NIU Physics PhD Candidacy Exam - Spring 2024

Quantum Mechanics

YOU MAY SOLVE ALL FOUR PROBLEMS! THE **three best graded** COUNT TOWARDS YOUR TOTAL SCORE.
(40 POINTS EACH; TOTAL POSSIBLE SCORE: 120 POINTS)

I. CHARGED PARTICLE IN A MAGNETIC FIELD *[(5+15+20) PTS]*

The Hamiltonian for a spinless charged particle in a magnet field $\vec{B}$ is given by

$$\hat{\mathcal{H}} = \left(\hat{\vec{p}} - \frac{q}{c}\vec{A}(\hat{\vec{r}})\right)^2 ,$$

where q is the charge of the particle, $\vec{p} = (p_x, p_y, p_z)$ its momentum conjugate to the particle's position $\vec{r}$, and $\vec{A}$ the magnetic vector potential. Here we assume for the vector potential $\vec{A} = -B_0 y \vec{e}_x$, where $\vec{e}_x$ is the unit vector in x-direction.

a) Write down the Schrödinger equation for the particle. What is $\vec{B}$?

b) Prove that $\hat{p}_x$ and $\hat{p}_z$ are constants of motion. What does this imply for the energy eigenfunctions of this problem?
Hint: An observable $\hat{A}$ is a constant of motion if its expectation value $\left\langle \hat{A} \right\rangle = \langle\psi| \hat{A} |\psi\rangle$ is time independent.

c) Calculate the energy eigenvalues of the system.
Hints: Use b)! Show that the energy eigenvalues do not depend on p_x.

II. SPIN OPERATORS *[(8+12+20) PTS]*

Consider a quantum system with two independent spin-1/2 operators, $\vec{S}_1$ and $\vec{S}_2$, so that the state space is spanned by a basis of S_{1z} and S_{2z} eigenstates $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$. In each ket, the first entry labels states with S_{1z} eigenvalue $\pm\hbar/2$, and the second entry labels states with S_{2z} eigenvalue $\pm\hbar/2$. The Hamiltonian for the system is

$$H = \frac{a}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2 + \frac{b}{\hbar}(S_{1z} + S_{2z}).$$

The system is in the state

$$|\psi\rangle = c\left(4\,|\uparrow\uparrow\rangle + 3i\,|\uparrow\downarrow\rangle + 2\,|\downarrow\uparrow\rangle + i\,|\downarrow\downarrow\rangle\right),$$

where c is a real positive constant. You should solve for and eliminate c from your answers.

a) Suppose that S_{1z} is measured. What is the probability that the outcome is $+\hbar/2$?

b) Suppose that the measurement in part a) was made, and that the outcome was indeed $+\hbar/2$. If the x-component of spin 2 (that is, S_{2x}) is measured, what are the possible outcomes and their probabilities?

c) Now suppose that instead of the measurements in parts a) and b), the energy of the original state was measured. What are the possible outcomes, and their probabilities?

III. HARMONIC OSCILLATOR *[(3+7+10+15+5) PTS]*

Consider a spinless particle of charge q and mass m constrained to move in the xy plane under the influence of the two-dimensional harmonic oscillator potential

$$V(x, y) = \frac{1}{2} m\omega^2 (x^2 + y^2).$$

a) Give the Hamilton operator H for the two-dimensional harmonic oscillator in rectangular coordinates.

b) Give the stationary states and the corresponding energies for the Hamilton operator H.

   Remark: you may use without proof the stationary states $|n\rangle$ and the corresponding energies for a one-dimensional harmonic oscillator.

Now we turn on a weak magnetic field $\boldsymbol{B} = (0, 0, B)$ pointing in the z-direction, so that (to first order in B) the Hamilton operator acquires an extra term

$$H' = -\boldsymbol{\mu} \cdot \boldsymbol{B} = -\frac{q}{2m}(\boldsymbol{L} \cdot \boldsymbol{B}) = -\frac{qB}{2m}(x p_y - y p_x)$$

that we treat as a perturbation to H.

c) Find the first-order correction to the energy of the ground state.

d) Find the first-order corrections to the first excited state energies.

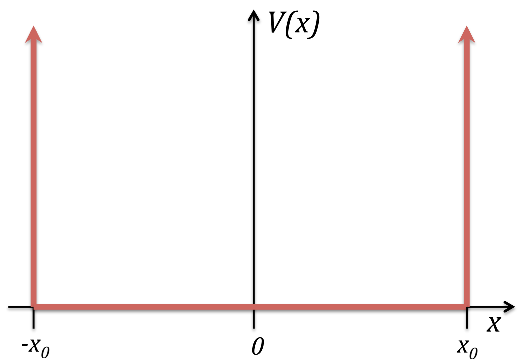
e) Interpret your results for parts c) and d). Your answer should be brief, not more than two or three sentences.

Hint: The ladder operators for the one-dimensional harmonic oscillator are

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{i p_x}{m\omega} \right) \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{i p_x}{m\omega} \right)$$

IV. POTENTIAL WELL *[(8+10+10+12) PTS]*

Let us consider a particle of mass m in an infinite square potential well $V(x)$ [0 for $|x| < x_0$, ∞ otherwise]. The potential well is depicted in the figure.



a) Redraw the potential well and sketch the lowest three eigenfunctions (for $n = 1, 2, 3$ with n being the quantum number).

b) Calculate the energy eigenenergies E_n of the particle.

Next, we consider the wave function

$$\psi(x) = \frac{\gamma}{\sqrt{x_0}} \left[\cos\left(\pi x/(2x_0)\right) + 2 \sin\left(\pi x / x_0\right) \right] ,$$

which is a linear combination of the lowest two eigenfunctions.

c) Find the value of γ, such that $\psi(x)$ is normalized.

d) Calculate the expectation value of the kinetic energy.

