PHYS 600 **Homework 8** **Due Friday, November 21, 2025**

Reading assignment: pages 173-184 and 197-203 of Fetter and Walecka.

Problem 1. A relativistic particle with mass m and position x moving in a static potential $V(x)$ in one dimension is described by the Lagrangian $L(x, \dot{x}) = -mc^2\sqrt{1 - \dot{x}^2/c^2} - V(x)$, where c is the speed of light.
(a) Find the Euler-Lagrange equation for $x(t)$.
(b) Find the canonical momentum p and write a simplified expression for the Hamiltonian $H(x, p)$. Is H a constant of the motion?

Problem 2. A particle with mass m, position $\vec{r}$, and momentum $\vec{p}$ has angular momentum $\vec{L} = \vec{r} \times \vec{p}$. It is useful to define the antisymmetric symbol ϵ_{ijk} as

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \qquad \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1, \tag{1}$$

and all other components $\epsilon_{ijk} = 0$. It obeys the identities

$$\begin{aligned} \epsilon_{ijk}\epsilon_{lnk} &= \delta_{il}\delta_{jn} - \delta_{in}\delta_{lj}, \\ \epsilon_{ijk}\epsilon_{ljk} &= 2\delta_{il} \\ \epsilon_{ijk}\epsilon_{ijk} &= 6. \end{aligned}$$

Now the components of the angular momentum can be written as

$$L_i = \epsilon_{ijk} r_j p_k.$$

Note that the Einstein summation convention is in effect: repeated indices are summed over except when they appear on both sides of an equation. Evaluate the following Poisson brackets in fully simplified form:
(a) $[r_j, L_k]_{\rm PB} = ?$
(b) $[p_j, L_k]_{\rm PB} = ?$
(c) $[L_j, L_k]_{\rm PB} = ?$
(d) $[r_j, L^2]_{\rm PB} = ?$

Problem 3. A particle of mass m and charge q moves in a background magnetic field $\vec{B}$.
(a) Show that $[m\dot{r}_i, m\dot{r}_j]_{\rm PB} = q\epsilon_{ijk}B_k$. Note that this means that the Poisson brackets of two *kinematic momenta* (also known as mechanical momenta) can be non-zero in the presence of a magnetic field orthogonal to them both, even though the Poisson bracket of any two *canonical momenta* is always zero.
(b) Compute $[m\dot{r}_i, r_j]_{\rm PB} = ?$

Problem 4. Consider a system with phase space coordinates q_1, q_2 and their canonical conjugate momenta p_1, p_2, with Hamiltonian $H = \frac{1}{2}(q_1^2+q_2^2+p_1^2+p_2^2)$. Consider the new coordinates Q_1, Q_2 and P_1, P_2 on phase space, defined by

$$\begin{aligned} q_1 &= Q_1 \cos\alpha + P_2 \sin\alpha, \\ q_2 &= Q_2 \cos\beta + P_1 \sin\beta, \\ p_1 &= -Q_2 \sin\beta + P_1 \cos\beta, \\ p_2 &= -Q_1 \sin\alpha + P_2 \cos\alpha, \end{aligned}$$

where α and β are constants.

(a) Find the inverse relations, solving for Q_1, Q_2, P_1, P_2 in terms of q_1, q_2, p_1, p_2. Make sure your answers are in fully simplified form.

(b) Compute all of the Poisson brackets of every pair from Q_1, Q_2, P_1, P_2. By requiring that the relation is a canonical transformation, solve for β in terms of α.

(c) Express each of p_1, p_2, P_1, and P_2 in terms of q_1, q_2, Q_1, Q_2. Use these results to find the Type 1 generating function $F(q_1, q_2, Q_1, Q_2)$ for the canonical transformation. [For the definition of this type of generating function, see Fetter and Walecka, eqs. (34.9)-(34.11).]

(d) Express each of p_1, p_2, Q_1, and Q_2 in terms of q_1, q_2, P_1, P_2. Use these results to find the Type 2 generating function $F_2(q_1, q_2, P_1, P_2)$ for the canonical transformation, which means that

$$p_n = \frac{\partial F_2}{\partial q_n}, \qquad Q_n = \frac{\partial F_2}{\partial P_n}.$$

(e) Find the new Hamiltonian $H(Q_1, Q_2, P_1, P_2)$ in fully simplified form. What are the Hamilton equations of motion for Q_1, Q_2, P_1, P_2?

(f) Use your results to find the solutions for q_1, q_2, p_1, p_2 as functions of the time t, under the constraints that $Q_2 = 0$ and $P_2 = 0$ for all times, and with the boundary conditions $q_1(0) = x_0$ and $\dot{q}_1(0) = v_0$. [You do not need to introduce a Lagrange multiplier for the constraints. Your answers should depend only on x_0, v_0, α, and t.]

Problem 5. A group of particles, all with the same mass m, have initial heights z and initial vertical momenta p lying in the rectangle $-a < z < a$ and $-b < p < b$. The particles then fall freely for a time t in a uniform gravitational field with acceleration g downwards. Find the region in phase space that they occupy at time t, sketch it, and show by direct calculation (not appealing to any fancy general theorems) that the phase space area is still $4ab$. This is an illustration of Liouville's Theorem, which says that areas in phase space are preserved under Hamiltonian time evolution.