NIU Physics PhD Candidacy Exam - Spring 2026

Quantum Mechanics

YOU MAY SOLVE ALL FOUR PROBLEMS! THE **three best graded** COUNT TOWARDS YOUR TOTAL SCORE.
(40 POINTS EACH; TOTAL POSSIBLE SCORE: 120 POINTS)

I. PARTICLE NEAR A WALL IN 1D *[(5+25+10) PTS]*

An approximate model for a particle of mass m near a wall is to consider a particle moving under the influence of the one-dimensional potential given by

$$V(x) = \begin{cases} -V_0\delta(x), & \text{for } x > -d \\ \infty, & \text{else} \end{cases},$$

for a distance $d > 0$, $V_0 > 0$, and $\delta(x)$ the delta function.

a) Sketch the potential $V(x)$.

b) Find the modification of the bound-state energy caused by the wall, when it is far away. Explain how far is "far away".

c) What is the exact condition for V_0 and d for the existence of at least one bound state?

Hint for b): In the transcendental equation for k, use the large-k solution, which you also need for defining the inequality for "far away".

II. HARMONIC OSCILLATOR *[(10+10+10+10) PTS]*

Consider a spherical potential well with

$$V(r) = \begin{cases} 0 & r \leq a \\ \infty & r > a \end{cases}$$

a) Give the radial part of the Hamiltonian for $r < a$.

   Hint: The formula sheet may help you with spherical coordinates.

b) We are now only interested in the wavefunction $R(r)$ for $l = 0$. Give the general solution of the differential equation by trying a solution $R(r) = u(r)/r$. (Normalization is not necessary.)

c) Determine the behavior of the wavefunction for $r \to 0$. What does the condition that the wavefunction should not diverge at $r = 0$ imply?

d) Determine the energy eigenvalues.

III. SPIN PRECESSION *[(8+8+8+8+8) PTS]*

We consider a spin-$1/2$ particle in an external magnetic field B_z, i.e., the Hamiltonian is $H = -\mu B_z S_z$, where μ is the spin magnetic moment and S_z is the z component of the spin operator.

a) Suppose the particle is initially (time $t = 0$) in an eigenstate $|x, +\rangle$ of spin S_x with eigenvalue $+\hbar/2$. Express $|x, +\rangle$ in terms of the eigenstates $|z, +\rangle$ and $|z, -\rangle$ of spin S_z.

b) Evaluate the time evolution of the state $|x, +\rangle$.

c) Evaluate the time evolution of the expectation value $\langle S_x \rangle$ in the Schrödinger picture.

d) Solve the Heisenberg equation of motion for the operator $S_x^H(t)$ in the Heisenberg picture.

e) Show that the time-dependent expectation value $\left\langle S_x^H(t) \right\rangle$ in the Heisenberg picture equals your result from (c) obtained in the Schrödinger picture.

IV. PERTURBED HARMONIC OSCILLATOR *[(8+16+16) PTS]*

A perturbed two-dimensional harmonic oscillator has Hamiltonian

$$H = \frac{1}{2m}(P_x^2 + P_y^2) + \frac{1}{2}m\omega^2(X^2 + Y^2) + \lambda X^2 Y^2.$$

a) First treating $\lambda = 0$, what are the energies and the degeneracies of the lowest three energy levels?

b) Now treating λ as a small perturbation, find the first-order correction to the ground-state energy.

c) Still treating λ as a small perturbation, find the first-order corrections to the energies of the other states (namely, the first and second excited energy levels of the unperturbed problem) that you identified in part a).

Useful information: for a one-dimensional harmonic oscillator with mass m and angular frequency ω and position and momentum operators X and P, the ladder operators are

$$a = \sqrt{\frac{m\omega}{2\hbar}}X + i\frac{1}{\sqrt{2\hbar\omega m}}P,$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\frac{1}{\sqrt{2\hbar\omega m}}P.$$

They obey $[a, a^\dagger] = 1$, and act on the energy eigenstates according to

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle\,,$$
$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,.$$

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I. TWO FERMIONS *[(10+10+20) PTS]*

Suppose that two **identical** spin-1/2 fermions, each of mass m, interact only via the potential

$$V(r) = \frac{4}{3}\frac{a^2}{\hbar^2}\frac{\vec{S}_1 \cdot \vec{S}_2}{r}$$

where r is the distance between the particles, and $\vec{S}_1$ and $\vec{S}_2$ are the spin operators of the two particles, and a is a constant.

a) For bound states of the system, the total spin angular momentum squared $S^2 = (\vec{S}_1 + \vec{S}_2)^2$ can only have one value. What is it, and why?

b) What values of the orbital angular momentum quantum number are allowed for bound states?

c) Find the energies and degeneracies of the ground state and the first two excited states of the system.

