Hints on HW1 Problem 3:

Rocket

$\leftarrow m_0\left(1-\frac{t}{T}\right)$

$\leftarrow V_{ex}$, mass $\frac{m_0}{T}\Delta t$ in time Δt

Solve for height (and velocity) of rocket as a function of t.

At time t, $\quad P_{rocket} = m_0\left(1-\frac{t}{T}\right) v(t)$.

At time $t+\Delta t$,

$$P(t+\Delta t)\begin{cases} P_{rocket} = m_0\left(1-\frac{t}{T}-\frac{\Delta t}{T}\right) v(t+\Delta t) \\ \qquad\quad = m_0\left(1-\frac{t}{T}\right) v(t+\Delta t) - m_0 \frac{\Delta t}{T}\, v(t) \\ P_{exhaust} = m_0 \frac{\Delta t}{T}\left(v(t) - V_{ex}\right) \end{cases}$$

replaced $v(t+\Delta t)$ by $v(t)$, difference of order Δt.

$$F = -mg = \frac{dp}{dt} = \frac{\Delta p}{\Delta t}$$

Take it from there...

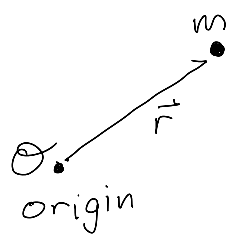
Central Forces

$$\vec{F} = \hat{r}\, f(r)$$

unit vector

spherical coordinate

no θ, ϕ dependence, and no $\hat{\theta}, \hat{\phi}$ direction



Newton's gravity potential = special case.

$$\vec{\nabla} \times \vec{F} = 0 \iff \vec{F} \text{ is conservative} \iff$$

$$\vec{F} = -\vec{\nabla} V = -\frac{dV}{dr} \qquad V(r)$$

also no θ, ϕ dependence

Torque about origin: $\vec{N} = (\vec{r} \times \vec{r})\frac{f(r)}{r} = 0.$

So, $\vec{L} = \vec{r}\times\vec{p}$ = constant of motion. Also,

$$\vec{r}\cdot\vec{L} = \vec{r}\cdot(\vec{r}\times\vec{p}) = 0$$
$$\vec{p}\cdot\vec{L} = \vec{p}\cdot(\vec{r}\times\vec{p}) = 0$$

All motion is in the plane $\perp$ to $\vec{L}$.

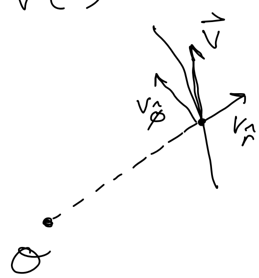
Without loss of generality, choose plane of motion to be $z=0$. Coordinates are

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}\right\} \Longrightarrow \begin{aligned} \dot{x} &= \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\ \dot{y} &= \dot{r}\sin\phi + r\dot{\phi}\cos\phi \end{aligned}$$

Now $E = T + V = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + V(r)$

constant

$$\boxed{E = \frac{1}{2}m\,(\underbrace{\dot{r}^2}_{(V_{\hat{r}})^2} + \underbrace{r^2\dot{\phi}^2}_{(V_{\hat{\phi}})^2}) + V(r)}$$



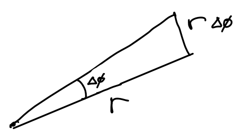
$$\vec{L} = \vec{r}\times m\vec{v} = m\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & 0 \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = \hat{z}\,m\,\underbrace{(x\dot{y}-\dot{x}y)}_{r^2\dot{\phi}}$$

$$\boxed{\vec{L} = \hat{z}\,m r^2\dot{\phi} = \text{constant vector} = \hat{z}\,\ell} \qquad \boxed{\ell = mr^2\dot{\phi}}$$

So, if $\begin{cases} r \text{ increases}, & \dot{\phi} \text{ must decrease} \\ r \text{ decreases}, & \dot{\phi} \text{ must increase} \end{cases}$

$\dot{\phi}$ can never change sign (mr^2 = always positive)

Area swept out in time Δt:

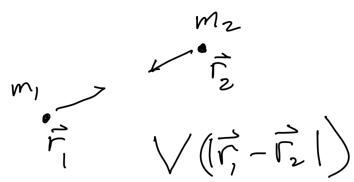


$$\Delta A = \frac{1}{2}r(r\Delta\phi) = \frac{1}{2}r^2\frac{d\phi}{dt}\Delta t = \frac{\ell}{2m}\Delta t$$

So $\frac{dA}{dt} = \frac{\ell}{2m} = \text{constant}$ Kepler's 2nd Law of planetary motion.

Found $\underbrace{\text{empirically}}_{\text{experiment}}$ for Newton's gravity, but general for <u>any</u> central force.

