Physics 485 **Homework 8** **Due April 8, 2003**

Problem 1. Write each of the following functions in terms of spherical harmonics and the spherical coordinate r:
(a) $f(x,y,z) = y$, (b) $f(x,y,z) = xy$, (c) $f(x,y,z) = x^2 - y^2$, (d) $f(x,y,z) = x^3$.

Problem 2. In Quantum Mechanics, the angular momentum raising and lowering operators can be represented as differential operators,

$$\begin{aligned} L_+ = L_x + iL_y &= \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right), \\ L_- = L_x - iL_y &= -\hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right). \end{aligned}$$

Show that, acting on spherical harmonics,

$$\begin{aligned} L_+ Y_\ell^m(\theta,\phi) &= \hbar\sqrt{(\ell-m)(\ell+m+1)}Y_\ell^{m+1}(\theta,\phi), \\ L_- Y_\ell^m(\theta,\phi) &= \hbar\sqrt{(\ell+m)(\ell-m+1)}Y_\ell^{m-1}(\theta,\phi). \end{aligned}$$

Problem 3. A hydrogen electron in a $2p$ orbital has a charge distribution:

$$\rho(r',\theta',\phi') = \frac{q_e}{64\pi a_0^5} r'^2 e^{-r'/a_0}\sin^2\theta'$$

where a_0 is the Bohr radius. Use the Green's function method to find the electrostatic potential $V(r,\theta,\phi)$ corresponding to this charge distribution.

[Hint: Write the charge density in terms of $Y_\ell^m(\theta',\phi')$. You can then do the angular part of the integral, over θ',ϕ', using the orthogonality of spherical harmonics. Notice that the radial integrals split up into two parts because of the $r_<$, $r_>$ issue. You may write the result (specifically the part coming from the r' radial integrals) in terms of the incomplete gamma functions:

$$\begin{aligned} \Gamma(a,x) &\equiv \int_x^\infty e^{-t}t^{a-1}dt, \\ \gamma(a,x) &\equiv \int_0^x e^{-t}t^{a-1}dt, \end{aligned}$$

and in terms of spherical harmonics of θ,ϕ.]