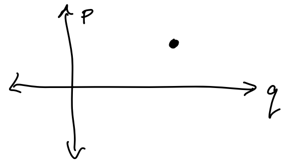
Distributions in Phase Space

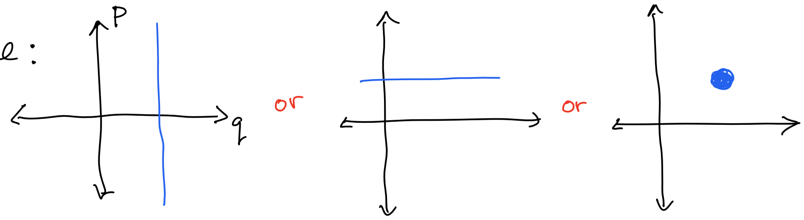
Perfect classical knowledge
= point in phase space



Perfect quantum knowledge:

$(\Delta q)(\Delta p) \geq \hbar/2$

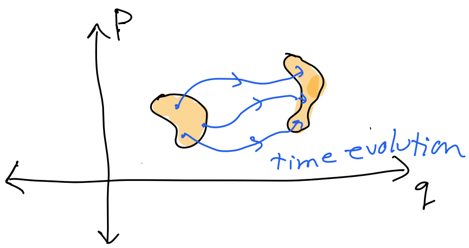
uncertainties



Imperfect classical knowledge:

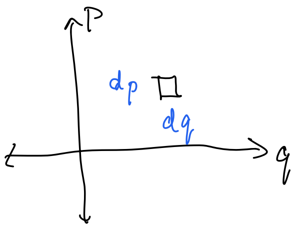
Describe with:

- areas
- probability distributions



Liouville's Theorem: Following trajectories in phase space, the volume of a region stays constant. (Shape changes!)

Proof: Consider an infinitesimal volume, and an infinitesimal time



$$dV = \underbrace{dq_1\, dq_2 \cdots dq_N\; dp_1\, dp_2 \cdots dp_N}_{\text{volume of 2N-dimensional hypercube}}$$

In time dt:

$$q_n \rightarrow q_n + \frac{dq_n}{dt}\, dt = q_n + \frac{\partial H}{\partial p_n}\, dt = Q_n$$

$$p_n \rightarrow p_n + \frac{dp_n}{dt}\, dt = p_n - \frac{\partial H}{\partial q_n}\, dt = P_n$$

New volume: $d\tilde{V} = dQ_1\, dQ_2 \cdots dQ_N\, dP_1 \cdots dP_N = (\det J)\, dV,$

where J = Jacobian matrix (2n x 2n) $= \left(\begin{array}{c|c} \frac{\partial Q_i}{\partial q_j} & \frac{\partial Q_i}{\partial p_j} \\ \hline \frac{\partial P_i}{\partial q_j} & \frac{\partial P_i}{\partial p_j} \end{array} \right)$

$$J = \left(\begin{array}{c|c} \delta_{ij} + dt\,\dfrac{\partial^2 H}{\partial p_i \partial q_j} & dt\,\dfrac{\partial^2 H}{\partial p_i \partial p_j} \\ \hline -dt\,\dfrac{\partial^2 H}{\partial q_i \partial q_j} & \delta_{ij} - dt\,\dfrac{\partial^2 H}{\partial q_i \partial p_j} \end{array}\right) = \underset{\substack{\uparrow \\ 2n\times 2n \\ \text{identity}}}{I} + dt\, A$$

where $A = \left(\begin{array}{c|c} \dfrac{\partial^2 H}{\partial p_i \partial q_j} & \sim\sim \\ \hline \sim\sim & -\dfrac{\partial^2 H}{\partial q_i \partial p_j} \end{array}\right)$.

Now: $\det(I + \epsilon A) = 1 + \epsilon\, \mathrm{Tr}(A) + \mathcal{O}(\epsilon^2)$

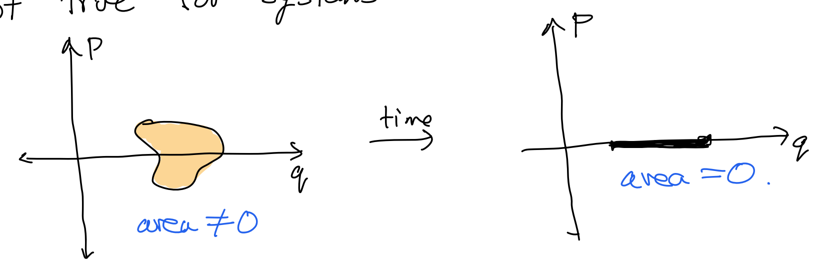
So $\det J = 1 + dt \sum_i \underbrace{\left(\dfrac{\partial^2 H}{\partial p_i \partial q_i} - \dfrac{\partial^2 H}{\partial p_i \partial q_i}\right)}_{0} + \cancel{\mathcal{O}(dt^2)} = 1.$

So $d\tilde{V} = dV$ ✓

Classical systems with Hamiltonian $\Longleftrightarrow$ incompressible fluid in phase space

(even if energy not conserved).