II. MYSTERY POTENTIAL *[(10+10+6+14) PTS]*

The wave function for a particle of mass m moving in one dimension, subject to a potential $U(x)$ is given by

$$\psi(x,t) = \Theta(x) x e^{-\kappa x - \imath\omega t},$$

where $\Theta(x)$ is the Heaviside step function:

$$\Theta(x) = \begin{cases} 1 & , x \geq 0 \\ 0 & , \text{else} \end{cases} .$$

κ & ω are real constants such that $\psi(x,t)$ is a properly normalized wave function that obeys the time-dependent Schrödinger equation for potential $U(x)$.

a) Sketch this wave function at time $t = 0$. Mark and label any significant features, e.g., zeros and extrema.

b) Using what you know about ψ, make a qualitative sketch of the potential $U(x)$ governing this system, indicating in particular any classically forbidden regions and classical turning points.

c) Is the particle in a state corresponding to a definite energy? If so, what is the energy (in terms of κ and/or ω); if not, why not?

d) Derive an expression for the potential $U(x)$ in terms of κ, ω, m, and $\hbar$. Does your result agree with your sketch?

III. TWO ELECTRONS IN A BOX *[(25+10+5) PTS]*

Consider two non-interacting electrons confined within a 1D box. Define the single electron eigenvalue states as

$$|0\rangle\,, |1\rangle\,, |2\rangle\,, \text{etc.}$$

for the ground, first, second, etc. excited states of each electron in the box. Denote the spin via: $|\uparrow\rangle$ or $|\downarrow\rangle$.
In this notation, the ground state is given by:

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle\,|\downarrow\rangle - |\downarrow\rangle\,|\uparrow\rangle\right)|0\rangle\,|0\rangle$$

Consider the lowest energy excited states for this system. There are four antisymmetric states corresponding to this energy level. These states can be distinguished by their quantum numbers for the total spin angular momentum: magnitude j and z-component m.

a) Write down each of these states in terms of the individual electron energy and spin states listed above (as shown for the ground state). Make sure to properly normalize the states.

b) List the value of the z-component of the total spin angular momentum m for each of these states.

c) You should have found two states with $m = 0$. For spin states with the same j, the symmetry or antisymmetry under interchange of the two spins is independent of the m value. Using this argument, identify which of the two $m = 0$ states corresponds to $j = 1$ and which to $j = 0$.

IV. HARMONIC OSCILLATOR *[(14+14+12) PTS]*

Consider a harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

Do the following algebraically, that is, without using wave functions.

a) Construct a linear combination $|\psi\rangle$ of the ground state $|0\rangle$ and the first excited state $|1\rangle$ such that the expectation value $\langle x\rangle = \langle\psi|\,x\,|\psi\rangle$ is as large as possible.

b) Suppose at $t = 0$ the oscillator is in the state constructed in a). What is the state vector for $t > 0$? Evaluate the expectation value $\langle x\rangle$ as a function of time for $t > 0$.

c) Evaluate the variance $\Delta^2 x = \left\langle (x - \langle x\rangle)^2\right\rangle$ as a function of time for the state constructed in a).

You may use

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right) \qquad \text{and} \qquad p = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right),$$

where a and $a^\dagger$ are the annihilation and creation operators for the oscillator eigenstates.

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(40 points each; total possible score: 120 points)

I. ASYMMETRIC 1D POTENTIAL WELL *[(10+22+8) PTS]*

Here we consider a particle of mass m in the 1D potential well $V(x)$ defined by

$$V(x) = \begin{cases} \infty, & x < 0 \\ 0, & 0 \leq x \leq a \\ V_0, & x > a \end{cases} .$$

Here $a > 0$ and $V_0 > 0$.

a) Write down the Schrödinger equation and boundary conditions for the wave function $\psi(x)$ of the particle. Under what condition for the eigenenergy E do you expect a bound state?

b) Show that the bound state energies are defined through the transcendental equation

$$\tan\left(\gamma\sqrt{E}\right) = -\sqrt{\frac{E}{V_0 - E}},$$

with some constant γ you need to find.

c) Without further calculations, sketch the ground state wave function and the potential $V(x)$.

