsilon^2} \Big) \int \frac{dx_0}{C} \underbrace{\exp\left(\frac{i}{\hbar} \frac{m}{2\epsilon} (x-x_0)^2 \right) \psi(x_0, t)}$$

Key point: there will be almost complete cancellation for $\epsilon \to 0$, since this varies rapidly except when $x \approx x_0$. Let $x_0 - x = u$, $dx_0 = du$.

$$RHS = \left(1 - i \frac{\epsilon}{\hbar} V(x)\right) \int_{-\infty}^{\infty} \frac{du}{C} \exp\left(\frac{i}{\hbar} \frac{m}{2\epsilon} u^2 \right) \psi(u+x, t)$$

Region of importance is $u \approx 0$. So $\psi(u+x) = \psi(x) + \cancelto{0}{u \frac{\partial \psi}{\partial x}} + \frac{u^2}{2} \frac{\partial^2 \psi}{\partial x^2} + \dots$ odd function of u

So $LHS = RHS$ gives

$$\psi(x,t) + \epsilon \frac{\partial \psi(x,t)}{\partial t} = \left(1 - i \frac{\epsilon}{\hbar} V(x)\right) \int_{-\infty}^{\infty} \frac{du}{C}\, e^{imu^2/2\hbar\epsilon} \left(\psi(x,t) + \frac{u^2}{2} \frac{\partial^2 \psi(x,t)}{\partial x^2} \right)$$

Expand, and match powers of ϵ:

$\underline{\epsilon^0}$:
$$\cancel{\psi(x,t)} = \frac{1}{C}\underbrace{\int_{-\infty}^{\infty} du\, \exp\left(\frac{imu^2}{2\hbar\epsilon}\right)}_{\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{1/2}} \cancel{\psi(x,t)} \quad\Rightarrow\quad \boxed{C = \left(\frac{2\pi i\epsilon\hbar}{m}\right)^{1/2}}$$

Now
$$\int_{-\infty}^{\infty} \frac{du}{C} \exp\left(\frac{imu^2}{2\hbar\epsilon}\right) u^2 = \frac{i\epsilon\hbar}{m}.$$
So,

$\underline{\epsilon^1}$:
$$\epsilon \frac{\partial\psi}{\partial t}(x,t) = -\frac{i\epsilon}{\hbar} V(x)\,\psi(x,t) + \frac{i\epsilon\hbar}{m}\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x,t)$$

Rearrange:
$$\boxed{i\hbar\frac{\partial}{\partial t}\psi(x,t) = \underbrace{\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\psi(x,t)\right]}_{H}\psi(x,t)}$$

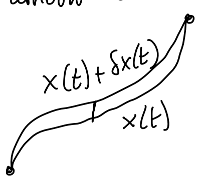
This is Schrodinger's equation. ✓

Now take the classical limit $\hbar \to 0$. Almost complete cancellation in path integral $\propto \exp\left[iS(x(t))/\hbar\right]$

fast oscillations, nearby paths efficiently cancel

Intensity enhancement only for stationary paths, such that $\delta S = 0$ for $\underline{\text{all}}$ $\delta x(t)$. (Equivalently, $\frac{\delta S}{\delta x(t)} = 0$.)

Like rainbow enhancement, $\delta\alpha = 0$.



To a good approximation, as $\hbar \to 0$, only $\underline{\text{one}}$ path contributes. Which one?

<u>Derivation of equations of motion:</u>

Consider a change of path $x(t) \to x(t) + \delta x(t)$, with boundary conditions $\delta x(t_0) = 0$ and $\delta x(t_f) = 0$. Need to calculate change in action, $\delta S = ?$

First, generalize: instead of one coordinate $x(t) \to q_k(t)$
$k=1,\dots,N$

This allows:
1) Motion in 3-d
2) More than 1 particle
3) Modification to include relativistic effects
4) General kinetic terms, dependence on $\dot{q}_k(t)$

$$S = \int_{t_0}^{t_f} dt \; L(q_k, \dot{q}_k, t)$$

Change in path: $q_k(t) \to q_k(t) + \delta q_k(t)$

Then $$\delta S = \int_{t_0}^{t_f} dt \sum_{k=1}^{N} \left[\delta q_k(t) \frac{\partial L}{\partial q_k} + \delta \dot{q}_k(t) \frac{\partial L}{\partial \dot{q}_k} \right]$$

Here $\delta \dot{q}_k(t) = \frac{d}{dt} q_k(t)$. Note $q_k(t)$ and $\dot{q}_k(t)$ are treated as independent.

Generalization 5) k could be continuous, so $\sum_{k=1}^{N} \longrightarrow \int dk$

Otherwise, use summation convention: repeated indices are always summed over (unless they appear on both sides of the equation).

Integrate last term by parts:

$$\int_{t_0}^{t_f} dt \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt}\left(\delta q_k(t)\right) = \underbrace{\frac{\partial L}{\partial \dot{q}_k} \delta q_k(t) \Big|_{t=t_i}^{t=t_f}}_{0} - \int_{t_0}^{t_f} dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$$

So $$\delta S = \int_{t_0}^{t_f} dt \; \delta q_k(t) \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \right]$$

To have $\delta S = 0$ for all $\delta q_k(t)$, the stationary path must satisfy Lagrange's equations of motion:

$$\boxed{\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0}$$

Know the Lagrangian, know the physics.

The Lagrangian is not unique.

Take $L \to L' = L + \frac{df}{dt}$, where

Not a function of $\dot{q}_k$

$f = f(q_k, t)$. Doesn't affect equations of motion.

Easy proof: $S \to S' = S + \int_{t_0}^{t_f} dt \frac{df}{dt} = f(q_k(t_f), t_f) - f(q_k(t_0), t_0)$

= constant (same for every path).