Two-body Problem



$$E = \frac{1}{2} m_1 \underbrace{\left(\frac{d\vec{r}_1}{dt}\right)^2}_{\vec{v}_1 \cdot \vec{v}_1 = v_1^2} + \frac{1}{2} m_2 \underbrace{\left(\frac{d\vec{r}_2}{dt}\right)^2}_{v_2^2} + V(|\vec{r}_1 - \vec{r}_2|)$$

<u>Define</u> $\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$ $\qquad M = m_1 + m_2$

$\vec{r} = \vec{r}_1 - \vec{r}_2$ $\qquad \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}}$ (reduced mass)

Can invert: $\vec{r}_1 = \vec{R} + \frac{m_2 \vec{r}}{M}$, $\qquad$ Also, $\vec{r} = \vec{r}_1{}' - \vec{r}_2{}'$ COM frame.

$\vec{r}_2 = \vec{R} - \frac{m_1 \vec{r}}{M}$.

Now: $$E = \underbrace{\frac{1}{2} M \left(\frac{d\vec{R}}{dt}\right)^2}_{\text{free particle with mass } M} + \underbrace{\frac{1}{2}\mu\left(\frac{d\vec{r}}{dt}\right)^2 + V(r)}_{\text{single "particle" with mass } \mu = \frac{m_1 m_2}{m_1 + m_2}}$$

Can reduce the problem to that of a single particle, with central force potential $V(r)$. After solving, can change back to coordinates $\vec{r}_1$, $\vec{r}_2$. In the limit $m_2 \gg m_1$, then

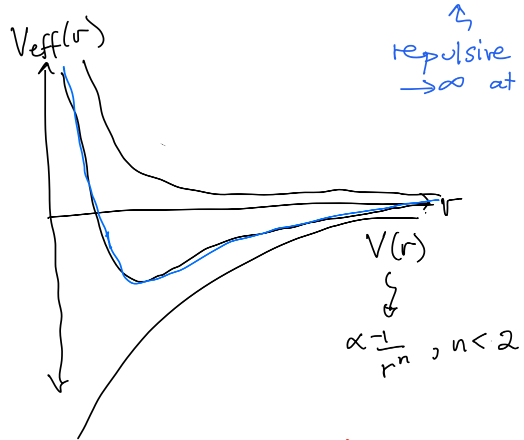
$\mu \approx m_1$ and $\vec{r}_1 \approx \vec{r} + \vec{R}$ and $\vec{r}_2 \approx \vec{R}$.

Types of Central Force Orbits

$$E = \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{1}{2} m r^2 \dot{\phi}^2}_{\frac{1}{2} m r^2 \left(\frac{\ell}{m r^2}\right)^2} + V(r)$$

So, define $V_{eff}(r) = V(r) + \underbrace{\dfrac{\ell^2}{2 m r^2}}_{\text{repulsive core. } \to \infty \text{ at } r=0}$, $\quad \underbrace{E = \frac{1}{2} m \dot{r}^2 + V_{eff}(r)}_{\text{Particle moving in 1 dimension, } r.}$

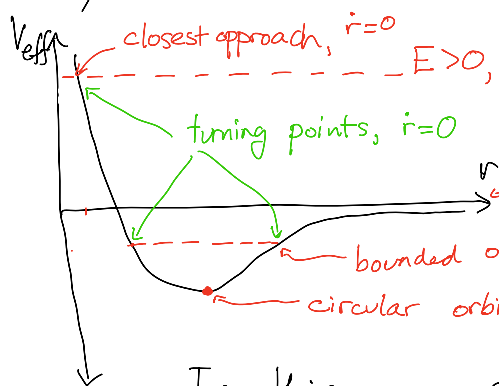
For example:



Note $\frac{1}{2} m \dot{r}^2 \geq 0$, so

$$E \geq \min\left(V_{eff}(r)\right)$$

Possibilities:



$E>0$, unbounded $r \to \infty$ with $v>0$

$E=0$, barely escapes to $r=\infty$ with 0 velocity

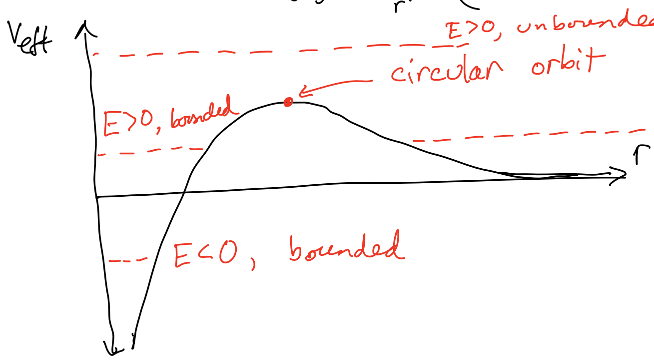
bounded orbit $E<0$

circular orbit $r = r_{min}$ $\quad E = E_{min}$

In this case, circular orbit is stable,

$$\frac{d}{dr} V_{eff}(r)\Big|_{r=r_c} = 0 \qquad \frac{d^2}{dr^2} V_{eff}(r)\Big|_{r=r_c} > 0.$$