II. A SPIN 1/2 PARTICLE *[(10+10+10+10) PTS]*

A particle of spin 1/2 is in the spin state:

$$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Note, that $\binom{1}{0}$ and $\binom{0}{1}$ are the eigenstates of $\hat{S}_z$.

a) If you measured $\hat{S}_x$, what values might you get, and what is the probability of each?

b) If you measured $\hat{S}_y$, what values might you get, and what is the probability of each?

c) If you measured $\hat{S}_z$, what values might you get, and what is the probability of each?

d) Is the state an eigenvector of $\hat{S}_x^2$? If so, what is its eigenvalue?

III. HYDROGEN ATOM *[(25+15) PTS]*

In studying the hydrogen atom, one usually takes the proton to be a point charge. The ground-state wavefunction is proportional to e^{-r/a_0}, where a_0 is the Bohr radius, and its energy is -13.6 eV. Suppose that instead all of the proton's charge is distributed uniformly over the surface of a spherical shell of radius R, where $R/a_0 \ll 1$.

a) Find the shift in the energy of the ground state, at first order in perturbation theory compared to the usual treatment. You should give the answer in the approximation of the lowest non-trivial order in R/a_0.

b) For some unperturbed states, the shift in the energy is much smaller, in the sense of being proportional to a higher power of R/a_0 than the result of the previous part. For how many states with energy lower than -1 eV is this true? Give a complete listing of these states. (For the purposes of this problem, assume that the electron and the proton have no spin.)

IV. PARTICLES IN A HARMONIC-OSCILLATOR POTENTIAL *[(15+5+10+10) PTS]*

Two particles, each of mass m, are bound in a one-dimensional harmonic-oscillator potential $V_i = \frac{1}{2}kx_i^2$ and interact with each other through an attractive harmonic force $F_{12} = -K(x_1 - x_2)$, where x_i are the individual positions. You may take K to be small.

a) Provide a general expression for the eigenvalues and eigenfunctions of the system based on single-particle harmonic-oscillator solutions.

b) What are the energies of the three lowest states of this system?

c) If the particles are identical and spinless, which of the states of part b) are allowed?

d) If the particles are identical and have spin $1/2$, what is the total spin of each of the states of part b)?

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I. 1D HARMONIC OSCILLATOR WITH PERTURBATION *[(10+24+6) PTS]*

A 1-dimensional quantum harmonic oscillator has mass m and angular frequency ω. The usual raising and lowering operators are $a^\dagger$ and a, where

$$a = \sqrt{\frac{m\omega}{2\hbar}} X + i \frac{1}{\sqrt{2\hbar\omega m}} P, \tag{1}$$

and $[a, a^\dagger] = 1$. The eigenstates of the unperturbed Hamiltonian are $|n\rangle$, with $a|n\rangle = \sqrt{n}\,|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$. Consider the operator

$$W = \lambda x^4, \tag{2}$$

with λ a positive real constant.

a) Find the complete selection rules for the matrix elements $\langle k| W |n\rangle$. In other words, what are necessary and sufficient conditions on k and n for this matrix element to be non-zero? (You do not need to compute the matrix elements in this part.)

b) If W is added to the Hamiltonian as a perturbation, find all of the energy eigenvalues, at first order in perturbation theory.

c) Give a rough estimate for how large λ can be such that perturbation theory can be trusted for the ground state energy.

II. STEP POTENTIAL *[(20+20) PTS]*

In one dimension a particle of mass m is traveling from the left to the right. It encounters a potential step at $x = 0$, i.e., the potential can be written as $V(x) = V_0\Theta(x)$, where $\Theta(x)$ is the unit step function and $V_0 > 0$. Solve the Schrödinger equation and calculate the probability that the particle is scattered backwards by the potential in these two cases:

a) For energy $E < V_0$.

b) For energy $E > V_0$.

III. SPIN 1/2 PARTICLE *[(10+6+20+4) PTS]*

a) What are the eigenstates and eigenvalues of $\hat{S}_y$ and $\hat{S}_z$? (You can just write these down.)

b) Write down the squares and commutation relation for the Pauli matrices.