Hard proof: $\Delta L = \frac{\partial f}{\partial t} + \dot{q}_k \frac{\partial f}{\partial q_k}$

Plug into equations of motion.

The Lagrangian for a particle gives Newton's Second Law

$$T = \frac{1}{2} m \left(\frac{d\vec{r}}{dt}\right)^2 \qquad V = V(\vec{r})$$

$$L = \frac{1}{2} m \left(\frac{d\vec{r}}{dt}\right)^2 - V(\vec{r}).$$

So $\begin{cases} \frac{\partial L}{\partial \dot{r}_k} = m \frac{dr_k}{dt} \\ \frac{\partial L}{\partial r_k} = -\frac{\partial V}{\partial r_k} \end{cases} \Rightarrow$

So $$\underbrace{\frac{d}{dt}\left(m \frac{dr_k}{dt}\right)}_{m \frac{d^2 r_k}{dt^2} = ma} = -\frac{\partial V}{\partial r_k} = \text{force}$$

Can use generalized coordinates

Let $Q_J = Q_J(q_k, t)$, and inverse $q_k = q_k(Q_J, t)$.

Note: no time derivatives

Condition for invertibility: $\text{Det}\left|\frac{\partial Q_J}{\partial q_k}\right| \neq 0$

$$L(q_k, \dot{q}_k, t) = L(Q_J, \dot{Q}_J, t)$$

Strategy: start with $\frac{\partial L}{\partial Q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j} = 0$, write in terms of q_k, $\dot{q}_k$.

Compute: $\dot{q}_k = \frac{d}{dt} q_k = \frac{dQ_j}{dt}\frac{\partial q_k}{\partial Q_j} + \frac{\partial q_k}{\partial t} = \dot{Q}_L \frac{\partial q_k}{\partial Q_L} + \frac{\partial q_k}{\partial t}$.

$$\frac{\partial L}{\partial Q_j} = \frac{\partial L}{\partial q_k}\frac{\partial q_k}{\partial Q_j} + \frac{\partial L}{\partial \dot{q}_k}\underbrace{\frac{\partial \dot{q}_k}{\partial Q_j}}_{\frac{\partial^2 q_k}{\partial Q_j \partial Q_L}\dot{Q}_L + \frac{\partial^2 q_k}{\partial Q_j \partial t}}$$

$$\frac{\partial L}{\partial \dot{Q}_j} = \frac{\partial L}{\partial \dot{q}_k}\underbrace{\frac{\partial \dot{q}_k}{\partial \dot{Q}_j}}_{\frac{\partial q_k}{\partial Q_j}}$$

So $$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}\right)\frac{\partial q_k}{\partial Q_j} + \frac{\partial L}{\partial \dot{q}_k}\underbrace{\frac{d}{dt}\left(\frac{\partial q_k}{\partial Q_j}\right)}_{\frac{\partial^2 q_k}{\partial Q_j \partial t} + \frac{\partial^2 q_k}{\partial Q_j \partial Q_L}\underbrace{\frac{dQ_L}{dt}}_{\dot{Q}_L}}$$

So $$\frac{\partial L}{\partial Q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j} = \underbrace{\frac{\partial q_k}{\partial Q_j}\left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}\right)}_{\text{Note sum over all } k.}$$ for each j.

If the eqns of motion for each q_k are satisfied, then RHS $= 0$, so eqns of motion for Q_j are also satisfied!

<u>Can use any coordinates you want!</u>

Some systems don't have a Lagrangian

Non-conservative forces, for example friction $\vec{F} = -\gamma \dot{x}$

<u>Example: Spherical coordinates</u>

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x,y,z)$$

EOM: $\quad -\dfrac{\partial V}{\partial x} - \dfrac{d}{dt}(m\dot{x}) = 0 \quad \Longrightarrow \quad m\ddot{x} = -\dfrac{\partial V}{\partial x}$, and

similarly $\quad m\ddot{y} = -\dfrac{\partial V}{\partial y}$ and $\quad m\ddot{z} = -\dfrac{\partial V}{\partial z}$.

Instead of (x, y, z), can use (r, θ, ϕ).

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - V(r,\theta,\phi)$$

This is especially useful if V only depends on r.

First, θ:

$$\frac{\partial L}{\partial \theta} = mr^2\sin\theta\cos\theta\,\dot{\phi}^2 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\left(mr^2\dot{\theta}\right)$$

Solution: $\boxed{\theta = \underline{\text{constant}} = \pi/2,}$ so $\sin\theta = 1$, $\cos\theta = 0$.

Now, ϕ:

$$\frac{\partial L}{\partial \phi} = 0, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt}\Big(mr^2\underbrace{\sin^2\theta}_{1}\,\dot{\phi}\Big)$$

So $\quad \dfrac{d}{dt}\left(mr^2\dot{\phi}\right) = 0 \quad \Longrightarrow \quad \boxed{mr^2\dot{\phi} = \ell = \text{constant}}$

Finally, r:

$$\frac{\partial L}{\partial r} = mr\Big(\cancelto{0}{\dot{\theta}^2} + \cancelto{1}{\sin^2\theta}\,\dot{\phi}^2\Big) - \frac{\partial V}{\partial r} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{d}{dt}(m\dot{r}) = m\ddot{r}$$

So $\quad mr\underbrace{\dot{\phi}^2}_{\left(\frac{\ell}{mr^2}\right)^2} - \dfrac{\partial V}{\partial r} = m\ddot{r}$

So $\quad \boxed{m\ddot{r} = \dfrac{\ell^2}{mr^3} - \dfrac{\partial V}{\partial r}}$

2nd order diff. eq.,
compare to 1st order eq. from $E = \text{const}$