Another case: $V(r) \propto \frac{-1}{r^n}$ $(n>2)$



$E>0$, unbounded, hits origin

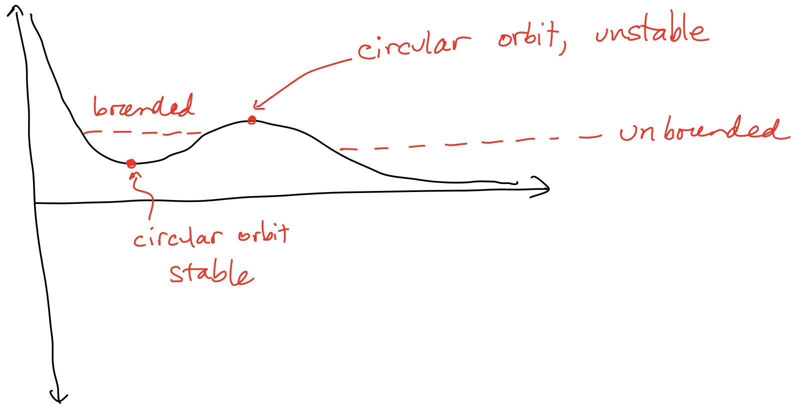
circular orbit $r = r_{max}$, $E = \max(V_{eff}(r))$, unstable

$E>0$, unbounded, doesn't hit origin

Unstable circular orbit

$$\frac{d^2}{dr^2} V_{eff}(r)\Big|_{r=r_c} < 0$$

Yet another case:



Condition for circular orbit: $\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_c} = 0 \Rightarrow \frac{\partial V}{\partial r} + \frac{\partial}{\partial r}\left(\frac{\ell^2}{2mr^2}\right) = 0$

So $F_r = \frac{-\partial V}{\partial r} = -\frac{\ell^2}{mr^3}$. Define "centripetal force" $F_c = \frac{\ell^2}{mr^3}$,

then $\left.(F_r + F_c)\right|_{r=r_c} = 0$ for a circular orbit.

For a stable circular orbit, $\frac{\partial^2 V_{eff}}{\partial r^2} > 0 \Rightarrow \frac{\partial^2 V}{\partial r^2} + \frac{\partial^2}{\partial r^2}\left(\frac{\ell^2}{2mr^2}\right) > 0$

So need $\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_c} + \frac{3\ell^2}{mr_c^4} > 0$

Suppose $V = \frac{k}{r^n}$, then $\frac{\partial V}{\partial r} = \frac{-nk}{r^{n+1}}$, so need

$$\frac{-nk}{r_c^{n+1}} - \frac{\ell^2}{mr_c^3} = 0 \Rightarrow nk = \frac{-\ell^2}{m r_c^{2-n}}$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{n(n+1)k}{r^{n+2}}, \quad \text{so} \quad \left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_c} = \frac{kn(n+1)}{r_c^{n+2}} = \frac{-\ell^2(n+1)}{m r_c^4}.$$

So stability implies $\frac{\ell^2}{mr_c^4}[3-(n+1)] > 0$ so $2-n>0 \Rightarrow \boxed{n<2}$

This includes the case $n=1$, Newtonian gravity = Kepler problem.

Central Potential: Solving for orbits

Two constants of motion:

$$\begin{cases} E = \frac{1}{2} m \dot{r}^2 + V_{eff}(r) \\ \ell = m r^2 \dot{\phi} \end{cases} \qquad V_{eff}(r) = V(r) + \frac{\ell^2}{2mr^2}$$

To find $r(t)$: $\left(\frac{dr}{dt}\right)^2 = \frac{2}{m}\left[E - V_{eff}(r)\right]$, so

$$dt = \pm \frac{1}{\sqrt{\frac{2}{m}[E - V_{eff}(r)]}}\, dr$$

Integrate both sides:

$$\int_{t_0}^{t} dt = \pm \int_{r_0}^{r} dr \frac{1}{\sqrt{\frac{2}{m}[E - V_{eff}(r)]}}$$

$$\boxed{t - t_0 = \pm \int_{r_0}^{r} dr \frac{1}{\sqrt{\frac{2}{m}[E - V_{eff}(r)]}}} \longrightarrow$$

Reduced to quadratures
Implicit solution for $r(t)$

To find $\phi(t)$: $\frac{d\phi}{dt} = \frac{\ell}{mr^2} \Longrightarrow d\phi = dt \frac{\ell}{mr(t)^2}$. So

$$\boxed{\phi - \phi_0 = \frac{\ell}{m} \int_{t_0}^{t} dt \frac{1}{r(t)^2}} \longrightarrow \phi(t) \text{ implicit solution.}$$