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I. POTENTIAL BARRIER *[(10+10+10+10) PTS]*

A particle with energy E, traveling from the left, encounters a potential barrier of width a and finite energy V_0. The potential as a function of x is therefore:

$$V(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ V_0 & \text{for} \quad 0 < x < a \\ 0 & \text{for} \quad x > a \end{cases}$$

a) Consider these two cases.

- $E > V_0$ where the energy of the particle is higher than the barrier
- $E < V_0$ where the energy of the particle is lower than the barrier

Write down the Schrödinger equation and find an Ansatz for the wave function in the different regions, including the wave number.

b) In both cases, give the corresponding boundary conditions and write down the system of equations that define the unknown parameters of the Ansatz in a).

c) Consider the case for $a \to \infty$. Redo parts a) and b) under this condition. The modified potential is given by:

$$V(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ V_0 & \text{for} \quad x > 0 \end{cases}$$

d) Find the transmission probability T for the potential described in c) and $E > V_0$.

II. TWO INTERACTING SPINS *[(12+12+8+8) PTS]*

Two **identical** particles of mass m move in 3 dimensions in a harmonic oscillator potential $V(r) = \frac{1}{2}m\omega^2 r^2$, where r is the spherical radial coordinate. They interact with each other only through a spin-spin interaction Hamiltonian $H_{\text{int}} = c\vec{S}_1 \cdot \vec{S}_2$, where $i = 1, 2$ label the two particles and c is a constant.

a) Suppose that the identical particles have spin 1/2, and that $c = 0$. What are the lowest two energy eigenvalues and their degeneracies?

b) For the states in part a), but now taking $c \neq 0$, what are the exact energy eigenvalues and their degeneracies?

c) Now suppose that the identical particles have spin 1, and that $c = 0$. What is the lowest energy eigenvalue and its degeneracy?

d) For the states in part c), but now taking $c \neq 0$, what are the exact energy eigenvalues and their degeneracies?

III. HARMONIC OSCILLATOR *[(4+7+7+15+7) PTS]*

Given a one-dimensional harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

and a wavefunction $\psi(x)$ which is a mixture of the $n = 0$ and $n = 1$ states

$$\psi(x) = N\left[\psi_0(x) - 2\psi_1(x)\right]$$

where $\psi_0(x)$ and $\psi_1(x)$ are the normalized eigenfunctions of the lowest two energy eigenstates and N is the normalization constant. Note that

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

a) Draw $\psi(x)$.

b) Find the normalization constant N.

c) What is $\langle E\rangle$ in terms of m and ω?

d) What are $\langle x\rangle$, $\langle x^2\rangle$, and Δx (the uncertainty of x)?

e) What is $\langle p\rangle$?

IV. TWO CONFINED PARTICLES *[(5+20+15) PTS]*

In one dimension, a particle of mass m is attracted to the origin by a linear force $-kx$. Its Schrödinger equation has eigenfunctions

$$\psi_n(\xi) = H_n(\xi)\exp(-\xi^2/2)\,,$$

where $\xi = (mk/\hbar^2)^{1/4}x$ and H_n is the Hermite polynomial of order n (you do not need the explicit expressions for those in this problem). The eigenvalues are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega\,, \quad \text{where} \quad \omega = \left(\frac{k}{m}\right)^{1/2}.$$

a) Write down the potential for the particle and sketch it.

Next, we consider two non-interacting distinguishable particles ($i = 1, 2$), each of mass m, each attracted to the origin by a force $-kx_i$. Write down expressions for the Hamiltonian, eigenfunctions, eigenvalues, and their degeneracies for the two-particle system using each of the following coordinate systems:

b) single-particle coordinates x_1 and x_2.

c) relative $[x = x_2 - x_1]$ and center-of-mass $[X = (x_1 + x_2)/2]$ coordinates.

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I. RELATIVISTIC OSCILLATOR *[10+10+20 PTS]*

A particle with (resting) mass m_0 moves in the harmonic potential $\frac{1}{2}m_0\omega^2x^2$ in one dimension. In the non-relativistic limit, where the kinetic energy T and momentum p are related by $T = \frac{p^2}{2m_0}$, the ground state and its energy are well known. The latter is $E_0 = \frac{1}{2}\hbar\omega$.
Now we allow for relativistic corrections in the relation between T and p.

a) Derive the relativistic kinetic energy T_r as function of p starting with $T_r = mc^2 - m_0c^2$, where the relativistic mass is given by $m = m_0(1 - v^2/c^2)^{-1/2}$. You need to express T_r in terms of p, m_0, and c.

b) Expand T_r in small $p/(m_0c)$ to order c^{-2} (c = speed of light).

c) Calculate the ground state energy shift ΔE due to the correction $T_r - T$ calculated in b), which is $\propto p^4$. Use first order perturbation theory.