Now we consider a particular quantum state of a spin-$1/2$ particle, described by the spinor

$$\frac{1}{\sqrt{5}}\begin{pmatrix} i \\ 2 \end{pmatrix},$$

which is given in the basis of the eigenstates of $\hat{S}_z$.

c) Calculate $\left\langle \hat{S}_z \right\rangle$, $\left\langle \hat{S}_y \right\rangle$, $\left\langle \hat{S}_y^2 \right\rangle$, and $(\Delta S_y)^2 = \left\langle \left(\hat{S}_y - \left\langle \hat{S}_y \right\rangle \right)^2 \right\rangle$ for the above spinor.

d) What is the probability that a measurement of $\hat{S}_y$ gives the value $\hbar/2$?

Hint: The Pauli matrices are on the formula sheet.

IV. ROTATIONAL INVARIANCE *[(10+10+10+10) PTS]*

In many systems, the Hamiltonian is invariant under rotations. An example is the hydrogen atom where the potential $V(r)$ in the Hamiltonian

$$H = \frac{p^2}{2m} + V(r),$$

depends only on the distance r to the origin.
An infinitesimally small rotation of a wavefunction $\psi(x, y, z)$ about the z-axis by an angle $d\phi$ can be represented by an operator $R_{z,d\phi}$ with

$$R_{z,d\phi}\,\psi(x, y, z) = \psi(x - yd\phi, y + xd\phi, z),$$

a) Show that the operator $R_{z,d\phi}$ can be expressed in terms of the operator of angular momentum component L_z.

b) Starting from the expression of L_z in Cartesian coordinates, show that L_z can be related to the derivative with respect to ϕ in spherical coordinates. Derive the ϕ-dependent part of the wavefunction corresponding to an eigenstate of L_z.

c) Show that if $R_{z,d\phi}$ commutes with the Hamiltonian, then there exist eigenfunctions of H that are also eigenfunctions of $R_{z,d\phi}$.

d) Using the fact that the components of angular momentum L_i with $i = x, y, z$ commute with the Hamiltonian, show that L^2 commutes with the Hamiltonian.

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I. CHARGED PARTICLE IN A MAGNETIC FIELD *[(5+15+20) PTS]*

The Hamiltonian for a spinless charged particle in a magnet field $\vec{B}$ is given by

$$\hat{\mathcal{H}} = \left(\hat{\vec{p}} - \frac{q}{c}\vec{A}(\hat{\vec{r}})\right)^2 ,$$

where q is the charge of the particle, $\vec{p} = (p_x, p_y, p_z)$ its momentum conjugate to the particle's position $\vec{r}$, and $\vec{A}$ the magnetic vector potential. Here we assume for the vector potential $\vec{A} = -B_0 y \vec{e}_x$, where $\vec{e}_x$ is the unit vector in x-direction.

a) Write down the Schrödinger equation for the particle. What is $\vec{B}$?

b) Prove that $\hat{p}_x$ and $\hat{p}_z$ are constants of motion. What does this imply for the energy eigenfunctions of this problem?
Hint: An observable $\hat{A}$ is a constant of motion if its expectation value $\left\langle \hat{A} \right\rangle = \langle\psi|\, \hat{A}\, |\psi\rangle$ is time independent.

c) Calculate the energy eigenvalues of the system.
Hints: Use b)! Show that the energy eigenvalues do not depend on p_x.

II. SPIN OPERATORS *[(8+12+20) PTS]*

Consider a quantum system with two independent spin-1/2 operators, $\vec{S}_1$ and $\vec{S}_2$, so that the state space is spanned by a basis of S_{1z} and S_{2z} eigenstates $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$. In each ket, the first entry labels states with S_{1z} eigenvalue $\pm\hbar/2$, and the second entry labels states with S_{2z} eigenvalue $\pm\hbar/2$. The Hamiltonian for the system is

$$H = \frac{a}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2 + \frac{b}{\hbar}(S_{1z} + S_{2z}).$$

The system is in the state

$$|\psi\rangle = c\left(4\,|\uparrow\uparrow\rangle + 3i\,|\uparrow\downarrow\rangle + 2\,|\downarrow\uparrow\rangle + i\,|\downarrow\downarrow\rangle\right),$$

where c is a real positive constant. You should solve for and eliminate c from your answers.

a) Suppose that S_{1z} is measured. What is the probability that the outcome is $+\hbar/2$?

b) Suppose that the measurement in part a) was made, and that the outcome was indeed $+\hbar/2$. If the x-component of spin 2 (that is, S_{2x}) is measured, what are the possible outcomes and their probabilities?

c) Now suppose that instead of the measurements in parts a) and b), the energy of the original state was measured. What are the possible outcomes, and their probabilities?