If we just want shape of orbit, eliminate dt.

$$E = \frac{1}{2} m \underbrace{\left(\frac{dr}{dt}\right)^2}_{\left(\frac{dr}{d\phi}\frac{d\phi}{dt}\right)^2 = \left(\frac{dr}{d\phi}\right)^2 \left(\frac{\ell}{mr^2}\right)^2} + V_{eff}(r) = \frac{\ell^2}{2mr^4}\left(\frac{dr}{d\phi}\right)^2 + V_{eff}(r)$$

So $\boxed{\dfrac{dr}{d\phi} = \pm \dfrac{r^2}{\ell}\sqrt{2m(E-V_{eff})}}$ or

$$d\phi = \frac{\ell}{r^2\sqrt{2m(E-V_{eff})}}\,dr \qquad \text{or}$$

$$\boxed{\phi - \phi_0 = \pm \ell \int_{r_0}^{r} \frac{dr}{r^2\sqrt{2m(E-V_{eff})}}}$$

Implicitly gives $r(\phi)$

Kepler problem

$V = -\dfrac{GMm}{r}$ $\qquad F_r = -\dfrac{GMm}{r^2}$

$$V_{eff} = -\frac{GMm}{r} + \frac{\ell^2}{2mr^2}. \qquad E = \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} + \frac{\ell^2}{2mr^2}$$

$= \text{constant}.$

Solve for orbit relation $\phi(r)$.

$$\phi - \phi_0 = \pm \frac{\ell}{\sqrt{2m}} \int_{r_0}^{r} dr \, \frac{1}{r^2} \frac{1}{\sqrt{E + \frac{GMm}{r} - \frac{\ell^2}{2mr^2}}}$$

Easier than doing the integral: retreat to diff eq.

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{2m(E + GMm/r - \frac{\ell^2}{2mr^2})}{\ell^2} = \frac{2mE}{\ell^2} + \frac{2GMm^2}{\ell^2 r} - \frac{1}{r^2}$$

Let $u = 1/r$, $\quad du = -\frac{1}{r^2}dr.$ $\qquad$ So

$$\left(\frac{du}{d\phi}\right)^2 = \frac{2mE}{\ell^2} + \frac{2GMm^2}{\ell^2}u - u^2.$$

Try: $\boxed{u = C\left[1 - \epsilon\cos(\phi - \phi_0)\right]}$ $\qquad \dfrac{du}{d\phi} = C\epsilon\sin(\phi-\phi_0)$

constants

$$\left(\frac{du}{d\phi}\right)^2 = C^2\epsilon^2\left(1-\cos^2(\phi-\phi_0)\right)$$

Plug in: $$C^2\epsilon^2\left(1-\cos^2\right) = \frac{2mE}{\ell^2} + \frac{2GMm^2}{\ell^2}\,C\left[1-\epsilon\cos\right] - C^2\left[1-\epsilon\cos\right]^2$$

Expand: $\cos^2$ terms cancel

$\cos^1$ terms $\Rightarrow$ $C = \dfrac{GMm^2}{\ell^2}$

$\cos^0$ terms $\Rightarrow$ $\epsilon = \sqrt{1+\dfrac{2E\ell^2}{G^2M^2m^3}}$

ϕ_0 = anything = initial angle

$$\boxed{\frac{1}{r} = C\left[1-\epsilon\cos\phi\right]}$$

conic sections!

$\phi=\pi \Rightarrow$ maximum $\frac{1}{r}$ $\Leftrightarrow$ minimum r

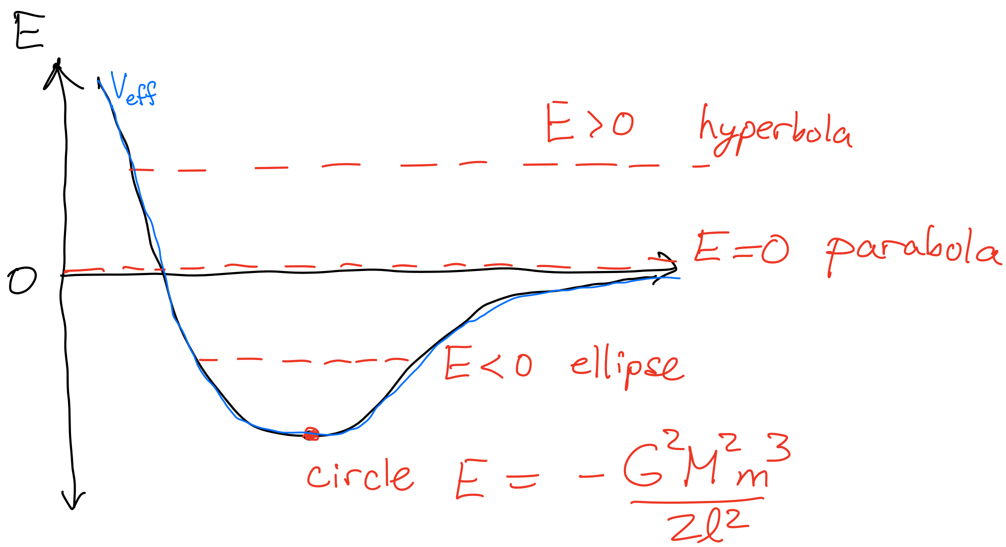
= closest approach.