Useful formulas: ladder operators for the harmonic oscillator: $\hat{a} = \sqrt{\frac{m_0\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m_0\omega}\hat{p}\right)$, $\hat{a}^\dagger = \sqrt{\frac{m_0\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m_0\omega}\hat{p}\right)$; Gauss integral: $\int_{-\infty}^{\infty} e^{-a(x+b)^2}\,dx = \sqrt{\frac{\pi}{a}}$ for $a > 0$

II. PERTURBED HYDROGEN *[(16+10+10+4) PTS]*

Suppose that a Hydrogen atom is initially in an energy eigenstate $|\Psi_0\rangle$ with energy E_0. Then at time $t = 0$, a light wave of (angular) frequency ω and electric field amplitude $\mathcal{E}$ shines on the atom. According to first order perturbation theory, the probability amplitude for finding the electron in a new energy state $|\Psi_n\rangle$ (energy E_n) is given by

$$\langle\psi_n|\,\psi_0\rangle = \frac{e^{-\imath E_n t/\hbar}}{\imath h}\int_0^t e^{\imath(E_n-E_0)\tau/\hbar}\,\langle\Psi_n|\,V(x,\tau)\,|\Psi_0\rangle\,d\tau \tag{1}$$

If the potential $V(x,\tau)$ for the light wave is given by

$$V(x,\tau) = \frac{\imath\mathcal{E}}{k}e^{\imath(px-\hbar\omega\tau)/\hbar} = V(x)e^{-\imath\omega\tau} \tag{2}$$

a) Show that the probability, $P_{0\to n}(t)$, that the electron makes a transition from E_0 to E_n is given by

$$P_{0\to n}(t) = C\frac{\sin^2(\Delta E t/(2\hbar))}{\Delta E^2}|\,\langle\Psi_n|\,V(x)\,|\Psi_0\rangle\,|^2\,, \tag{3}$$

with $\Delta E = E_n - E_0 - \hbar\omega$ and a numerical constant C (calculate its value).

b) For a fixed time, sketch the graph of $P_{0\to n}(t)$ as function of ΔE assuming the spatial integral, $\langle\Psi_n|\,V(x)\,|\Psi_0\rangle$ is constant with respect to E_n and ω and indicate the values of ΔE where $P_{0\to n}(t)$ goes to 0.

c) Explain how this plot, or its analytical expression, shows that energy conservation can and will be violated by an amount ΔE as long as $\Delta E \le h/t$.

d) Explain how Eq. (3) can also be interpreted as a special case of the uncertainty principle, $(\Delta E)t \ge h$.

Hint: $\sin^2(\theta/2) = [1 - \cos\theta]/2$

III. LINEAR POTENTIAL *[(24+16) PTS]*

A spinless particle moves in 3 dimensions in a linear potential $V(r) = ar$, where r is the spherical radial coordinate and a is a positive constant.

a) Use a trial wavefunction proportional to e^{kr} to estimate the energy of the ground state.

b) Use an appropriately modified trial wavefunction to estimate the lowest energy with *non-zero* angular momentum

IV. SPIN FLIP *[(4+8+20+4+4) PTS]*

In the presence of a magnetic field $\boldsymbol{B} = (B_x, B_y, B_z)$, the dynamics of the spin $1/2$ of an electron is characterized by the Hamiltonian $H = -\mu_B \,\boldsymbol{\sigma} \cdot \boldsymbol{B}$, where μ_B is the Bohr magneton and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli spin matrices. (The orbital part of the electron dynamics is completely ignored.)

a) Give an explicit matrix representation for H.

In the following, we investigate the time-dependent two-component wave function $\psi(t) = \binom{a(t)}{b(t)}$ characterizing the dynamics of the electron spin.

b) We assume that for $t < 0$ the magnetic field $\boldsymbol{B}$ is parallel to the z axis, $\boldsymbol{B}(t < 0) = (0, 0, B_z)$ and constant in time. Solve the time-dependent Schrödinger equation to obtain $\psi(t)$ such that $\psi(t = 0) = \binom{1}{0}$.

c) At $t = 0$ an additional magnetic field in x direction is switched on so that we have $\boldsymbol{B}(t \geq 0) = (B_x, 0, B_z)$. Solve the time-dependent Schrödinger equation for $t \geq 0$ using the ansatz

$$\psi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} a_1 \cos\omega t + a_2 \sin\omega t \\ b_1 \cos\omega t + b_2 \sin\omega t \end{pmatrix}$$

Hint: The boundary condition $\psi(t = 0) = \binom{1}{0}$ simplifies the calculation of the frequency ω and the coefficients a_1, a_2, b_1, and b_2.

Note also that in order to get a solution $\psi(t)$ valid for all times $t \geq 0$, we may split the coupled equations into equations proportional to $\sin\omega t$ and $\cos\omega t$.

d) Verify the normalization condition $|a(t)|^2 + |b(t)|^2 = 1$.

e) Interpret your result for $|b(t)|^2$ by considering the limiting cases $B_x \ll B_z$ and $B_x \gg B_z$.

Hint: The Pauli spin matrices are on the formula sheet.

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I. POSITRONIUM ATOM *[(12+16+12) PTS]*

A positronium atom (e^+e^- bound state) is in the orbital 1S state with an external magnetic field along the z axis, so that the effective Hamiltonian is

$$H = \frac{a}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2 + \frac{b}{\hbar}(S_{1z} - S_{2z})$$

where a and b are positive constants, and $\vec{S}_1$ and $\vec{S}_2$ are the spin operators for the electron and positron respectively.

a) Find a matrix representation for the Hamiltonian H, being sure to specify your choice of basis.

b) Find the eigenvalues of the Hamiltonian H, and their degeneracies. (Do not use perturbation theory.)

c) If the system is in the ground state, what is the probability that a measurement of the total angular momentum will give 0? Give your answer in terms of a and b.

II. ONE DIMENSIONAL BOUND PARTICLES *[(15+10+15) PTS]*

Consider a one-dimensional bound particle.

a) Show that

$$\frac{d}{dt}\int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)dx = 0\,,$$

using the Schrödinger equation. ($\psi(x,t)$ does not need to be a stationary state.)

b) Show that, if the particle is in a stationary state at a given time, then it will always remain in a stationary state.

c) If at $t = 0$ the wave function is constant, C, in the region $-a < x < a$ and zero elsewhere, calculate C and express the complete wave function at a subsequent time in terms of the energy eigenstates of the system.
Hint: The actual form of the Hamiltonian is not important, but use the fact that its eigenstates, $|n\rangle$, with eigenenergies, E_n, form a complete orthonormal basis.