III. HARMONIC OSCILLATOR *[(3+7+10+15+5) PTS]*

Consider a spinless particle of charge q and mass m constrained to move in the xy plane under the influence of the two-dimensional harmonic oscillator potential

$$V(x,y) = \frac{1}{2}m\omega^2(x^2+y^2).$$

a) Give the Hamilton operator H for the two-dimensional harmonic oscillator in rectangular coordinates.

b) Give the stationary states and the corresponding energies for the Hamilton operator H.

   Remark: you may use without proof the stationary states $|n\rangle$ and the corresponding energies for a one-dimensional harmonic oscillator.

Now we turn on a weak magnetic field $\boldsymbol{B} = (0,0,B)$ pointing in the z-direction, so that (to first order in B) the Hamilton operator acquires an extra term

$$H' = -\boldsymbol{\mu}\cdot\boldsymbol{B} = -\frac{q}{2m}(\boldsymbol{L}\cdot\boldsymbol{B}) = -\frac{qB}{2m}(xp_y - yp_x)$$

that we treat as a perturbation to H.

c) Find the first-order correction to the energy of the ground state.

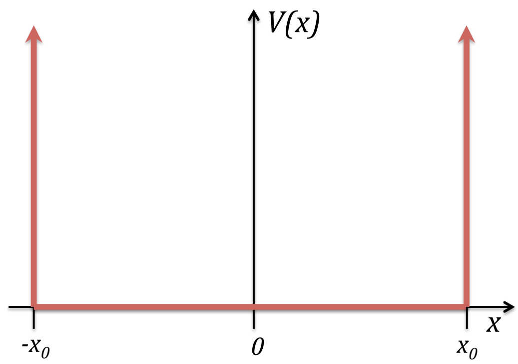
d) Find the first-order corrections to the first excited state energies.

e) Interpret your results for parts c) and d). Your answer should be brief, not more than two or three sentences.

Hint: The ladder operators for the one-dimensional harmonic oscillator are

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip_x}{m\omega}\right) \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip_x}{m\omega}\right)$$

IV. POTENTIAL WELL *[(8+10+10+12) PTS]*



Let us consider a particle of mass m in an infinite square potential well $V(x)$ [0 for $|x| < x_0$, ∞ otherwise]. The potential well is depicted in the figure.

a) Redraw the potential well and sketch the lowest three eigenfunctions (for $n = 1,2,3$ with n being the quantum number).

b) Calculate the energy eigenenergies E_n of the particle.

Next, we consider the wave function

$$\psi(x) = \frac{\gamma}{\sqrt{x_0}}\left[\cos\left(\pi x/(2x_0)\right) + 2\sin\left(\pi x/x_0\right)\right],$$

which is a linear combination of the lowest two eigenfunctions.

c) Find the value of γ, such that $\psi(x)$ is normalized.

d) Calculate the expectation value of the kinetic energy.

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I. POTENTIAL BARRIER *[(10+10+10+10) PTS]*

A particle with energy E, traveling from the left, encounters a potential barrier of width a and finite energy V_0. The potential as a function of x is therefore:

$$V(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ V_0 & \text{for} \quad 0 < x < a \\ 0 & \text{for} \quad x > a \end{cases}$$

a) Consider these two cases.

- $E > V_0$ where the energy of the particle is higher than the barrier
- $E < V_0$ where the energy of the particle is lower than the barrier

Write down the Schrödinger equation and find an Ansatz for the wave function in the different regions, including the wave number.

b) In both cases, give the corresponding boundary conditions and write down the system of equations that define the unknown parameters of the Ansatz in a).

c) Consider the case for $a \to \infty$. Redo parts a) and b) under this condition. The modified potential is given by:

$$V(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ V_0 & \text{for} \quad x > 0 \end{cases}$$

d) Find the transmission probability T for the potential described in c) and $E > V_0$.

