PHYS 600 Homework 9 Due Friday, December 5, 2025

To facilitate study for the final exam on Wednesday December 10, solutions for this homework set will be distributed the morning of Saturday December 6. Therefore, it will not be accepted late.

Reading assignment: pages 184-197 of Fetter and Walecka.

Problem 1. (This problem can be done entirely with energy conservation; it does not require Hamilton-Jacobi theory or action-angle variables. However, we will see how to make contact with the latter in class.) A particle of mass m moves in one dimension in a potential $V(x) = kx^4$, which is a steeper cousin of the harmonic oscillator.
(a) If the particle is released from rest at $x = x_0$, what is the period of oscillation? (There is a definite integral to be done, but it can and should be done analytically.)
(b) Rewrite your answer in terms of the conserved energy of the particle.

Problem 2. Consider the one-dimensional motion of a particle of mass m subject to a time-dependent force that doesn't depend on its position, $F = At$, where A is a constant.
(a) Find the Hamiltonian of the system. Is energy conserved? Is momentum conserved?
(b) Write the corresponding Hamilton-Jacobi equation in terms of Hamilton's principal function S.
(c) Try a solution of the form

$$S = \frac{1}{2}At^2x + \alpha x - \phi(t),$$

where α is a constant and $\phi(t)$ is a function of time. Solve the resulting equation for $\phi(t)$.
(d) Use the results above and $\partial S/\partial\alpha =$ constant to find the general solution for the position and momentum as a function of time.

Problem 3. A particle of mass m moves in a central potential $V(r)$ in three dimensions.
(a) Using spherical coordinates, find the Hamiltonian $H(p_r, p_\theta, p_\phi, r, \theta, \phi)$ and the resulting six equations of Hamilton's equations of motion.
(b) Check that $\theta = \pi/2$, $p_\theta = 0$ allows a solution to the equations of motion.
(c) Now assume that θ has been eliminated by setting it equal to $\pi/2$ for all time, as in the previous part. Write down the Hamilton-Jacobi equation for $S(r, \phi, t)$.
(d) Try a solution $S(r, \phi, t) = W(r) + \alpha\phi - E(t - t_0)$. What is the differential equation that $W(r)$ must obey? Find its solution as a definite integral, and use the fact that $\partial S/\partial E$ and $\partial S/\partial\alpha$ are constants to obtain general solutions that implicitly determine $r(t)$ and $\phi(r)$ as quadratures. How is α related to the angular momentum?

Problem 4. A charged particle of mass m moves in three dimensions in a potential $V(r) = (k\cos\theta)/r^2$. (This is the form of the potential that would result from the interaction of the charge with a fixed electric dipole at the origin. Note that it is not spherically symmetric.)

(a) Find the Hamiltonian $H(p_r, p_\theta, p_\phi, r, \theta, \phi)$ and the resulting Hamilton's equations of motion. There are two conserved quantities. What are they, and why?

(b) Find the Hamilton-Jacobi equation satisfied by Hamilton's principal function $S(r, \theta, \phi, t)$, and the resulting partial differential equation satisfied by Hamilton's characteristic function $W(r, \theta, \phi) = S + E(t - t_0)$.

(c) Assume that $W(r, \theta, \phi) = W_r(r) + W_\theta(\theta) + p_\phi \phi$, and find the first-order differential equations satisfied by W_r and W_θ. Integrate them to express

$$
\begin{aligned}
W_r(r) &= \int_{r_0}^{r} dr\, f(r, E, \alpha), \\
W_\theta(\theta) &= \int_{\theta_0}^{\theta} d\theta\, g(\theta, p_\phi, \alpha),
\end{aligned}
$$

where f and g are certain functions that you will discover in terms of a separation constant α. (You don't need to do the integrals!) Write down the result for $S(r, \theta, \phi, t)$.

(d) From requiring each of $\partial S/\partial E$ and $\partial S/\partial \alpha$ and $\partial S/\partial p_\phi$ to be constants, find three coupled integral equations that govern the motion. They should be of the form:

$$
\begin{aligned}
t - t_0 &= \int_{r_0}^{r} dr\, F_1(r), \\
\phi - \phi_0 &= \int_{\theta_0}^{\theta} d\theta\, F_2(\theta), \\
\int_{r_0}^{r} dr\, F_3(r) &= \int_{\theta_0}^{\theta} d\theta\, F_4(\theta),
\end{aligned}
$$

where you will discover the four functions $F_1(r)$, $F_2(\theta)$, $F_3(r)$, and $F_4(\theta)$.