$\epsilon=0 \Rightarrow r = \frac{1}{C}$ = constant = circle, $E = \underbrace{-\dfrac{G^2M^2m^3}{2\ell^2}}_{\text{minimum value of } V_{eff}\text{ stable}}$

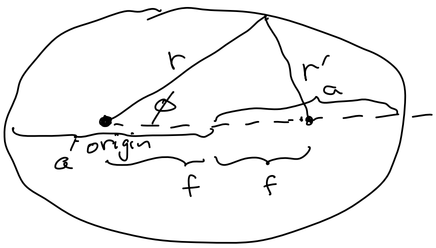
$\epsilon<1 \Rightarrow E<0$ ellipse (eccentricity ϵ)

$\epsilon=1 \Rightarrow E=0$ parabola ("tuned")

$\epsilon>1 \Rightarrow E>0$ hyperbola



Ellipse:



Definition of ellipse:

$$\text{constant} = r + r' = (a+f) + (a-f)$$
$$= 2a$$
$$= \text{major axis}$$

$2f$ = distance between foci

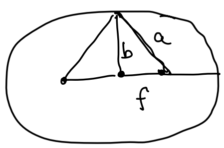
Law of cosines:
$$\underbrace{\underbrace{r'^2}_{(2a-r)^2}}_{\cancel{r^2} - 4ar + 4a^2} = \cancel{r^2} + (2f)^2 - 2r(2f)\cos\phi$$

So $$4(a^2 - f^2) = r\, 4a\left(1 - \frac{f}{a}\cos\phi\right) \quad \Rightarrow \quad \frac{1}{r} = \frac{a}{a^2 - f^2}\left(1 - \frac{f}{a}\cos\phi\right)$$

<u>Kepler's First Law: Planets follow ellipses with sun at one focus.</u>

Also, $$\epsilon = \frac{f}{a} = \sqrt{1 + \frac{2E\ell^2}{G^2M^2m^3}} \quad \text{and} \quad C = \frac{GMm^2}{\ell^2} = \frac{a}{a^2\left(1 - \left(1 + \frac{2E\ell^2}{G^2M^2m^3}\right)\right)}$$

or $$E = -\frac{GMm}{2a} \quad \Leftrightarrow \quad a = -\frac{GMm}{2E} > 0$$



$$b = \text{semi-minor axis} = \sqrt{a^2 - f^2} = a\sqrt{1 - \epsilon^2}$$

$$\text{Area of orbit} = \pi a b = \pi a^2\sqrt{1-\epsilon^2} = \pi a^2 \sqrt{\frac{-2E\ell^2}{G^2M^2m^3}}$$
$$= \pi a^2 \sqrt{\frac{\ell^2}{GMm^2 a}} = \pi a^{3/2} \frac{\ell}{m\sqrt{GM}}$$

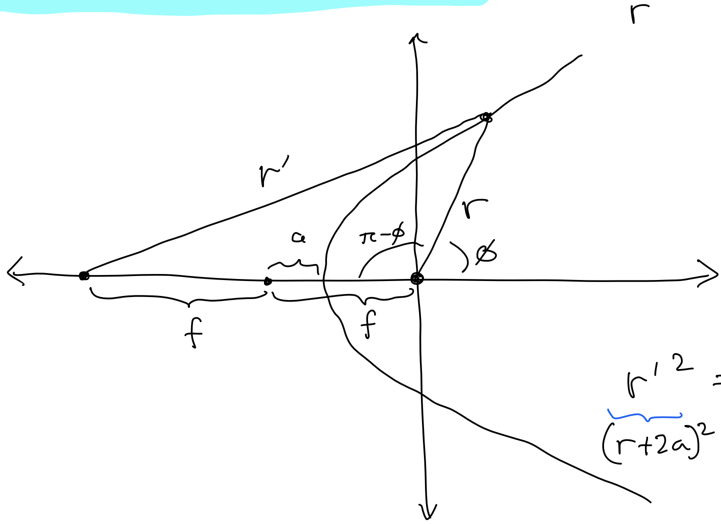
Recall $\frac{dA}{dt} = \frac{\ell}{2m}$, so Area of orbit $= \frac{\ell}{2m}\tau$ ← period of orbit.

