Hamilton-Jacobi Theory, Review:

Recall: $\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q_k}, q_k, t\right) = 0$ for $H(p_k, q_k, t)$.

S has three roles: 1) Type-2 generating function $S(q_k, P_k, t)$ so that $p_n = \frac{\partial}{\partial q_n} S(q_k, P_k, t)$, $Q_n = \frac{\partial}{\partial P_n} S(q_k, P_k, t)$,

$$\tilde{H}(P_k, Q_k, t) = 0, \quad \dot{P}_k = 0, \quad \dot{Q}_k = 0.$$

2) S = action, when evaluated "on-shell" = eqns of motion satisfied

3) Solution to H-J equation $S(q_k, \alpha_k, t)$ = Hamilton's Principal Function

constants of integration, often E, and/or P_k and/or Q_k.

Then $\frac{\partial S}{\partial \alpha_k}$ = constant for each α_k.

Example (F+W, problem 6.9, done two ways)

$$T = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right)\left(q_1^2 + q_2^2\right) \qquad V = \frac{k}{(q_1^2 + q_2^2)} \implies \begin{aligned} p_1 &= \dot{x}_1\,(x_1^2 + x_2^2) \\ p_2 &= \dot{x}_2\,(x_1^2 + x_2^2) \end{aligned}$$

$\implies H = \frac{p_1^2 + p_2^2 + 2k}{2(x_1^2 + x_2^2)}$. So, the H-J eqn is:

$$\frac{\partial S}{\partial t} + \frac{1}{2(x_1^2 + x_2^2)}\left[\left(\frac{\partial S}{\partial x_1}\right)^2 + \left(\frac{\partial S}{\partial x_2}\right)^2 + 2k\right] = 0. \quad \text{Try:}$$

$$S = -E(t - t_0) + W(x_1, x_2)$$ ← Hamilton's Characteristic Function

Then $\left(\frac{\partial W}{\partial x_1}\right)^2 + \left(\frac{\partial W}{\partial x_2}\right)^2 + 2k = 2E(x_1^2 + x_2^2)$.

Separation of variables: $W(x_1, x_2) = W_1(x_1) + W_2(x_2)$.

$$\underbrace{\left(\frac{\partial W_1}{\partial x_1}\right)^2 - 2Ex_1^2}_{\text{Not a function of } x_2\text{, call it } \alpha} + \underbrace{\left(\frac{\partial W_2}{\partial x_2}\right)^2 - 2Ex_2^2 + 2k}_{\text{Not a function of } x_1 \text{ must } = -\alpha.} = 0$$

Now α must be constant (no x_1 or x_2 dependence).

So $\dfrac{\partial W_1}{\partial x_1} = \sqrt{\alpha + 2Ex_1^2} \Rightarrow W_1 = \int_{x_{10}}^{x_1} dx_1 \sqrt{\alpha + 2Ex_1^2}$, and

$$\frac{\partial W_2}{\partial x_2} = \sqrt{-2k-\alpha+2Ex_2^2} \Rightarrow W_2 = \int_{x_{20}}^{x_2} dx_2 \sqrt{-2k-\alpha+2Ex_2^2}$$

So: $$S(x_1, x_2, t) = -E(t-t_0) + \int_{x_{10}}^{x_1} du \sqrt{2Eu^2+\alpha} + \int_{x_{20}}^{x_2} dv \sqrt{2Ev^2-2k-\alpha}$$

Count integration constants: x_{10}, x_{20}, E, α

4 = 2 initial positions, 2 initial velocities, OR

= 2 new coordinates $Q_1\, Q_2$, 2 new momenta P_1, P_2.

$$\frac{\partial S}{\partial E} = \text{constant} \Rightarrow (t-t_0) = \int_{x_{10}}^{x_1} du \frac{u^2}{\sqrt{2Eu^2+\alpha}} + \int_{x_{20}}^{x_2} dv \frac{v^2}{\sqrt{2Ev^2-\alpha-2k}}$$

$$\frac{\partial S}{\partial \alpha} = \text{constant} \Rightarrow \int_{x_{10}}^{x_1} du \frac{1}{\sqrt{2Eu^2+\alpha}} = \int_{x_{20}}^{x_2} dv \frac{1}{\sqrt{2Ev^2-\alpha-2k}}$$

Integrals are doable, not very enlightening, algebraic equations not solvable in general.