III. STATES OF THE HARMONIC OSCILLATOR *[(15+25) PTS]*

The Hamiltonian for a one-dimensional harmonic oscillator can be written in dimensionless units ($m = \hbar = \omega = 1$) as

$$\hat{H} = \hat{a}^\dagger \hat{a} + 1/2\,,$$

with the ladder operators

$$\hat{a} = (\hat{x} + \imath\hat{p})/\sqrt{2}\,, \quad \hat{a}^\dagger = (\hat{x} - \imath\hat{p})/\sqrt{2}\,.$$

Given is the energy eigenfunction of $\hat{H}$

$$\psi_k(x) = \mathcal{N}(2x^3 - 3x)\exp(-x^2/2)\,.$$

($\mathcal{N} = 1/\sqrt{3\sqrt{\pi}}$ is the normalization constant).

a) Using the ladder operators (or Hamiltonian) find the quantum number k for this energy eigenstate.

b) Find the two eigenfunctions of $\hat{H}$ which are closest in energy to ψ_k.

IV. TWO-DIMENSIONAL INFINITE SQUARE WELL *[(5+5+12+18) PTS]*

Consider a particle of mass m in a two-dimensional infinite square well of width a

$$V_0(x,y) = \begin{cases} 0 & 0 \le x \le a,\ 0 \le y \le a \\ \infty & \text{otherwise} \end{cases}$$

a) Write down the time-independent Schrödinger equation for this problem.

b) Write down the energy eigenfunctions and corresponding energy eigenvalues for the ground and first excited states. (You need not derive the answer if you know it.)

We now add a time-independent perturbation

$$V_1(x,y) = \begin{cases} \lambda xy & 0 \le x \le a,\ 0 \le y \le a \\ 0 & \text{otherwise} \end{cases}$$

c) Obtain the first-order energy shift for the ground state.

d) Obtain the zeroth-order energy eigenfunctions and the first-order energy shifts for the first excited states.

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I. HARMONIC OSCILLATOR *[(20+20) PTS]*

Given is a 2D harmonic oscillator with Hamiltonian

$$\hat{H} = \hat{\mathbf{p}}^2/(2m) + \frac{m\omega^2}{2}(\hat{x}^2 + \hat{y}^2) + km\hat{x}\hat{y}\,,$$

with $\hat{\mathbf{p}} = (\hat{p}_x, \hat{p}_y)$.

a) For $k = 0$, what are the energies of the ground state and first and second excited states? What are the degeneracies of each state?

b) For $k > 0$, using first order perturbation theory, what are the energy shifts of the ground state and the first excited states?

II. RECTANGULAR POTENTIAL BARRIER *[(5+20+10+5) PTS]*

A particle with energy E encounters a rectangular potential barrier in one dimension, $V(x) = V_0$ for $0 \leq x \leq a$ and $V(x) = 0$ for $x < 0$ and $x > a$, coming from the left. Here $V_0 > 0$ and $a > 0$.

a) Write down the Schrödinger equation, make an Ansatz for the wave function including the wave numbers, and give the boundary conditions at $x = 0$ and $x = a$.

b) Calculate the transmission *coefficient*, t, as a function of the *two* wave numbers and a.

c) The related transmission *probability* $T = |t|^2$ is

$$\left(1 + \frac{\sin^2(k_{\mathrm{II}}a)}{4\epsilon(\epsilon - 1)}\right)^{-1},$$

with $\epsilon = E/V_0$ for $\epsilon > 1$ and k_{II} the wave numbers for $0 < x < a$. What is the lowest energy at which the transmission probability becomes 1? What would this energy be for a classical particle? What is the tunneling probability for $E < V_0$ (the same expression as above holds, but write T as a function of real quantities only!)?

d) Sketch T as function of ϵ such that it shows the correct physical behavior for $\epsilon < 1$, $\epsilon = 1$, and $\epsilon > 1$.

Hint for b: The boundary conditions give you four equations for the six unknown coefficients of the wave functions. However, the definition of the problem (particle comes from the left!) defines two of the coefficients (explain in your answer)! Then you only need to solve the linear equation system for the transmission coefficient.

III. COUPLED ANGULAR MOMENTA *[(14+13+13) PTS]*

Let us consider two angular momenta $J_1 = J_2 = 1$ that interact via the Hamiltonian $H = \alpha\, \boldsymbol{J}_1 \cdot \boldsymbol{J}_2$. The basis set is denoted by $|J_1M_1, J_2M_2\rangle$, where M_i is the z component of $\boldsymbol{J}_i$ with $i = 1, 2$.

a) Calculate the matrix element $\langle 11, 11|H|11, 11\rangle$. Is this an eigenenergy (explain)?

Hint: You may use

$$J_\pm|JM\rangle = \sqrt{(J \mp M)(J \pm M + 1)}\, \hbar\, |J, M \pm 1\rangle \qquad \text{where} \quad J_\pm \equiv J_x \pm iJ_y$$

b) Calculate the matrix elements $\langle 11, 10|H|11, 10\rangle$, $\langle 10, 11|H|10, 11\rangle$, and $\langle 11, 10|H|10, 11\rangle$. Use these matrix elements to derive the eigenenergies and eigenfunctions for $M_1 + M_2 = 1$.

c) An alternative way to derive the eigenenergies of H is to express $\boldsymbol{J}_1 \cdot \boldsymbol{J}_2$ in terms of $\boldsymbol{J}_1^2$, $\boldsymbol{J}_2^2$, and $\boldsymbol{J}^2$, where $\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2$ is the total angular momentum. Derive this expression and determine the eigenenergies of H for all possible eigenstates $|JM\rangle$ of $\boldsymbol{J}$ (consistent with $J_1 = J_2 = 1$).