So, $$\boxed{\tau = \frac{2\pi}{\sqrt{GM}} a^{3/2}}$$ ← Independent of m (planet mass)

Kepler's 3rd Law $$\tau^2 = \left(\frac{4\pi^2}{GM}\right) a^3$$

Hyperbola case $E > 0$

$$\frac{1}{r} = C(1 - \epsilon \cos\phi)$$



Definition of hyperbola:

$$r' - r = (f+a) - (f-a) = 2a$$

Law of cosines:

$$\underbrace{r'^2}_{(r+2a)^2 = r^2 + 4ar + 4a^2} = r^2 + (2f)^2 - 2r(2f)\underbrace{\cos(\pi - \phi)}_{-\cos\phi}$$

So $\displaystyle \frac{1}{r} = \frac{a}{f^2 - a^2}\left(1 - \frac{f}{a}\cos\phi\right)$

$$C = \frac{GMm^2}{\ell^2} = \frac{a}{f^2 - a^2} > 0, \qquad \epsilon = \frac{f}{a} = \sqrt{1 + \frac{2E\ell^2}{G^2M^2m^3}} > 1.$$

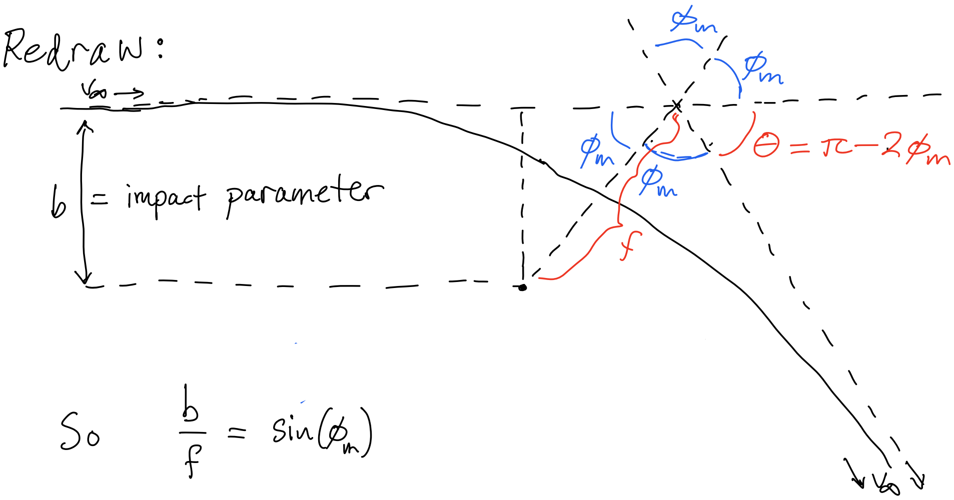
Closest approach: $r_{min} = f - a = a(\epsilon - 1)$.

Since $\frac{1}{r} > 0$, we also need $\frac{f}{a}\cos\phi < 1$, or

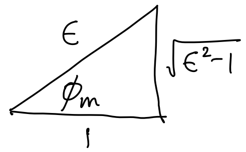
$$\cos\phi < \frac{1}{\epsilon} = \frac{1}{\sqrt{1 + \frac{2E\ell^2}{G^2M^2m^3}}}.$$

Asymptotic angle $\phi_m = \arccos\left(\frac{1}{\epsilon}\right)$

Redraw:



So $\displaystyle \frac{b}{f} = \sin(\phi_m)$



so $\sin(\phi_m) = \dfrac{\sqrt{\epsilon^2-1}}{\epsilon} = \dfrac{b}{a\epsilon} \Rightarrow b = \sqrt{\epsilon^2-1}\; a$

So $r_{min} = a(\epsilon-1) = b\sqrt{\dfrac{\epsilon-1}{\epsilon+1}}$.

For large r: $E = \frac{1}{2}mv_\infty^2 - \cancel{\dfrac{GMm}{r}} - \cancel{\dfrac{\ell^2}{2mr^2}} \Rightarrow \boxed{E = \frac{1}{2}mv_\infty^2}$

and $\vec{L} = (b\hat{y}) \times (mv_\infty \hat{x}) = -mbv_\infty \hat{z} \Rightarrow \boxed{\ell = mbv_\infty}$

So $\epsilon = \sqrt{1 + \dfrac{b^2 v_\infty^4}{G^2M^2}}$.

Total deflection angle $\theta = \pi - 2\phi_m = \pi - 2\arccos\left(\frac{1}{\epsilon}\right)$
$= \pi - 2\arctan\left(\sqrt{\epsilon^2-1}\right)$

So $\dfrac{\theta}{2} = \dfrac{\pi}{2} - \arctan\left(\sqrt{\epsilon^2-1}\right)$. Trig identity:

$\boxed{\cot\left(\frac{\theta}{2}\right) = \sqrt{\epsilon^2-1} = \dfrac{b v_\infty^2}{GM}}$ or $\boxed{\tan\left(\frac{\theta}{2}\right) = \dfrac{GM}{b v_\infty^2}}$

Small scattering angle $\theta \to 0$ for small GM, or large b, or large v_∞^2.

