Problem 1. Consider the infinite series:

$$\sum_{n=3}^{\infty} \frac{1}{n \ln(n) \, [\ln(\ln(n))]^p}$$

For what values of p does this series converge?

Hint: Make use of the fact that $\frac{d}{dx}\left([\ln(\ln(x))]^{1-p}\right) = \frac{1-p}{x \ln(x) \, [\ln(\ln(x))]^p}$.

Problem 2. According to the theory of Special Relativity, if a particle with mass M starting at rest at $x = 0$ at time $t = 0$ is acted on by a constant force F, then its displacement at time t is given by:

$$x = \frac{Mc^2}{F}\left[\left(1 + \frac{F^2 t^2}{M^2 c^2}\right)^{1/2} - 1\right],$$

where c is the speed of light. Find the resulting displacement x as a power series in t, up to and including terms of order t^5. Compare your answer with the Newtonian (non-relativistic) result.

Problem 3. Expand $\sin(x)$ in a Taylor series about the point $x = \pi/4$. Keep terms up to and including $(x - \pi/4)^3$.

Problem 4. The *dilogarithm function* (also known as the *Spence function*) is denoted $\mathrm{Li}_2(x)$. It appears in high-energy physics and statistical mechanics calculations. It can be defined as:

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t).$$

(a) Expand the above expression in an infinite power series in x, of the form $\sum_{n=1}^{\infty} c_n x^n$. (Give a general expression for the c_n in terms of n, not just the first few terms.) This infinite series is an alternative definition for the dilogarithm.

(b) Use an appropriate convergence test to determine: for what real values of x is your answer to part (a) absolutely convergent?

(c) Compute the integral

$$I(a,x) = \int_0^1 \frac{dt}{t} \ln(1 - atx),$$

as an infinite power series in x. By comparing it to your result for part (a), express it in terms of the dilogarithm function.

(d) Using your result in part (a), evaluate $\text{Li}_2(1)$. [Hint: you may use the result for a special infinite series mentioned in class.]

(e) An approximation for the function $\text{Li}_2(\frac{x}{1+2x})$ when x is small is: $x + c_2 x^2 + c_3 x^3$. Find c_2 and c_3.