Remark: Part (c) is independent of parts (a) and (b).

IV. SPIN PRECESSION *[(4+8+18+6+4) PTS]*

In the presence of a magnetic field $\boldsymbol{B} = (B_x, B_y, B_z)$, the dynamics of the spin $1/2$ of an electron is characterized by the Hamiltonian $H = -\mu_B\, \boldsymbol{\sigma} \cdot \boldsymbol{B}$ where μ_B is the Bohr magneton and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli spin matrices.

a) Give an explicit matrix representation for H.

In the following, we investigate the time-dependent two-component wave function $\psi(t) = \binom{a(t)}{b(t)}$ characterizing the dynamics of the electron spin. (The orbital part of the electron dynamics is completely ignored.)

b) We assume that for $t < 0$ the magnetic field $\boldsymbol{B}$ is parallel to the z axis, $\boldsymbol{B}(t < 0) = (0, 0, B_z)$ and constant in time. From the time-dependent Schrödinger equation, calculate $\psi(t)$ such that $\psi(t = 0) = \binom{1}{0}$.

c) At $t = 0$ an additional magnetic field in x direction is switched on so that we have $\boldsymbol{B}(t \geq 0) = (B_x, 0, B_z)$. Solve the time-dependent Schrödinger equation for $t \geq 0$ using the ansatz

$$\psi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} a_1 \cos\omega t + a_2 \sin\omega t \\ b_1 \cos\omega t + b_2 \sin\omega t \end{pmatrix}$$

Hints: The boundary condition $\psi(t = 0) = \binom{1}{0}$ simplifies the calculation of the frequency ω and the coefficients a_1, a_2, b_1, and b_2.

Note also that in order to get a solution $\psi(t)$ valid for all times $t \geq 0$, we may split the coupled equations into equations proportional to $\sin\omega t$ and $\cos\omega t$.

d) Verify the normalization condition $|a(t)|^2 + |b(t)|^2 = 1$.

e) Interpret your result for $|b(t)|^2$ by considering the limiting cases $B_x \ll B_z$ and $B_x \gg B_z$.

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I. TWO SPINS *[(8+16+16) PTS]*

Consider a quantum system consisting of two independent spin-1/2 operators, $\vec{S_1}$ and $\vec{S_2}$, so that the state space is spanned by orthonormal S_{1z} and S_{2z} eigenstates $|+,+\rangle$, $|+,-\rangle$, $|-,+\rangle$, and $|-,-\rangle$. Here the first $\pm$ labels states with S_{1z} eigenvalue $\pm\hbar/2$, and the second $\pm$ labels states with S_{2z} eigenvalue $\pm\hbar/2$. At time $t=0$, the system happens to be in the state:

$$|\psi(0)\rangle = A\Big(|+,+\rangle + 2|+,-\rangle + 3|-,+\rangle + 4|-,-\rangle\Big),$$

where A is a constant that should not appear in your answers.

a) At time $t=0$, we measure the z-component of spin 1, S_{1z}. What is the probability of finding $+\hbar/2$? What is the normalized state immediately after this measurement?

b) Suppose the measurement mentioned in the previous part has happened, and the result was indeed $+\hbar/2$. If we then measure the x-component (note: not the z-component!) of spin 2, S_{2x}, what results can be found, and with what probabilities?

c) Instead of performing the above measurements, we let the system evolve under the influence of the Hamiltonian $H = \omega S_{1z}$. Find the state of the system at time t. Use this to find the expectation value of the x-component of spin 1, $\langle S_{1x}\rangle$, as a function of time.

II. ENTANGLED STATES *[(5+5+5+15+10) PTS]*

Here we consider two-level systems (e.g., Qubits) having eigen-basis states $|0\rangle$ and $|1\rangle$. An arbitrary state $|\psi\rangle$ in such a system can be written as superposition $|\psi\rangle = \alpha_0\,|0\rangle + \alpha_1\,|1\rangle$ with complex coefficients $\alpha_{0,1}$. Now, two of these systems — system A and system B — are brought into contact forming the composite system C. A general state in system C is given by

$$|\psi\rangle_C = \sum_{i,j\in\{0,1\}} c_{ij}\,|i\rangle_A\,|j\rangle_B \ .$$

If $|\psi\rangle_C$ can be written in form $|\psi\rangle_C = |\psi_A\rangle_A\,|\psi_B\rangle_B$ it is called a product state. However, not all states are product states and a state is called *entangled* if one c_{ij} exists, which cannot be written as product of the basis coefficients $\alpha_{0,1}^{(A,B)}$. Let us study a particular state:

$$|\phi\rangle_C \equiv \gamma\left(|0\rangle_A\,|1\rangle_B - |1\rangle_A\,|0\rangle_B\right) .$$

a) Calculate the coefficient γ in $|\phi\rangle_C$.

b) Prove that $|\phi\rangle_C$ is entangled.

c) Write down another entangled state of system C with $c_{00} \neq 0$ (properly normalized).

d) The degree to which a state $|\phi\rangle_C$ is entangled is measured by the entropy $S = -\mathrm{Tr}(\hat{\rho}_{A,B}\ln(\hat{\rho}_{A,B}))$ with either of the operators $\hat{\rho}_{A,B} = \mathrm{Tr}_{B,A}\hat{\rho}_C$ (these are called density operators). Calculate this entropy for $|\phi\rangle_C$ (see hint). Justify that it is maximum by comparing it to pure states of one of the sub-systems.