Method 2: polar coordinates

Define $x_1 = r\cos\phi$
$x_2 = r\sin\phi$.

Then $T = \frac{1}{2}r^2(\dot r^2 + \dot\phi^2 r^2)$ $\qquad V = k/r^2$

$P_r = \dot r r^2 \qquad P_\phi = \dot\phi r^4$, $\qquad$ So:

$$H = p_r \dot{r} + p_\phi \dot{\phi} - T + V$$

$$= p_r\left(\frac{p_r}{r^2}\right) + p_\phi\left(\frac{p_\phi}{r^4}\right) - \frac{1}{2} r^2 \left(\left(\frac{p_r}{r^2}\right)^2 + \left(\frac{p_\phi}{r^4}\right)^2 r^2\right) + \frac{k}{r^2}$$

$$= \frac{p_r^2}{2r^2} + \frac{p_\phi^2}{2r^4} + \frac{k}{r^2}$$

(Aside: $\{r, p_r\} = \{\phi, p_\phi\} = 1$, all other Poisson brackets 0.)

Note ϕ is cyclic: $p_\phi = \text{constant}$

H-J equation: $\frac{\partial S}{\partial t} + \frac{1}{2r^2}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2r^4}\left(\frac{\partial S}{\partial \phi}\right)^2 + \frac{k}{r^2} = 0.$

Try: $S = -t(E - E_0) + \underbrace{W(r, \phi)}_{W_r(r) + W_\phi(\phi)}$

This gives $E = \frac{1}{2r^2}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{2r^4}\underbrace{\left(\frac{\partial W_\phi}{\partial \phi}\right)^2}_{p_\phi^2} + \frac{k}{r^2}$

So $W_\phi = p_\phi(\phi - \phi_0)$ integration constants

Still need to solve W_r.

$\left(\frac{\partial W_r}{\partial r}\right)^2 = 2r^2 E - \frac{p_\phi^2}{r^2} - 2k.$ So

$$\frac{\partial W_r}{\partial r} = \sqrt{2Er^2 - \frac{p_\phi^2}{r^2} - 2k} \implies W_r = \pm \int_{r_0}^{r} du \sqrt{2Eu^2 - \frac{p_\phi^2}{u^2} - 2k}$$

So $S = -\underbrace{E(t - t_0) + p_\phi(\phi - \phi_0)}_{\text{conserved quantities = simplicity}} \pm \int_{r_0}^{r} du \sqrt{2Eu^2 - \frac{p_\phi^2}{u^2} - 2k}$

$$\frac{\partial S}{\partial E} = \text{constant} \implies t - t_0 = \pm \int_{r_0}^{r} du \frac{u^2}{\sqrt{2Eu^2 - \frac{p_\phi^2}{u^2} - 2k}} \implies r(t)$$

$$\frac{\partial S}{\partial p_\phi} = \text{constant} \implies \phi - \phi_0 = \pm \int_{r_0}^{r} du \frac{p_\phi / 2u^2}{\sqrt{2Eu^2 - \frac{p_\phi^2}{u^2} - 2k}}$$

Allowed motion: $Q(r) = 2Er^2 - \frac{p_\phi^2}{r^2} - 2k > 0.$

Turning points: $2Er^4 - 2kr^2 - p_\phi^2 = 0 \Rightarrow$

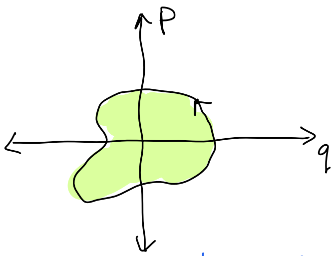
$$r_\pm^2 = \frac{2k \pm \sqrt{4k^2 + 8Ep_\phi^2}}{4E} = \frac{k \pm \sqrt{k^2 + 2Ep_\phi^2}}{2E}$$

If $k>0$, need $E>0$, and $r > r_+$.

