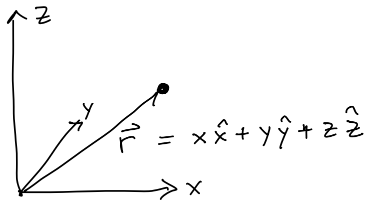
PHYS 600

Newtonian mechanics of a particle. Choose a $\underbrace{\text{coordinate system}}_{\text{"frame"}}$.



Not all frames are equally good!

Inertial frame: Newton's 1st law: an object at rest will remain at rest unless acted on by a force.

At time t, can specify $\vec{r}$ and $\underbrace{\vec{v} = \frac{d\vec{r}}{dt}}_{\text{velocity}} = \dot{\vec{r}}$ for each particle.

Particle has mass m $\longrightarrow$ momentum $= m\frac{d\vec{r}}{dt} = \vec{p}$ (p for Latin "pulsus", to push or to strike)

Newton's Second Law: In an inertial frame, motion described by diff. eq.

$$\boxed{\frac{d\vec{p}}{dt} = \vec{F}} \quad \text{or} \quad \vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}$$

if $m=$ constant   acceleration.

$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}}$. So 2nd-order diff. eq. Given $\vec{r}$ and $\vec{v}$ at time $t=t_0$, and $\vec{F}$ at all times, can solve for $\vec{r}(t)$.

What is $\vec{F}(t)$? Depends on the problem.

Galilean Invariance Given an inertial frame, the following are equally good inertial frames:

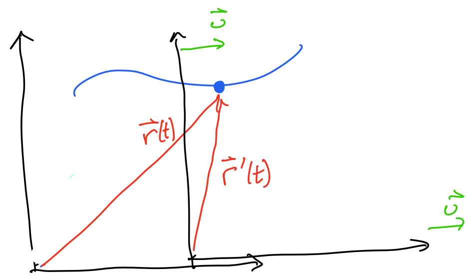
Translations $\vec{r} \to \vec{r}\,' = \vec{r} + \vec{c}$ ← constant vector $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix}$

Rotations: $\vec{r} \to \vec{r}\,' = \Omega\,\vec{r}$ ← constant matrix $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \Omega_{xx} & \Omega_{xy} & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Boosts: $\vec{r} \to \vec{r}\,' + t\vec{u}$ ← constant velocity

Time translation: $t \to t + c$ ← constant

Proof (for boosts): Suppose in two different frames, particle has trajectories $\vec{r}(t)$, $\vec{r}\,'(t)$



So $\vec{r}'(t) = \vec{r} - t\vec{v}$

$$\frac{d^2\vec{r}'}{dt^2} = \frac{d^2}{dt^2}(\vec{r} - t\vec{v}) = \frac{d^2\vec{r}}{dt^2} - \cancel{\vec{v}\frac{d^2}{dt^2}(t)} \quad 0$$

So if $\dfrac{d^2\vec{r}}{dt^2} = \vec{F}$, then $\dfrac{d^2\vec{r}'}{dt^2} = \vec{F}$ also.

Force is the same in inertial frames.

Not an inertial frame: $\vec{r}' = \vec{r} + \frac{1}{2}t^2\vec{a}$ ($\vec{r}'$ = coordinates of particle in accelerated frame)

$\vec{r}' = \Omega\vec{r}$

not constant in time.

Angular Momentum of a particle First, choose a frame, with an origin.

Then $\vec{L} = \vec{r} \times \vec{p}$ (depends on frame!)

How does this change?

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}.$$

product rule

Assume $m =$ constant,

$$= m\left(\cancel{\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt}}\right) + m\,\vec{r} \times \frac{d^2\vec{r}}{dt^2} = \vec{r} \times \vec{F}$$

0

So, define torque acting on a particle: $\vec{\Gamma} = \vec{N} = \vec{T} = \vec{r} \times \vec{F}$,

So that $\boxed{\dfrac{d\vec{L}}{dt} = \vec{N}}$

Conservation Laws: If $\vec{F} = 0$, then $\vec{p}$ = constant

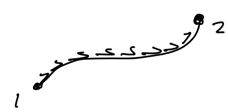
If $\vec{N} = 0$, then $\vec{L}$ = constant