Rewrite orbit equation:

$$\frac{1}{r} = \frac{GMm^2}{\ell^2}\left(1 - \sqrt{1 + \frac{2E\ell^2}{G^2M^2m^3}}\,\cos\phi\right)$$

$$= \frac{GM}{b^2 v_\infty^2}\left(1 - \sqrt{1 + \frac{b^2 v_\infty^4}{G^2M^2}}\,\cos\phi\right)$$

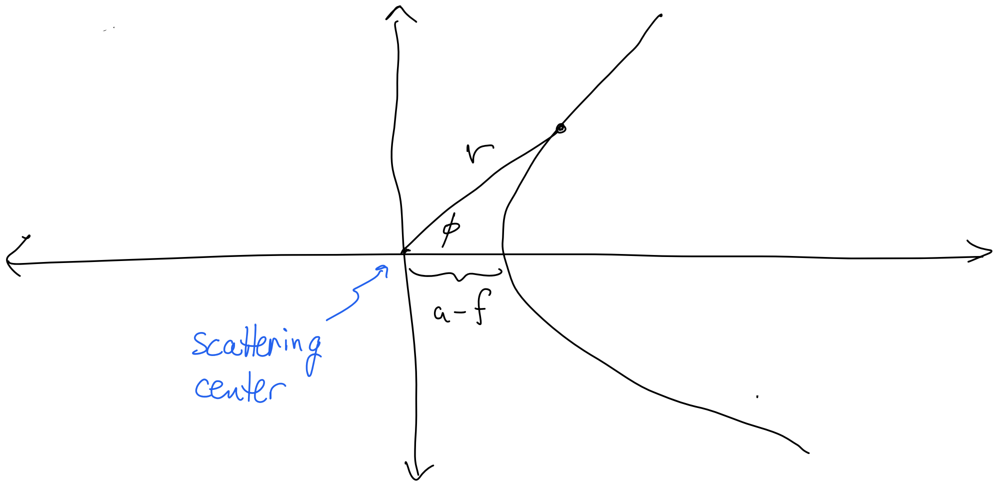
This assumed an attractive gravitational force $GMm > 0$.

What if $GMm \to -k$ (repulsive Coulomb force between like charges.)

$\quad\quad\quad\quad\quad\quad\quad$ positive k

Then $$\frac{1}{r} = -\frac{mk}{\ell^2}\Big(1 - \underbrace{\sqrt{1+\frac{2E\ell^2}{k^2 m}}}_{\epsilon}\cos\phi\Big).$$

In the preceding, $f < a < 0$, $\quad a = \dfrac{-\ell^2}{mk(\epsilon^2-1)}$



In this, Need $1-\epsilon\cos\phi < 0$, so $\cos\phi > 1/\epsilon$, and

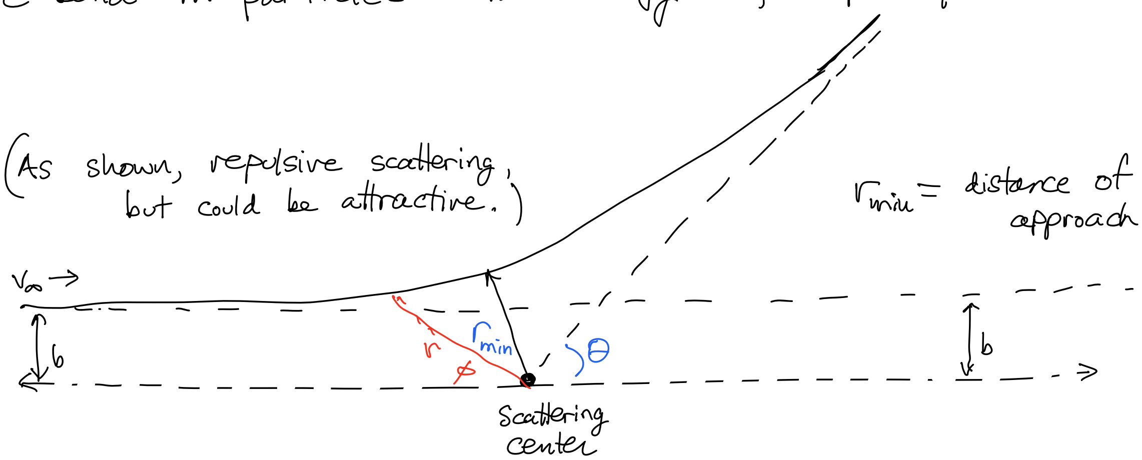
$\phi_m = \arccos\left(\frac{1}{\epsilon}\right)$ again.

Scattering Kinematics & Cross-Sections

We send in particles with energy E, impact parameter b.