Action-Angle variables

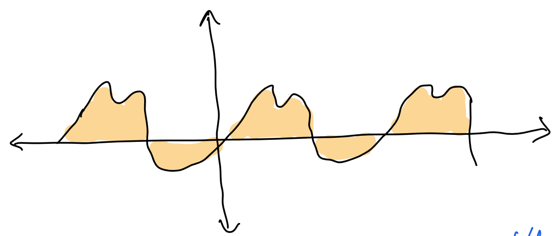
Consider problems with:

1) Conserved $H(p_k, q_k)$ (no explicit t dependence)
2) Separable H-J equation $W(q_1, q_2, \ldots, q_N) = W_1(q_1) + \ldots + W_N(q_N)$.
3) Periodic dynamics: p_k's return to same values and:
   - (a) so do q_k's, <u>or</u>
   - (b) q_k's get incremented by constant

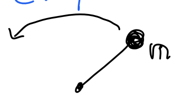


Example: harmonic oscillator (HO)

or



Example: pendulum with large E



Action variables: $J_n = \oint dq_n\, p_n$ = <u>area</u> in phase space, for one period, for each (q_n, p_n) pair.

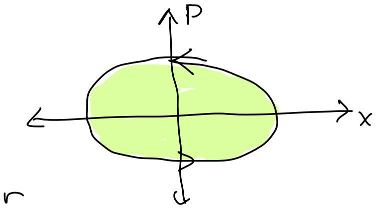
one complete period

Alternate version: $I_n = \frac{1}{2\pi} \oint dq_n\, p_n$

Example: HO $\quad H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 = E.$

Range of x is $x_{max} = \sqrt{\frac{2E}{m\omega^2}}$ $\quad x_{min} = -\sqrt{\frac{2E}{m\omega^2}}$.

For each x: $\quad p = \sqrt{2mE - m^2\omega^2 x^2}$.



So $J = \text{area} = \underset{\substack{\uparrow \\ \text{both} \\ \text{directions}}}{2} \int_{-\sqrt{\frac{2mE}{m\omega^2}}}^{\sqrt{\frac{2mE}{m\omega^2}}} dx \sqrt{2mE - m\omega^2 x^2}$, or

Let $v = \sqrt{\frac{m\omega^2}{2E}}\, x$, $\quad dx = \sqrt{\frac{2E}{m\omega^2}}\, dv$. Then

$$I = \frac{1}{2\pi}\, 2\left(\frac{2E}{\omega}\right) \underbrace{\int_{-1}^{1} dv \sqrt{1-v^2}}_{\pi/2} = \frac{E}{\omega}. \quad \Rightarrow \quad H = I\omega$$

More generally, $H = \omega_1 I_1 + \omega_2 I_2 + \dots \omega_N I_N$

Angle variables $\{\theta_n, I_k\}_{PB} = \delta_{nk}$, and

$$\dot{\theta}_n = \frac{\partial}{\partial I_n} H(I_1, I_2, \dots, I_N) = \omega_n$$

Note: θ_n is usually not a geometric angle.

Periods of oscillations (normal modes) are $T_N = \frac{2\pi}{\omega_N}$.

(Use of J_n: $\quad \nu_n = \frac{\partial H}{\partial J_n}$ = ordinary (not angular) frequencies.)

Example: 3-d HO $\quad H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$

H-J eqn: $\quad \frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right] + \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right) = E$

$W = W_x(x) + W_y(y) + W_z(z) \quad \Rightarrow$

$$\underbrace{\frac{1}{2m}\left(\frac{dW_x}{dx}\right)^2 + \frac{1}{2}m\omega_x^2 x^2}_{\alpha_x} + \underbrace{\frac{1}{2m}\left(\frac{dW_y}{dy}\right)^2 + \frac{1}{2}m\omega_y^2 y^2}_{\alpha_y} + \underbrace{\frac{1}{2m}\left(\frac{dW_z}{dz}\right)^2 + \frac{1}{2}m\omega_z^2 z^2 - E}_{\alpha_z - E} = 0$$

So $\alpha_x + \alpha_y + \alpha_z = E = H$

For each of $q_k = x, y, z$:

$$p_k = \frac{\partial S}{\partial q_k} = \frac{\partial W_k}{\partial q_k} = \pm\sqrt{2m\left(\alpha_k - \tfrac{1}{2} m\omega_k^2 q_k^2\right)} .$$

Define: new variables θ_k (= angle variables) by

$$q_k = \sqrt{\frac{2\alpha_k}{m\omega_k^2}} \sin\theta_k , \quad dq_k = \sqrt{\frac{2\alpha_k}{m\omega_k^2}} \cos\theta_k \, d\theta_k , \quad p_k = \sqrt{2m\alpha_k} \cos\theta_k .$$

Now $I_k = \frac{1}{2\pi} \oint dq_k \, p_k = \frac{1}{2\pi} \frac{2\alpha_k}{\omega_k} \underbrace{\int_0^{2\pi} d\theta_k \cos^2\theta_k}_{\pi} = \frac{\alpha_k}{\omega_k}$, or

$\alpha_k = \omega_k I_k$. So $H = \sum_k \alpha_k = \omega_x I_x + \omega_y I_y + \omega_z I_z$

Normal mode frequencies: $\omega_k = \frac{\partial H}{\partial I_k}$.