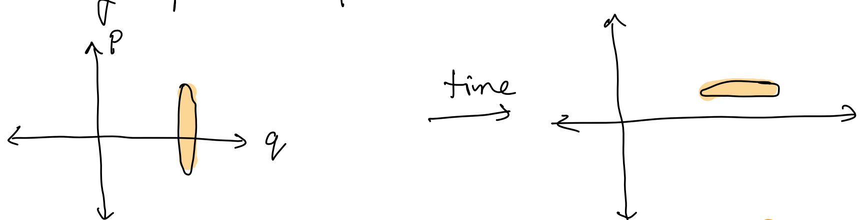
Not true for systems with friction:



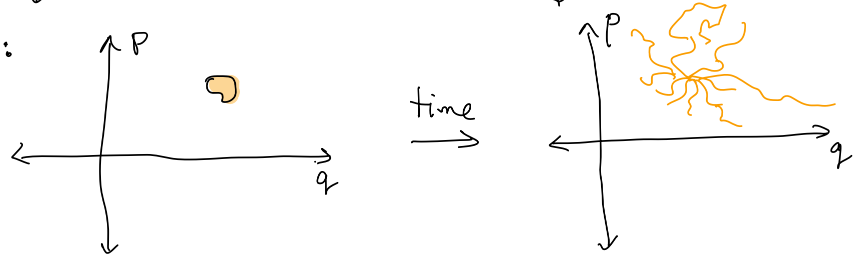
So:

friction $\Longleftrightarrow$ no Hamiltonian

Reducing spread in p $\Longleftrightarrow$ increasing spread in q:



Chaos:



Probability density description

$\rho(q_k, p_k, t)$ = density of probability in phase space

Conservation of probability:

$$\underbrace{\int dq_n \int dp_n}_{\text{2N-dimensional volume integral}} \rho(q_k, p_k, t) = 1 = \text{constant}$$

Local conservation law: $\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$ (recall from EM).

For an incompressible fluid: $\underbrace{\vec{J}}_{\text{2N-dim vector}} = \rho \vec{v}$ ← "velocity" field in phase space!

$$\vec{v} = \frac{d\vec{z}}{dt} = \left(\frac{dq_1}{dt}, \frac{dq_2}{dt}, \ldots \frac{dq_N}{dt}, \frac{dp_1}{dt}, \ldots, \frac{dp_N}{dt} \right).$$

So:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial z_i}\left(\rho \frac{dz_i}{dt}\right) = -\frac{\partial}{\partial q_i}(\rho \dot{q}_i) - \frac{\partial}{\partial p_i}(\rho \dot{p}_i)$$

$$= -\frac{\partial}{\partial q_i}\left(\rho \frac{\partial H}{\partial p_i}\right) - \frac{\partial}{\partial p_i}\left(-\rho \frac{\partial H}{\partial q_i}\right)$$

$$= \underbrace{-\frac{\partial \rho}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial \rho}{\partial p_i}\frac{\partial H}{\partial q_i}}_{\{H, \rho\}_{PB}} + \rho \underbrace{\left(-\frac{\partial^2 H}{\partial q_i \partial p_i} + \frac{\partial^2 H}{\partial p_i \partial q_i} \right)}_{0}$$

So, we have Liouville's Equation: $\frac{\partial \rho}{\partial t} = \{H, \rho\}_{PB}$

describes change in density with time.

Classical	Quantum	
$\{q_j, p_k\}_{PB} = \delta_{jk}$	$[q_j, p_k] = i\hbar \delta_{jk}$	
$\frac{dA}{dt} = \{H, A\}_{PB} + \frac{\partial A}{\partial t}$	$\frac{dA}{dt} = \frac{i}{\hbar}[H, A] + \frac{\partial A}{\partial t}$	(Heisenberg EOM)
$\frac{\partial \rho}{\partial t} = \{H, \rho\}_{PB}$	$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H, \rho]$	(ρ = density matrix operator)

Poincaré Recurrence Theorem

Consider a system with:

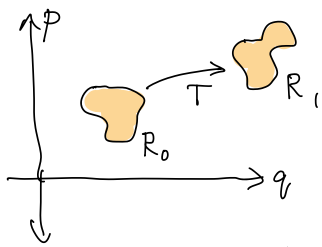
1) Time evolution governed by $H(p_k, q_k, t)$.
   (so, volumes are preserved)

2) All phase space orbits are bounded
   (for example, energy conserved, and initial conditions limit the energy, OR, phase space itself is bounded)

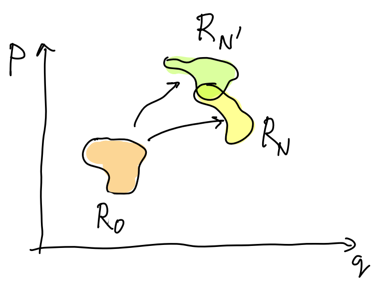
Then, for any region R_0 (no matter how small) in phase space, almost every point will return to R_0 after a finite time
might be very large.