e) Now we have two observers of Qubits A and B, Alice and Bob, respectively. Both measure in the eigen-basis states of their respective system, i.e., they apply operators $|i\rangle_{A,B}\langle i|_{A,B}$ to the composite state $|\phi\rangle_C$. Alice makes the measurement first and measures either state $|0\rangle_A$ or $|1\rangle_A$. What are the probabilities to measure these states and what are the resulting composite states of system C? After this measurement, what does Bob measure in either case (give the corresponding probabilities)? What does this mean for system B (even if systems A and B are spatially separated)? Does this mean that information can be transmitted instantaneously? (justify)

Hints: Read carefully and answer all sub-questions. The density operator related to $|\psi\rangle_C$ is given by $\hat{\rho}_C = |\psi\rangle_C \langle\psi|_C$. $\hat{\rho}_A = \mathrm{Tr}_B \hat{\rho}_C = \sum_{i\in\{0,1\}} \langle i|_B \hat{\rho}_C |i\rangle_B$. For an arbitrary state in a two-level system, the entropy can be written as $S = -\mathrm{Tr}(\hat{\rho}\ln(\hat{\rho})) = -\sum_{i\in\{0,1\}} \eta_i \ln \eta_i$, if $\hat{\rho}$ is written in eigen-decomposition $\hat{\rho} = \sum_{i\in\{0,1\}} \eta_i |i\rangle\langle i|$.

III. DRIVEN HARMONIC OSCILLATOR *[(8+8+8+8+8) PTS]*

Consider a particle of mass m and charge q in a three-dimensional harmonic oscillator described by the Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$$

with $\boldsymbol{p} = (p_x, p_y, p_z)$ and $\boldsymbol{r} = (x, y, z)$.

a) Show that the eigenstates of H_0 can be chosen as eigenstates of angular momentum L_z.

Similarly one can show that the eigenstates of H_0 may also be chosen as eigenstates of angular momentum L_x and L_y, so that the eigenstates of H_0 may be labeled $|n, \ell, m_z\rangle$, where $E_n = \hbar\omega(n + \frac{3}{2})$ with $n = 0, 1, 2, \ldots$ is the eigenvalue of H_0, $\hbar^2\ell(\ell+1)$ is the eigenvalue of $\boldsymbol{L}^2$ and m_z is the eigenvalue of L_z.
We assume that at time $t = -\infty$ the oscillator is in its ground state, $|0, 0, 0\rangle$ It is then acted upon by a spatially uniform but time dependent electric field

$$\boldsymbol{\mathcal{E}}(t) = \mathcal{E}_0 \exp(-t^2/\tau^2)\hat{\boldsymbol{z}}$$

(where $\mathcal{E}_0$ and τ are constant).

b) Show that, to first order in the perturbation, the only possible excited state the oscillator could end up in is the $|1, 1, 0\rangle$ state.

c) What is the probability for the oscillator to be found in this excited state at time $t = \infty$?
[Note that $\int_{-\infty}^{\infty} \exp[-(x-c)^2]\, dx = \sqrt{\pi}$ for any complex constant c.]

d) The probability you obtain should vanish for both $\tau \to 0$ and $\tau \to \infty$. Explain briefly why this is the case.

e) If instead the oscillator was in the $|1, 1, 0\rangle$ state at time $t = -\infty$, show that the probability that it ends up in the ground state at time $t = \infty$ is identical to what was found in part c).

IV. ATTRACTIVE POTENTIAL *[(30+10) PTS]*

Let us consider a particle with mass m in a one-dimensional square potential well with an attractive delta-potential at its center, i.e.,

$$U(x) = \begin{cases} \infty & x \leq -a \\ -U_0\delta(x) & -a < x < a \\ \infty & x \geq a \end{cases},$$

where $2a$ is the width of the square well and U_0 the strength of the delta potential.

a) Derive the equation determining the eigenenergy ϵ of the bound eigenstate of the delta potential.
Hint: This equation has the form $\tanh(ka) = \alpha\frac{k}{U_0}$ with a constant α you should find and wave number $\hbar k = \sqrt{2m|\epsilon|}$.

b) What is the minimum U_0 for which a state with $\epsilon < 0$ exists?

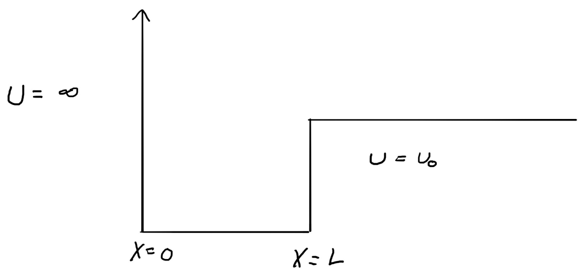
NIU Physics PhD Candidacy Exam - March 2021

Quantum Mechanics

You may solve ALL four problems! The **three best graded** count towards your total score.
(40 points each; total possible score: 120 points)

I. SEMI-INFINITE POTENTIAL *[(10+10+20) PTS]*

Consider a particle of mass m in the one-dimensional semi-infinite potential well shown in the figure. The potential is infinite for $x < 0$, zero for $0 \leq x \leq L$ and $U = U_0$ for $x > L$.



a) Find the approximate ground state energy E_0 in terms of m and L by assuming $U_0 \gg E_0$.

b) Obtain the transcendental equation that relates E to m, L, and U_0 for bound states of this potential.

b) Find the approximate change in the energy of the ground state of the particle relative to E_0 from the first part under the assumption that $U_0 \gg E$.