Period of oscillation for <u>whole system</u>: undefined, unless

$\frac{\omega_x}{\omega_z}$ and $\frac{\omega_y}{\omega_z}$ are <u>rational</u> numbers. (Example: isotropic $\omega_x = \omega_y = \omega_z = \omega$.)

Example: Kepler problem

$V(r) = -\frac{k}{r}$, ($k = GMm$ in gravity)

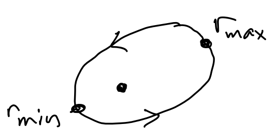
$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - \frac{k}{r}$. Two action variables.

$I_\phi = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, p_\phi = p_\phi$ = conserved angular momentum

To find I_r, first solve for $p_r = \sqrt{2m\left(E + \frac{k}{r}\right) - \frac{p_\phi^2}{r^2}}$

Then $I_r = \frac{1}{2\pi} \oint dr \, p_r = \frac{1}{2\pi} \, 2 \int_{r_{min}}^{r_{max}} dr \sqrt{2m\left(E + \frac{k}{r}\right) - \frac{I_\phi^2}{r^2}}$

motion from r_{min} to r_{max}, then from r_{max} to r_{min}



r_{max}, r_{min} are solutions to $\sqrt{\text{stuff}} = 0$

To do the integral, rewrite:

$$\sqrt{2m\left(E+\frac{k}{r}\right)-\frac{I_\phi^2}{r^2}} = \sqrt{-2mE}\sqrt{\left(1-\frac{r_{min}}{r}\right)\left(\frac{r_{max}}{r}-1\right)}$$

So
$$I_r = \frac{\sqrt{-2mE}}{\pi}\underbrace{\int_{r_{min}}^{r_{max}} dr \sqrt{\left(1-\frac{r_{min}}{r}\right)\left(\frac{r_{max}}{r}-1\right)}}_{\frac{\pi}{2}(r_{min}+r_{max})-\pi\sqrt{r_{min}r_{max}}}$$

Now $2mEr^2+2mkr-I_\phi^2=0 \Rightarrow \underbrace{r^2+\frac{k}{E}r-\frac{I_\phi^2}{2mE}}_{(r-r_{min})(r-r_{max})}=0 \Rightarrow$

$$r_{min}+r_{max} = -\frac{k}{E}$$

$$r_{min}r_{max} = -I_\phi^2/2mE$$

So
$$I_r = \frac{\sqrt{-2mE}}{\pi}\left(\frac{\pi}{2}\left(\frac{-k}{E}\right)-\pi\sqrt{\frac{I_\phi^2}{-2mE}}\right) = k\sqrt{\frac{m}{-2E}} - I_\phi$$

So $I_r + I_\phi = \sqrt{\frac{m}{-2E}}\,k$. So $(I_r+I_\phi)^2 = \frac{k^2 m}{-2E}$, or

$$H = E = \frac{-k^2 m}{2(I_r+I_\phi)^2}.$$

That was with $\theta = \frac{\pi}{2}$ fixed.

Repeat with general r, θ, ϕ:

$$\boxed{H = E = \frac{-k^2 m}{2(I_r+I_\phi+I_\theta)^2}.}$$

To find angular frequencies of the motion:

$$\omega_\phi = \frac{\partial H}{\partial I_\phi} = \frac{k^2 m}{(I_r+I_\phi+I_\theta)^3}.$$

But, because of special form of H:

$$\boxed{\omega_r = \omega_\theta = \omega_\phi}.$$

Closed orbits.

Period of revolution $T = \frac{2\pi}{\omega_\phi}$ = Period of radial oscillation.

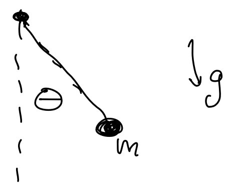
This is a special property of Kepler potential $V = -\frac{k}{r}$.

Only other central force $V = k r^n$ with this property is $n=2$ (isotropic 3-d HO).