Proof: I will assume for simplicity that $H(p_k, q_k)$ doesn't depend on t.
(Not necessary, but much simpler.)

Start with R_0 at $t=0$. It evolves to R_1 at $t=T$.
Assume they don't intersect.



Let R_n = region at time $t=nT$. Then there must be two integers N, N' such that R_N and $R_{N'}$ do intersect.
(Otherwise, the sum of volumes $R_0 + R_1 + R_2 + \ldots = \infty$, contradicts bounded volume assumption.)



$R_N \cap R_{N'}$ = at least one point

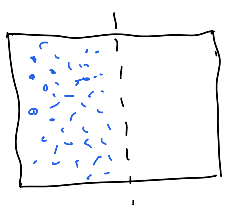
Assume $N' > N$. Now apply the inverse time evolution map for time TN, to find that $R_0 \cap R_{N'-N}$ = at least one point.

That point is in R_0 at $t=0$, and it is also in R_0 at time $t = T(N'-N)$. So we have at least one point in R_0 that "recurs."

Almost all points in R_0 have this property. To see why:
Let $N = \{$points in R_0 that never return to $R_0\}$. Suppose Volume$(N) > 0$. Now repeat steps above with R_0 replaced by N. There will be at least one point that returns to N. Contradiction!
So Volume$(N) = 0$. ← "set of measure 0", meaning of "almost".

Comments on Poincaré Recurrence

1) Some isolated points may never return.
2) Recurrence is not necessarily periodic
3) Even for very small R_0, some points may have much longer recurrence time than others.
4) Almost all points recur an infinite number of times, but the recurrence times can be vastly different.
5) Contradicts thermodynamics? (entropy increase = 2nd Law)



Gas of 10^{23} atoms. Start them in half volume. After a finite time, they will return (for almost all initial conditions).

Probability a given atom on L side: $\frac{1}{2}$.
" all atoms on L side: $\left(\frac{1}{2}\right)^{10^{23}}$

If time for atom to cross box is $\sim T$, then

$T_{\text{recurrence}} = 2^{(10^{23})} T$. Age of universe $= 2^{59}$ seconds

$2^{(10^{23})} \ggg 2^{59}$, no matter what T is.

Entropy increase = typical behavior

6) Often irrelevant for physics.

Hamilton–Jacobi Theory

Return to Type 2 Canonical Transform (would work just as well with Type 1, many books do)

Recall:
$$\begin{cases} p_n = \dfrac{\partial}{\partial q_n} S(q_k, P_k, t) \\ Q_n = \dfrac{\partial}{\partial P_n} S(q_k, P_k, t) \\ \tilde{H}(P_k, Q_k, t) = H(p_k, q_k, t) + \dfrac{\partial}{\partial t} S(q_k, P_k, t) \end{cases}$$

generating function, determines Q_k, P_k in terms of q_k, p_k

Ambitious idea: can we choose S so that $\tilde{H} = 0$?

If so, then $\dot{Q}_n = 0$, $\dot{P}_n = 0 \Rightarrow Q_n =$ constant, $P_n =$ constant

Yes! Need
$$\boxed{0 = H\left(\frac{\partial S}{\partial q_k}, q_k, t\right) + \frac{\partial S}{\partial t}}$$
H-J equation

1st order partial diff eq in $N+1$ variables $\begin{cases} q_k & (k=1,\dots,N) \\ t \end{cases}$

Solve for $S(q_k, \alpha_k, t)$. Often just write $S(q_k, t)$.

constants of integration

Consider $\dfrac{dS}{dt}$ when eqs of motion are obeyed:

$$\frac{dS}{dt} = \underbrace{\frac{\partial S}{\partial t}}_{-H} + \underbrace{\frac{\partial S}{\partial q_n}}_{p_n} \dot{q}_n + \frac{\partial S}{\partial P_n} \cancelto{0}{\dot{P}_n} = p_n \dot{q}_n - H(p_k, q_k, t) = L(q_k, \dot{q}_k, t)$$