Work done to move from position $\vec{r}$ to $\vec{r} + d\vec{r}$ is

$d\vec{r}$ = infinitesimal

$dW = \vec{F} \cdot d\vec{r}$. So, for a finite distance

("work = force times distance")



$$W_{1\to 2} = \int_1^2 d\vec{r} \cdot \vec{F}$$ Now use $\vec{F} = m \frac{d}{dt}\vec{v}$

$$W_{1\to 2} = m \int_1^2 \underbrace{d\vec{r}}_{dt\,\frac{d\vec{r}}{dt}} \cdot \frac{d\vec{v}}{dt} = m \int_1^2 dt \underbrace{\frac{d\vec{r}}{dt} \cdot \frac{d\vec{v}}{dt}}$$

$$\frac{d\vec{r}}{dt} \cdot \frac{d^2\vec{r}}{dt^2} = \frac{1}{2}\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}\right) = \frac{1}{2}\frac{d}{dt}(v^2)$$

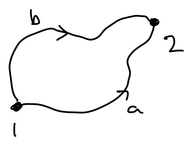
$$= \frac{1}{2} m \int_1^2 dt \frac{d}{dt}(v^2)$$

$$= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

Define <u>kinetic energy</u> $T = \frac{1}{2} m v^2$, then $\boxed{T_2 - T_1 = W_{1\to 2}}$

So far, haven't assumed anything about $\vec{F}$.

<u>Conservative forces</u>: Assume $\vec{F}(\vec{r})$ depends on position of particle (not time!) in such a way that $W_{1\to 2}$ doesn't depend on the path taken.



$$W_{1\to 2} = \int_{\vec{r}_1 \text{ path a}}^{\vec{r}_2} d\vec{r} \cdot \vec{F} = \int_{\vec{r}_1 \text{ path b}}^{\vec{r}_2} d\vec{r} \cdot \vec{F} = -\int_{\vec{r}_2 \text{ path b}}^{\vec{r}_1} d\vec{r} \cdot \vec{F}$$

reverse this path.

$$\oint_{\text{closed path}} d\vec{r} \cdot \vec{F} = 0 \;\underset{\text{math fact}}{\Longleftrightarrow}\; \vec{\nabla} \times \vec{F} = 0 \;\underset{\text{math fact}}{\Longleftrightarrow}\; \vec{F} = -\vec{\nabla} V$$ for some function V.

(In 1-d, $F = -\frac{\partial V}{\partial x}$.)

Then $$T_2 - T_1 = -\int_{\vec{r}_1}^{\vec{r}_2} d\vec{r} \cdot \vec{\nabla} V = -\left[V(\vec{r}_2) - V(\vec{r}_1)\right],$$ so

$$T_2 + V_2 = T_1 + V_1 = E = \text{total energy}$$

If forces are conservative, then energy is conserved, and $\vec{F}(\vec{r}) = -\vec{\nabla} V$

<u>Examples</u>: (1) Simple harmonic oscillator, in 1-d. $T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2$

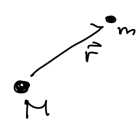
$F = -kx$ (restoring force) $\quad V(x) = \frac{1}{2} k x^2 \quad E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$

$= \text{Constant}$

(2) Damped H.O. $F = -kx - \gamma \dot{x}$ <u>Not</u> conservative!

(3) Particle in gravitational field of <u>much</u> heavier particle, mass M.

doesn't move



$\vec{F} = -GMm \frac{\hat{r}}{r^2}$ where $\hat{r} = \frac{\vec{r}}{r}$ = unit vector. $\quad V(\vec{r}) = -\frac{GMm}{r}$

Systems of Particles

N particles labeled by $i = 1, \dots, N$

masses are m_i. Positions, velocities, accelerations are $\vec{r}_i(t)$, $\frac{d\vec{r}_i}{dt}$, $\frac{d^2\vec{r}_i}{dt^2}$.

Total mass of system is $M = \sum_{i=1}^{N} m_i$.

Center of mass of system is $\vec{R} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i$ (weighted average)

What forces act on particle i?

$$\vec{F}_i = \underset{\text{external}}{\vec{F}_i^{(e)}} + \sum_{j \neq i} \vec{F}_{ji}$$

(Assume particle i doesn't act on itself!)