Adiabatic invariants Suppose a potential depends on parameters $\lambda(t)$ that can change in time:

$H = H(p_k, q_k, \lambda(t))$. Then E is not conserved. But, if $\lambda(t)$ changes slowly, there are still conserved quantities: the action variables $I_n = \frac{1}{2\pi}\oint dq_n\, p_n$. Proof in text.

Example: Pendulum $H = \frac{p_\theta^2}{2mL^2} + \frac{1}{2} m g L \theta^2$



$$I_\theta = \frac{1}{2\pi}\int_{\theta_{min}}^{\theta_{max}} d\theta \; p_\theta \quad \text{where} \begin{cases} p_\theta = \sqrt{2mL^2E - m^2 g L^3 \theta^2} \\ \theta_{max} = -\theta_{min} = \sqrt{\frac{2E}{mgL}} \end{cases}$$

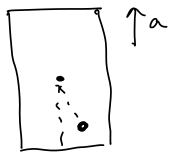
Let $\theta = \sqrt{\frac{2E}{mgL}}\, u \implies p_\theta = \sqrt{2mL^2E}\sqrt{1-u^2}$

$d\theta = \sqrt{\frac{2E}{mgL}}\, du.$ So

$$I_\theta = \frac{1}{2\pi}\sqrt{2mL^2E}\sqrt{\frac{2E}{mgL}}\underbrace{\int_{-1}^{1} du\,\sqrt{1-u^2}}_{\pi/2} = \frac{1}{2} E\sqrt{\frac{L}{g}}$$

$$= \frac{1}{4} m\sqrt{g}\, L^{3/2}\, \theta_{max}^2 = \text{constant if } g, L \text{ change adiabatically.}$$

Pendulum in an elevator:

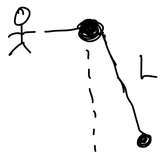


$g \to g+a$, slowly.
What happens to θ_{max}?

$$\sqrt{g}\,\theta_0^2 = \sqrt{g+a}\,\theta_\uparrow^2$$

So $\theta_\uparrow = \theta_0 \left(\frac{1}{1+a/g}\right)^{1/4}$. Smaller oscillation for positive a

Pendulum of variable length:



$L \to L+\Delta$, slowly.
What happens to θ_{max}?

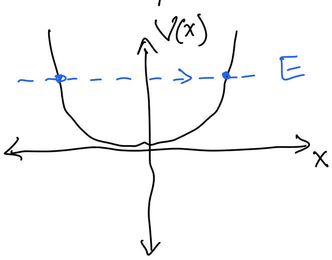
$$L^{3/2}\theta_0^2 = (L+\Delta)^{3/2}\theta_\Delta^2 .$$

So $\theta_\Delta = \theta_0 \left(\frac{1}{1+\Delta/L}\right)^{3/4}$ Smaller oscillation for longer length.

Example:

$H = \frac{p^2}{2m} + \lambda x^n$ (HW9, Problem 1 has $n=4$)

This is a steeper version of the harmonic oscillator:



$$p = \sqrt{2m\,(E - \lambda x^n)}$$

If $\lambda(t)$ depends slowly on t.

Action variable = adiabatic invariant $= \frac{1}{2\pi}\oint dx\, p$

$$I = \frac{2}{2\pi}\int_{x_{min}}^{x_{max}} dx\,\sqrt{2m(E-\lambda x^n)} \qquad x_{max} = -x_{min} = (E/\lambda)^{1/n}.$$

$$= \frac{\sqrt{2mE}}{\pi}\int_{-(E/\lambda)^{1/n}}^{(E/\lambda)^{1/n}} dx\,\sqrt{1-\frac{\lambda x^n}{E}} \qquad \text{Let } x = \left(\frac{E}{\lambda}\right)^{1/n} u$$

$$= \frac{\sqrt{2mE}}{\pi}\left(\frac{E}{\lambda}\right)^{1/n}\underbrace{\int_{-1}^{1} du\,\sqrt{1-u^n}}_{\text{dimensionless}} \qquad \propto E^{\frac{1}{2}+\frac{1}{n}}\lambda^{-1/n} = \text{adiabatic constant}$$

So $E \propto \lambda^{\frac{2}{n+2}} \implies E(t) = E(t_0)\left(\frac{\lambda(t)}{\lambda(t_0)}\right)^{\frac{2}{n+2}}$. Also, $\omega = \frac{\partial H}{\partial I}$