So, evaluated on the classical path:

$$S(t) = \int_{t_0}^{t} dt \; L(q_k, \dot{q}_k, t) + S(t_0) = \text{the action}$$

Also know as "Hamilton's Principal Function"

What good is it? Can solve for dynamics:

Key point: $\underbrace{S(q_k, \alpha_k, t)}_{\text{solution to H-J}} = \underbrace{S(q_k, P_k, t)}_{\text{definition}}$

So α_k are functions of the P_k, and vice versa.

$$\underbrace{Q_k}_{\text{constants}} = \frac{\partial S}{\partial P_k} = \frac{\partial S}{\partial \alpha_n} \underbrace{\frac{\partial \alpha_n}{\partial P_k}}_{\text{constant}} . \quad \text{So} \quad \boxed{\frac{\partial S}{\partial \alpha_n} = \text{constant} \quad \text{solve these!}}$$

If E conserved, it is one of the $\alpha_n \Rightarrow \frac{\partial S}{\partial E} = \text{constant}$

Comments: 1) Dimensions of S: (energy)(time)

2) If any p_n is conserved, then $\frac{\partial S}{\partial q_n} = p_n = \text{constant}$, so

$$S = p_n (q_n - q_{n0}) + (\text{stuff without } q_n)$$

3) If energy conserved, then $H = -\frac{\partial S}{\partial t} = E$, so

$$S = -E(t - t_0) + \underbrace{W(q_k, E)}_{\text{Hamilton's characteristic function}} \leftarrow \text{no time dependence}$$

Now E = constant of integration = fn of P_k. So:

$$\frac{\partial S}{\partial E} = 0 \Rightarrow \boxed{\frac{\partial W}{\partial E} = t - t_0} \quad \text{Integrate to find the motion}$$

Example: free particle in 3-d $\quad H = \frac{1}{2m}(P_x^2 + P_y^2 + P_z^2)$

H-J equation: $$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] = 0$$

Try additive separation of variables: $S = S_x(x) + S_y(y) + S_z(z) + S_t(t)$.

Plug in: $$\underbrace{\frac{\partial S_t}{\partial t}}_{\text{only a fn of } t} + \frac{1}{2m}\left[\underbrace{\left(\frac{\partial S_x}{\partial x}\right)^2}_{\text{only a fn of } x} + \underbrace{\left(\frac{\partial S_y}{\partial y}\right)^2}_{\text{only a fn of } y} + \underbrace{\left(\frac{\partial S_z}{\partial z}\right)^2}_{\text{only a function of } z.}\right] = 0$$

So, each term must be constant!

$$S_x = p_x (x - x_0) \qquad S_y = p_y (y - y_0) \qquad S_z = p_z (z - z_0)$$

Now view as constant! constant of integration

So $$\frac{\partial S_t}{\partial t} = -\frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) \Rightarrow S_t = -\frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right)(t - t_0)$$

All together: $$S = -\frac{\left(p_x^2 + p_y^2 + p_z^2\right)}{2m}(t - t_0) + p_x(x - x_0) + p_y(y - y_0) + p_z(z - z_0)$$

Now: $$\frac{\partial S}{\partial x_0} = \text{constant} \Rightarrow p_x = \text{constant} \qquad (\text{same for } p_y, p_z)$$

$$\frac{\partial S}{\partial p_x} = \text{constant} \Rightarrow -\frac{p_x}{m}(t - t_0) = x - x_0 \Rightarrow$$

$$x = x_0 - \frac{p_x}{m}(t - t_0) \checkmark \qquad (\text{same for } y, z)$$

$$\frac{\partial S}{\partial E} = \text{constant} \quad (\text{nothing new})$$

Example $$L = \frac{1}{2} m \dot{x}^2 - V(x) \qquad p = m\dot{x}$$

$$H = \frac{p^2}{2m} + V(x)$$

H-J equation: $$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x) = 0$$

Try $$S(x, E, t) = W(x, E) - \underbrace{E(t - t_0)}_{\text{always do this if energy conserved!}} \Rightarrow$$

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + V(x) = E \quad \Rightarrow \quad \frac{\partial W}{\partial x} = \pm\sqrt{2m}\,\left(E - V(x)\right)^{1/2}$$

So $W(x, E) = \pm\sqrt{2m}\int_{x_0}^{x} dx\,\left(E - V(x)\right)^{1/2}$. To solve for motion,

$$\frac{\partial S}{\partial E} = \text{constant} \Rightarrow t - t_0 = \pm\sqrt{2m}\int_{x_0}^{x} dx\,\frac{\partial}{\partial E}\left(E - V(x)\right)^{1/2} \quad \text{or}$$

$$\boxed{t - t_0 = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx}{\sqrt{E - V(x)}}}$$ "Reduced to quadratures"