II. INTERACTING SPINS *[(16+8+16) PTS]*

We consider two spin-1/2 particles interacting via the operator

$$f = a + b\, \boldsymbol{s}_1 \cdot \boldsymbol{s}_2$$

where a and b are constants and $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are the spin operators for particles 1 and 2. The total spin angular momentum is $\boldsymbol{j} = \boldsymbol{s}_1 + \boldsymbol{s}_2$.

a) Show that f, $\boldsymbol{j}^2$ and j_z can be measured simultaneously.

b) Derive the matrix representation for f in the $|j, m, s_1, s_2\rangle$ basis. (Label rows and columns of your matrix.)

c) Derive the matrix representation for f in the $|s_1, s_2, m_1, m_2\rangle$ basis. (Again, label rows and columns of your matrix.)

III. 1D HARMONIC OSCILLATOR *[(10+10+20) PTS]*

Given is a 1D harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

and a wave function which is a superposition of the $n = 0$ and $n = 1$ eigenstates

$$\psi(x) = \frac{1}{\sqrt{5}} \left[u_0(x) - 2u_1(x)\right] .$$

a) Sketch $\psi(x)$.

b) What is $\langle E \rangle$ in terms of m and ω?

c) What are $\langle \hat{x} \rangle$, $\langle \hat{x}^2 \rangle$, and δx?

IV. FLUX QUANTUM *[(5+10+20+5) PTS]*

A charged particle (charge $-e$) moves in a (three dimensional) space having an infinite, impenetrable cylinder of radius a along the z-axis in its center. ψ_0 shall be the solution of the stationary Schrödinger equation outside the cylinder without magnetic field.

a) Now we apply a magnetic field $\mathbf{B}$ to the system, which is determined by the vector potential, $\mathbf{A}$. Write down the corresponding Schrödinger equation.

b) Here the vector potential will be

$$\mathbf{A}(r, \varphi, z) = \begin{cases} \frac{1}{2} Br\hat{\varphi}, & r < a \\ \frac{a^2 B}{2r}\hat{\varphi}, & r > a \end{cases}$$

where $\hat{\varphi}$ is the angular unit vector. Calculate the magnetic field distribution, $\mathbf{B}$, from this vector potential.

c) Use the functional form $\psi = e^{-i\gamma\chi}\psi_0$ with

$$\chi(\mathbf{x}) = \int_{\mathbf{x_0}}^{\mathbf{x}} \mathbf{A} \cdot d\mathbf{s}$$

for the wave function and solve the Schrödinger equation corresponding to the above vector potential. Find the constant γ.

d) For which magnetic flux $\Phi = B\pi a^2$ in the cylinder is the wave function ψ unique when going around the cylinder? (πa^2 is the cross section of the cylinder.)

Hint: $\nabla = (\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z})$, $\nabla \times \mathbf{v} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\varphi} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ v_r & rv_\varphi & v_z \end{vmatrix}$.

NIU Physics PhD Candidacy Exam - Spring 2021

Quantum Mechanics

YOU MAY SOLVE ALL FOUR PROBLEMS! THE **three best graded** COUNT TOWARDS YOUR TOTAL SCORE.
(40 POINTS EACH; TOTAL POSSIBLE SCORE: 120 POINTS)

I. PARTICLE NEAR A WALL IN 1D *[(5+25+10) PTS]*

An approximate model for a particle of mass m near a wall is to consider a particle moving under the influence of the one-dimensional potential given by

$$V(x) = \begin{cases} -V_0\delta(x), & \text{for } x > -d \\ \infty, & \text{else} \end{cases},$$

for a distance $d > 0$, $V_0 > 0$, and $\delta(x)$ the delta function.

a) Sketch the potential $V(x)$.

b) Find the modification of the bound-state energy caused by the wall, when it is far away. Explain also how far is "far away".

c) What is the exact condition for V_0 and d for the existence of at least one bound state?

Hint for b): In the transcendental equation for k, use the large-k solution, which you also need for defining the inequality for "far away".

II. A SPIN-1/2 PARTICLE *[(10+15+15) PTS]*

A spin-1/2 particle has its spin along the $\hat{z}$ direction measured, and the result obtained is $S_z = +\hbar/2$.

a) An ideal measurement is then made of the spin along the direction $\hat{n} = \sin\theta\,\hat{x} + \cos\theta\,\hat{z}$. What is the probability that this measurement gives $+\hbar/2$? What is the probability that it gives $-\hbar/2$?

b) Suppose that Professor Bumblesort made the measurement described in part (a), but forgot to write down the result. Immediately afterwards, the spin along the $\hat{z}$ direction is measured again. What is the probability that the result is $S_z = +\hbar/2$?

c) Instead of part (b), suppose that the spin is in a magnetic field pointing along the $\hat{z}$ direction, so that the Hamiltonian is $H = -bS_z$, where b is a constant. If the spin along the $\hat{n}$ direction was measured to be $+\hbar/2$ at time $t = 0$, what is the probability that another measurement of the spin along the $\hat{n}$ direction at time $t = T$ is again $+\hbar/2$?

III. HARMONIC OSCILLATOR WAVE FUNCTION *[(5+8+12+15) PTS]*

At time $t = 0$ a particle in the potential $V(x) = m\omega^2 x^2/2$ shall be described by the wave function

$$\Psi(x,0) = A \sum_{n=0}^{\infty} 2^{-n/2} \psi_n(x)\,,$$

where $\psi_n(x)$ are the eigenstates with energy eigenvalues $E_n = (n+1/2)\hbar\omega$.

a) Find the normalization constant A.

b) Write an expression for $\Psi(x,t)$ for $t > 0$.

c) Show that $|\Psi(x,t)|^2$ is a periodic function of time and indicate the longest period τ.

d) Find the expectation value of the energy at $t = 0$.

