Coordinate Systems and Vector Derivatives Formula Sheet

Rectangular (Cartesian) Coordinates (x, y, z)

Line element: $d\vec{\ell} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$

Volume element: $d\tau = dx\,dy\,dz$

Gradient: $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$

Divergence: $\vec{\nabla} \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\vec{\nabla} \times \vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}$

Laplacian: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Spherical Coordinates (r, θ, ϕ)

Relations to rectangular (Cartesian) coordinates and unit vectors:

$$x = r\sin\theta\cos\phi \qquad \hat{x} = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$$
$$y = r\sin\theta\sin\phi \qquad \hat{y} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$$
$$z = r\cos\theta \qquad \hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$
$$\theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \qquad \hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$$
$$\phi = \tan^{-1}(y/x) \qquad \hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

Line element: $d\vec{\ell} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi$

Volume element: $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\,\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$

Divergence: $\vec{\nabla} \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\,\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\,v_\theta) + \frac{1}{r\,\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl:
$$\vec{\nabla} \times \vec{v} = \frac{1}{r\,\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r\,v_\phi)\right]\hat{\theta}$$
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi}$$

Laplacian: $\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\,\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\,\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$

Cylindrical Coordinates (r, ϕ, z)

Relations to rectangular (Cartesian) coordinates and unit vectors:

$$
\begin{aligned}
x &= r\cos\phi & \hat{x} &= \hat{r}\cos\phi - \hat{\phi}\sin\phi \\
y &= r\sin\phi & \hat{y} &= \hat{r}\sin\phi + \hat{\phi}\cos\phi \\
z &= z & \hat{z} &= \hat{z}
\end{aligned}
$$

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} & \hat{r} &= \hat{x}\cos\phi + \hat{y}\sin\phi \\
\phi &= \tan^{-1}(y/x) & \hat{\phi} &= -\hat{x}\sin\phi + \hat{y}\cos\phi \\
z &= z & \hat{z} &= \hat{z}
\end{aligned}
$$

Line element: $$d\vec{\ell} = \hat{r}\,dr + \hat{\phi}\,r\,d\phi + \hat{z}\,dz$$

Volume element: $$d\tau = r\,dr\,d\phi\,dz$$

Gradient: $$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$$

Divergence: $$\vec{\nabla}\cdot\vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\vec{\nabla}\times\vec{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial \phi}\right]\hat{z}$$

Laplacian: $$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$