So
$$m_i \frac{d^2\vec{r}_i}{dt^2} = \vec{F}_i^{(e)} + \sum_{j \neq i} \vec{F}_{ji}$$
(2nd order diff eq)

Sum over all i:

$$\frac{d^2}{dt^2} \sum_i m_i \vec{r}_i = \underbrace{\sum_i \vec{F}_i^{(e)}}_{\vec{F}^{(e)} = \text{ total external force}} + \underbrace{\sum_i \sum_{j \neq i} \vec{F}_{ji}}_{\frac{1}{2}\sum_i \sum_j \left(\vec{F}_{ij} + \vec{F}_{ji}\right)}$$

Use Newton's 3rd Law, weak version: $\vec{F}_{ji} = -\vec{F}_{ij}$.

(Consistent with assigning $\vec{F}_{ii} = 0$.)

So $\frac{d^2}{dt^2}\left(M\vec{R}\right) = \vec{F}^{(e)}$, or
$$\boxed{M \frac{d^2\vec{R}}{dt^2} = \vec{F}^{(e)}}$$

Center of mass $\vec{R}$ moves like a single particle with mass M, and force $\vec{F}^{(e)}$. Can also write $\vec{P} = \sum_i \vec{P}_i = \sum_i m_i \frac{d\vec{r}_i}{dt}$, $=$ total momentum

so $\frac{d\vec{P}}{dt} = \vec{F}^{(e)}$.

Conservation Law: If $\vec{F}^{(e)} = 0$, then $\vec{P} = $ constant, no matter what the internal forces $\vec{F}_{ji}$ are.

The position of the COM $\vec{R}$, moves as if the total external force $\vec{F}^{(e)}$ acted on the total mass M:

$$M\frac{d^2\vec{R}}{dt^2} = \vec{F}^{(e)}.$$

Define total angular momentum: $\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$.

Now compute: $$\frac{d\vec{L}}{dt} = \sum_i \left(\underbrace{\frac{d\vec{r}_i}{dt} \times \vec{p}_i}_{\text{0 just like for single particle}} + \vec{r}_i \times \underbrace{\frac{d\vec{p}_i}{dt}}_{\vec{F}_i^{(e)} + \sum_j \vec{F}_{ji}} \right)$$

So $$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{(e)} + \underbrace{\sum_{i,j} \vec{r}_i \times \vec{F}_{ji}}_{\frac{1}{2}\sum_{i,j}(\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \underbrace{\vec{F}_{ij}}_{-\vec{F}_{ji}})}$$

$$\underbrace{\frac{1}{2}\sum_{i,j}(\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij})}_{\frac{1}{2}\sum_{i,j}(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji}}$$

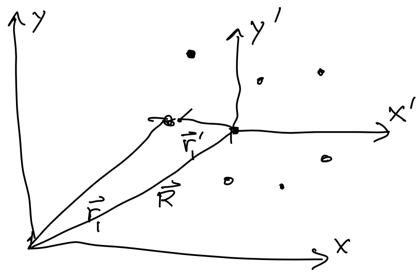
Now, if we assume the strong form of Newton's 3rd Law ($\vec{F}_{ij} = -\vec{F}_{ji}$ <u>and</u> both along the direction between i,j, so along $\vec{r}_i - \vec{r}_j$) then last term is 0. ($\vec{a} \times \vec{a} = 0$)

Then $$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{(e)} = \vec{N}^{(e)} = \begin{cases} \text{external torque} \\ = \\ \text{total applied torque} \end{cases}$$

With the assumption, internal forces do not contribute to total torque.