II. TWO INTERACTING SPINS *[(12+12+8+8) PTS]*

Two **identical** particles of mass m move in 3 dimensions in a harmonic oscillator potential $V(r) = \frac{1}{2}m\omega^2 r^2$, where r is the spherical radial coordinate. They interact with each other only through a spin-spin interaction Hamiltonian $H_{\text{int}} = c\vec{S}_1 \cdot \vec{S}_2$, where $i = 1, 2$ label the two particles and c is a constant.

a) Suppose that the identical particles have spin 1/2, and that $c = 0$. What are the lowest two energy eigenvalues and their degeneracies?

b) For the states in part a), but now taking $c \neq 0$, what are the exact energy eigenvalues and their degeneracies?

c) Now suppose that the identical particles have spin 1, and that $c = 0$. What is the lowest energy eigenvalue and its degeneracy?

d) For the states in part c), but now taking $c \neq 0$, what are the exact energy eigenvalues and their degeneracies?

III. HARMONIC OSCILLATOR *[(4+7+7+15+7) PTS]*

Given a one-dimensional harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

and a wavefunction $\psi(x)$ which is a mixture of the $n = 0$ and $n = 1$ states

$$\psi(x) = N\left[\psi_0(x) - 2\psi_1(x)\right]$$

where $\psi_0(x)$ and $\psi_1(x)$ are the normalized eigenfunctions of the lowest two energy eigenstates and N is the normalization constant. Note that

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

a) Draw $\psi(x)$.

b) Find the normalization constant N.

c) What is $\langle E\rangle$ in terms of m and ω?

d) What are $\langle x\rangle$, $\langle x^2\rangle$, and Δx (the uncertainty of x)?

e) What is $\langle p\rangle$?

IV. TWO CONFINED PARTICLES *[(5+20+15) PTS]*

In one dimension, a particle of mass m is attracted to the origin by a linear force $-kx$. Its Schrödinger equation has eigenfunctions

$$\psi_n(\xi) = H_n(\xi)\exp(-\xi^2/2)\,,$$

where $\xi = (mk/\hbar^2)^{1/4}x$ and H_n is the Hermite polynomial of order n (you do not need the explicit expressions for those in this problem). The eigenvalues are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega\,, \quad \text{where} \quad \omega = \left(\frac{k}{m}\right)^{1/2}.$$

a) Write down the potential for the particle and sketch it.

Next, we consider two non-interacting distinguishable particles ($i = 1, 2$), each of mass m, each attracted to the origin by a force $-kx_i$. Write down expressions for the Hamiltonian, eigenfunctions, eigenvalues, and their degeneracies for the two-particle system using each of the following coordinate systems:

b) single-particle coordinates x_1 and x_2.

c) relative $[x = x_2 - x_1]$ and center-of-mass $[X = (x_1 + x_2)/2]$ coordinates.

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I. RELATIVISTIC OSCILLATOR *[10+10+20 PTS]*

A particle with (resting) mass m_0 moves in the harmonic potential $\frac{1}{2}m_0\omega^2x^2$ in one dimension. In the non-relativistic limit, where the kinetic energy T and momentum p are related by $T = \frac{p^2}{2m_0}$, the ground state and its energy are well known. The latter is $E_0 = \frac{1}{2}\hbar\omega$.
Now we allow for relativistic corrections in the relation between T and p.

a) Derive the relativistic kinetic energy T_r as function of p starting with $T_r = mc^2 - m_0c^2$, where the relativistic mass is given by $m = m_0(1 - v^2/c^2)^{-1/2}$. You need to express T_r in terms of p, m_0, and c.

b) Expand T_r in small $p/(m_0c)$ to order c^{-2} (c = speed of light).

c) Calculate the ground state energy shift ΔE due to the correction $T_r - T$ calculated in b), which is $\propto p^4$. Use first order perturbation theory.

Useful formulas: ladder operators for the harmonic oscillator: $\hat{a} = \sqrt{\frac{m_0\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m_0\omega}\hat{p}\right)$, $\hat{a}^\dagger = \sqrt{\frac{m_0\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m_0\omega}\hat{p}\right)$; Gauss integral: $\int_{-\infty}^{\infty} e^{-a(x+b)^2}\,dx = \sqrt{\frac{\pi}{a}}$ for $a > 0$