(As shown, repulsive scattering, but could be attractive.)

r_{min} = distance of closest approach



Solve to find $\phi(r)$:

$$\frac{d\phi}{dr} = \frac{d\phi/dt}{dr/dt} = \frac{\frac{\ell}{mr^2}}{\sqrt{\frac{2}{m}}\sqrt{E - V(r) - \frac{\ell^2}{2mr^2}}}.$$

Then total deflection angle is

$\theta = |\pi - 2\phi|$. The orbit is symmetric about closest approach,

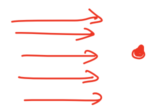
So $$\phi(r_{min}) = \int_{r_{min}}^{\infty} dr \frac{d\phi}{dr} = \int_{r_{min}}^{\infty} dr \frac{\frac{\ell}{mr^2}}{\sqrt{\frac{2}{m}}\sqrt{E - V(r) - \frac{\ell^2}{2mr^2}}}$$

What is r_{min}? Find where $\frac{dr}{dt} = 0 \iff$ denominator vanishes

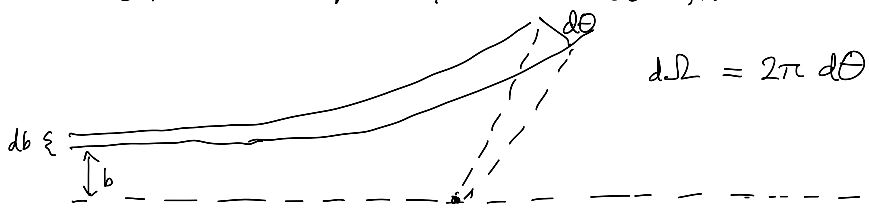
$E = \frac{1}{2}mv_\infty^2$, $\ell = mbv_\infty$.

Incident beam = particles with random b, flux

$$f = \frac{\text{Number of incident particles}}{(\text{area})\,(\text{time})}$$



<u>Scattering problem</u>: How many particles come out with angle between $\begin{cases} \theta \text{ and } \theta + d\theta \\ \varphi \text{ and } \varphi + d\varphi \end{cases}$? ← call this solid angle $d\Omega$



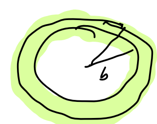
$d\Omega = 2\pi\, d\theta$ if no dependence on φ.

Define: $$d\sigma = \underbrace{\frac{\begin{pmatrix}\text{rate of particles} \\ \text{scattered in solid angle } d\Omega\end{pmatrix}}{f}}_{\frac{d\sigma}{d\Omega}} \underbrace{d\Omega}_{d\varphi\, d(\cos\theta)}$$

$\underbrace{d\varphi}$ integrates to 2π if azimuthal symmetry

Now θ is some function of b, write $\theta(b)$

Area of initial beam between b and $b+db$ = $2\pi b\, db$



Suppose there are n particles in that annulus in time t.

Then $f = \dfrac{n}{t(2\pi b\, db)}$, and $\dfrac{d\sigma}{d\Omega} = \dfrac{\dfrac{n}{t\, 2\pi\, d(\cos\theta)}}{\dfrac{n}{t\, 2\pi b\, db}}$

or $\dfrac{d\sigma}{d\Omega} = b\left|\dfrac{db}{d(\cos\theta)}\right| = \dfrac{b}{\sin\theta}\left|\dfrac{db}{d\theta}\right|$.

Can also define $\dfrac{d\sigma}{d(\cos\theta)} = \underbrace{\int_0^{2\pi} d\phi}_{2\pi}\, \underbrace{\dfrac{d\sigma}{d\phi\, d(\cos\theta)}}_{\text{doesn't depend on } \phi} = \dfrac{2\pi b}{\sin\theta}\left|\dfrac{db}{d\theta}\right|$,

and total cross-section $\sigma = \int_{-1}^{1} d(\cos\theta)\, \dfrac{d\sigma}{d(\cos\theta)}$.

Note: units of cross-section = (area).

Recipe for computing differential cross-section $\dfrac{d\sigma}{d\Omega}(\theta, E)$.

1) Given $V(r)$, find $\varphi(r_{min}) = \int_{r_{min}}^{\infty} dr \dfrac{\frac{\ell}{mr^2}}{\sqrt{\frac{2}{m}(E - V_{eff}(r))}}$, where

r_{min} = solution to denominator vanishing, for a given E, b.

2) Express $\theta(b, E) = |\pi - 2\varphi(r_{min})|$

3) Invert the relationship to find $b(\theta, E)$ (may get several solutions)

4) $\dfrac{d\sigma}{d\Omega} = \sum_k \dfrac{b_k}{\sin\theta}\left|\dfrac{db_k}{d\theta}\right|$

We will do three examples:

1) Hard-sphere scattering
2) Rutherford scattering
3) Rainbows $\leftarrow$ analogy for how classical mechanics emerges as an approximation to quantum mechanics