Choose a new inertial frame, the COM frame:

$$\vec{r}_i' = \vec{r}_i - \vec{R} \qquad \text{for each } i = 1, \ldots, N$$



Find useful expressions for angular momentum, work, kinetic energy.

$$\cancel{M\vec R} \underset{\substack{\text{definition}\\ \text{of } \vec R}}{=} \sum_i m_i \vec r_i \underset{\substack{\text{definition}\\ \text{of } \vec r_i'}}{=} \sum_i m_i (\vec R + \vec r_i') = \cancel{M\vec R} + \sum_i m_i \vec r_i'$$

So, we learn $\sum_i m_i \vec r_i' = 0$. Take $\frac{d}{dt}$, gives also $\sum_i m_i \frac{d\vec r_i'}{dt} = 0$

Now consider total angular momentum:

$$\vec L = \sum_i \vec r_i \times m \vec v_i = \sum_i (\vec R + \vec r_i') \times m_i \left(\frac{d\vec R}{dt} + \frac{d\vec r_i'}{dt} \right)$$

$$= \underbrace{\sum_i m_i}_{M} \vec R \times \underbrace{\frac{d\vec R}{dt}}_{\substack{\text{call this}\\ \vec V = \text{velocity}\\ \text{of COM}}} + \underbrace{\sum_i m_i \vec r_i'}_{=0} \times \frac{d\vec R}{dt} + \vec R \times \underbrace{\sum_i m_i \frac{d\vec r_i'}{dt}}_{=0} + \sum_i m_i \vec r_i \times \frac{d\vec r_i}{dt}$$

So $\vec L = \underbrace{\vec R \times \vec P}_{\vec L_{cm}} + \underbrace{\sum_i \vec r_i' \times \vec p_i'}_{\vec L'}$ In words:

Total angular momentum = angular momentum <u>of</u> COM + angular momentum <u>about</u> COM

relative to

Can now show (F+W page 8):

$$\frac{d}{dt} \vec L_{cm} = \vec R \times \vec F^{(e)}$$

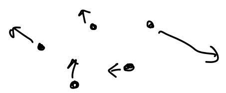
total net force $= \sum_i \vec F_i^{(e)}$

and,

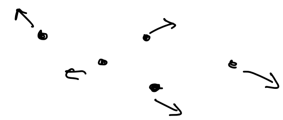
$$\underbrace{\frac{d}{dt}}_{\text{rate of change of angular momentum about COM.}}\vec{L}' = \sum_i \vec{r}_i{}' \times \vec{F}_i^{(e)} \quad = \text{external torque about COM}$$

Work for collection of particles

Configuration 1



configuration 2



Total work done:

$$W_{1\to 2} = \sum_i \int_1^2 d\vec{r}_i \cdot \vec{F}_i = \sum_i \int_1^2 d\vec{r}_i \cdot \vec{F}_i^{(e)} + \underbrace{\sum_{i,j} \int_1^2 d\vec{r}_i \cdot \vec{F}_{ji}}$$

$$= \sum_{i,j} \frac{1}{2} \int_1^2 d\vec{r}_i \cdot (\vec{F}_{ji} - \vec{F}_{ij})$$

relabel $i \Leftrightarrow j$ on 2nd term

$$= \frac{1}{2} \sum_{i,j} \int_1^2 (d\vec{r}_i - d\vec{r}_j) \cdot \vec{F}_{ji} = \frac{1}{2} \sum_{i,j} \int_1^2 d\vec{r}_{ij} \cdot \vec{F}_{ji}$$

Assume conservative forces: $\vec{F}_i^{(e)} = -\vec{\nabla}_i V^{(e)}(\vec{r}_i)$

$$\vec{F}_{ji} = -\vec{\nabla}_i V_{ij}(\vec{r}_{ij}) \qquad \vec{r}_{ij} = \vec{r}_i - \vec{r}_j$$

Then
$$W_{1\to 2} = \underbrace{-\sum_i \int_1^2 d\vec{r}_i \cdot \vec{\nabla}_i V_i^{(e)}}_{-\sum_i V_i^{(e)}(\vec{r}_i)\Big|_1^2} \; \underbrace{-\frac{1}{2} \sum_{i,j} \int_1^2 d\vec{r}_{ij} \cdot \vec{\nabla}_i V_{ij}(\vec{r})}_{-\frac{1}{2} \sum_{i,j} V_{ij}(\vec{r}_{ij})\Big|_1^2}$$

each term contributes twice, no $i=j$ term.