II. PERTURBED HYDROGEN *[(16+10+10+4) PTS]*

Suppose that a Hydrogen atom is initially in an energy eigenstate $|\Psi_0\rangle$ with energy E_0. Then at time $t = 0$, a light wave of (angular) frequency ω and electric field amplitude $\mathcal{E}$ shines on the atom. According to first order perturbation theory, the probability amplitude for finding the electron in a new energy state $|\Psi_n\rangle$ (energy E_n) is given by

$$\langle\psi_n|\,\psi_0\rangle = \frac{e^{-\imath E_n t/\hbar}}{\imath h}\int_0^t e^{\imath(E_n-E_0)\tau/\hbar}\,\langle\Psi_n|\,V(x,\tau)\,|\Psi_0\rangle\,d\tau \tag{1}$$

If the potential $V(x,\tau)$ for the light wave is given by

$$V(x,\tau) = \frac{\imath\mathcal{E}}{k}e^{\imath(px-\hbar\omega\tau)/\hbar} = V(x)e^{-\imath\omega\tau} \tag{2}$$

a) Show that the probability, $P_{0\to n}(t)$, that the electron makes a transition from E_0 to E_n is given by

$$P_{0\to n}(t) = C\frac{\sin^2(\Delta E t/(2\hbar))}{\Delta E^2}|\,\langle\Psi_n|\,V(x)\,|\Psi_0\rangle\,|^2\,, \tag{3}$$

with $\Delta E = E_n - E_0 - \hbar\omega$ and a numerical constant C (calculate its value).

b) For a fixed time, sketch the graph of $P_{0\to n}(t)$ as function of ΔE assuming the spatial integral, $\langle\Psi_n|\,V(x)\,|\Psi_0\rangle$ is constant with respect to E_n and ω and indicate the values of ΔE where $P_{0\to n}(t)$ goes to 0.

c) Explain how this plot, or its analytical expression, shows that energy conservation can and will be violated by an amount ΔE as long as $\Delta E \le h/t$.

d) Explain how Eq. (3) can also be interpreted as a special case of the uncertainty principle, $(\Delta E)t \ge h$.

Hint: $\sin^2(\theta/2) = [1 - \cos\theta]/2$

III. LINEAR POTENTIAL *[(24+16) PTS]*

A spinless particle moves in 3 dimensions in a linear potential $V(r) = ar$, where r is the spherical radial coordinate and a is a positive constant.

a) Use a trial wavefunction proportional to e^{kr} to estimate the energy of the ground state.

b) Use an appropriately modified trial wavefunction to estimate the lowest energy with *non-zero* angular momentum

IV. SPIN FLIP *[(4+8+20+4+4) PTS]*

In the presence of a magnetic field $\boldsymbol{B} = (B_x, B_y, B_z)$, the dynamics of the spin $1/2$ of an electron is characterized by the Hamiltonian $H = -\mu_B \, \boldsymbol{\sigma} \cdot \boldsymbol{B}$, where μ_B is the Bohr magneton and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli spin matrices. (The orbital part of the electron dynamics is completely ignored.)

a) Give an explicit matrix representation for H.

In the following, we investigate the time-dependent two-component wave function $\psi(t) = \binom{a(t)}{b(t)}$ characterizing the dynamics of the electron spin.

b) We assume that for $t < 0$ the magnetic field $\boldsymbol{B}$ is parallel to the z axis, $\boldsymbol{B}(t < 0) = (0, 0, B_z)$ and constant in time. Solve the time-dependent Schrödinger equation to obtain $\psi(t)$ such that $\psi(t = 0) = \binom{1}{0}$.

c) At $t = 0$ an additional magnetic field in x direction is switched on so that we have $\boldsymbol{B}(t \geq 0) = (B_x, 0, B_z)$. Solve the time-dependent Schrödinger equation for $t \geq 0$ using the ansatz

$$\psi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} a_1 \cos \omega t + a_2 \sin \omega t \\ b_1 \cos \omega t + b_2 \sin \omega t \end{pmatrix}$$

Hint: The boundary condition $\psi(t = 0) = \binom{1}{0}$ simplifies the calculation of the frequency ω and the coefficients a_1, a_2, b_1, and b_2.

Note also that in order to get a solution $\psi(t)$ valid for all times $t \geq 0$, we may split the coupled equations into equations proportional to $\sin \omega t$ and $\cos \omega t$.

d) Verify the normalization condition $|a(t)|^2 + |b(t)|^2 = 1$.

e) Interpret your result for $|b(t)|^2$ by considering the limiting cases $B_x \ll B_z$ and $B_x \gg B_z$.

Hint: The Pauli spin matrices are on the formula sheet.