So $W_{1\to 2} = V_1 - V_2$, where for each of configurations 1 and 2,

$$V = \sum_i V_i^{(e)}(\vec{r}_i) + \frac{1}{2} \sum_{i,j} V_{ij}(\vec{r}_i - \vec{r}_j) = \text{total potential energy.}$$

Also, $W_{1\to 2} = T_2 - T_1$, where for each of 1 and 2,

$T = \frac{1}{2}\sum_i m_i v_i^2$ = <u>total kinetic energy</u>

Then $T + V = E$ = total energy = unchanged.

Note: we assumed conservative forces!

Another way to write total kinetic energy:

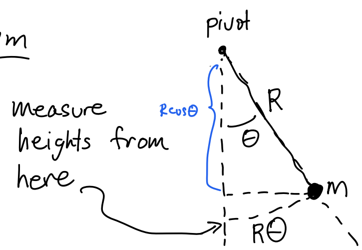
$$T = \frac{1}{2}\sum_i m_i (\vec{V} + \vec{v}_i') \cdot (\vec{V} + \vec{v}_i')$$

$$= \frac{1}{2}\underbrace{\sum_i m_i}_{M} V^2 + \frac{1}{2}\sum_i m_i v_i'^2 + \underbrace{\sum_i m_i \vec{V}\cdot\vec{v}_i'}_{\vec{V}\cdot \underbrace{\sum_i m_i \vec{v}_i'}_{0}}$$

So $T = \frac{1}{2}MV^2 + \frac{1}{2}\sum_i m_i v_i'^2$.

Kinetic energy splits into COM part, and kinetic energy of motions <u>relative</u> to COM frame.

<u>Example: pendulum</u>



$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{d(R\theta)}{dt}\right)^2 = \frac{1}{2}mR^2\dot{\theta}^2$$

$V = mgh$ (uniform gravitational field with acceleration $g \approx 9.8\ m/sec^2$)

$h = R - R\cos\theta$. So $E = \frac{1}{2}mR^2\dot{\theta}^2 + mgR(1-\cos\theta)$ = constant

If we release pendulum from rest at angle θ_0 at $t = 0$, what is its period of oscillation?

At time $t=0$, $E = \frac{1}{2}mR^2\cancel{\dot{\theta}^2}_{0} + mgR(1-\cos\theta_0) = mgR(1-\cos\theta_0)$

At time t, $E = \frac{1}{2}mR^2\left(\frac{d\theta}{dt}\right)^2 + mgR(1-\cos\theta)$

$\frac{1}{2}mR^2\left(\frac{d\theta}{dt}\right)^2 = mgR(\cos\theta - \cos\theta_0)$

or...

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{R}(\cos\theta - \cos\theta_0)} .\quad \text{So}\quad \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \pm\sqrt{\frac{2g}{R}}\, dt \qquad \text{Integrate both sides,}$$

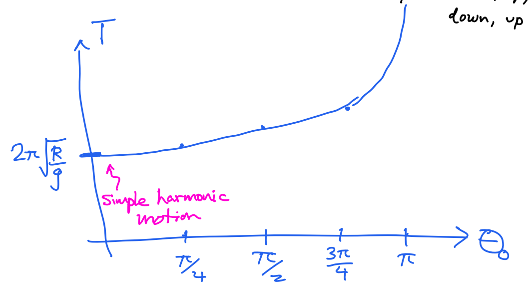
$$\underbrace{\int_{\theta_0}^{\theta} d\theta \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}}_{\text{"Reduced to quadratures"}} = \pm\sqrt{\frac{2g}{R}}\int_0^t dt = \pm\sqrt{\frac{2g}{R}}\, t$$

Elliptic integral $\Rightarrow$ $\theta(t)$.

Now suppose $\theta = 0$. Then $t = \frac{T}{4}$ ← period. ← down, up, down, up

So $T = 4\sqrt{\frac{R}{2g}} \underbrace{\int_0^{\theta_0} d\theta \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}}$

$\frac{\pi}{\sqrt{2}} \begin{cases} 1 & \text{for small } \theta_0 \\ 1.03997 & \text{for } \theta_0 = \pi/4 \\ 1.18034 & \text{for } \theta_0 = \pi/2 \\ 1.52795 & \text{for } \theta_0 = 3\pi/4 \\ \infty & \text{for } \theta_0 = 2\pi \end{cases}$



Lessons:
1) Conservation of energy is your friend
2) Small oscillations → simple harmonic motion