Hints: $\langle \psi_n \psi_m \rangle = \delta_{n,m}$ and $\sum_{n=0}^{\infty} x^{-n} = x/(x-1)$ (and use the x-derivative of the latter equality).

IV. ELECTRONS IN A RIGID-WALL POTENTIAL *[(6+8+8+8+10) PTS]*

We consider electrons in a rigid-wall potential of width a

$$V(x) = \begin{cases} 0 & 0 \le x \le a \\ \infty & \text{otherwise} \end{cases}$$

a) Give the stationary wave functions and energies of a single electron in the potential $V(x)$. (If you know the answer, you may write it without derivation.)

Next we consider two noninteracting electrons in the potential $V(x)$.

b) What is the lowest-energy wave function of the two-electron system if the two electrons are in the same spin state?

c) What is the lowest-energy wave function of the two-electron system if the two electrons are in opposite spin states?

d) What is the total spin angular momentum (magnitude and z component) of the two electrons in the cases (a) and (b)?

e) What are the first and second excited state for the two-electron system if the two electrons are in the same spin state and if the two electrons are in opposite spin states?

NIU Physics PhD Candidacy Exam - Fall 2020

Quantum Mechanics

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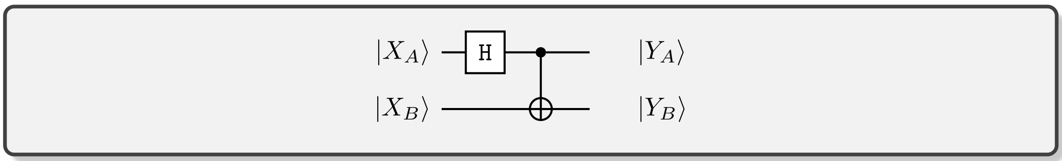
(40 POINTS EACH; TOTAL POSSIBLE SCORE: 120 POINTS)

I. TWO QUBITS *[(10+5+5+10+5+5) PTS]*

Consider two spin $1/2$ particles trapped in separate cells, called qubits A and B, each in a uniform magnetic field that creates two energy levels for each particle; $|0\rangle = \binom{1}{0}$ for spin up and $|1\rangle = \binom{0}{1}$ for spin down. Imagine the qubits are manipulated by two quantum operators as shown below (operators are also called *gates* in quantum logics). The first operator denoted by $\mathtt{H}$ (also called *Hadamard* gate), operates on the upper qubit as a matrix, $\mathtt{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and the second operator acts on the lower qubit as $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **only if** the upper qubit is in the state $|1\rangle$. The latter operator is also called *controlled not* (CNOT) gate and denoted as a $\oplus$ with the control (the upper) qubit denoted by $\bullet$ in the diagram below.



(for quantum operations shown here, the time axis goes from left to right.)

a) If the initial qubit states are in states $|X_A\rangle = |X_B\rangle = |0\rangle$, what are the final states $|Y_A\rangle$ and $|Y_B\rangle$ in the $\{|0\rangle, |1\rangle\}$ basis?

b) Using results in part a), what is the probability of finding qubit A, i.e. $|Y_A\rangle$, in a state $|0\rangle$, if $|Y_B\rangle$ is found in the state $|1\rangle$?

c) Using results of part a), what is the probability of finding $|Y_A\rangle$ in a state $|1\rangle$, if $|Y_B\rangle$ is found to be in the state $|1\rangle$?

d) Does $|Y_A\rangle$ depend on a measurement of qubit B? Explain why or why not, based on your results in part a).

e) If $|X_B\rangle$ is in eigenstate $|0\rangle$, find a state for $|X_A\rangle$, correctly normalized, such that only one eigenstate can be measured for $|Y_A\rangle$ and $|Y_B\rangle$.

f) Repeat e) for the case that $|X_B\rangle$ is in eigenstate $|1\rangle$.

II. TWO FERMIONS IN A BOX *[(30+10) PTS]*

Consider two non-interacting electrons confined within a 1D box. Define the single electron eigenvalue states as $|0\rangle, |1\rangle, |2\rangle$, etc. for the ground, first, second etc excited states of each electron in the box. Denote spin via: $|\uparrow\rangle$ or $|\downarrow\rangle$. In this notation, the ground state is given by:

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)|0\rangle|0\rangle$$

There are four antisymmetric wavefunctions for the lowest energy excited state of this two electron system. These two-electron states can can be distinguished by the quantum numbers for the total spin angular momentum (magnitude j and z-component m.) There is no orbital angular momentum.

a) Write down these four wavefunctions and their corresponding values of j and m. Make sure to properly normalize the wavefunctions.

b) Justify your assignment of $m = 0$ states to the appropriate state of total angular momentum j. You can use either a symmetry argument, or direct application of the lowering and raising operators for angular momentum.

III. MYSTERY POTENTIAL *[(10+10+6+14) PTS]*

The wave function for a particle of mass m moving in one dimension, subject to a potential $U(x)$ is given by

$$\psi(x,t) = \Theta(x) x e^{-Bx} e^{-\imath Ct/\hbar},$$

where $\Theta(x)$ is the Heaviside step function:

$$\Theta(x) = \begin{cases} 1 & , x \geq 0 \\ 0 & , \text{else} \end{cases} .$$

B & C are real constants such that $\psi(x,t)$ is a properly normalized wave function that obeys the time-dependent Schrödinger equation for potential $U(x)$.

a) Sketch this wave function at time $t = 0$. Mark and label any significant features, e.g., zeros and extrema.

b) Using what you know about ψ, make a qualitative sketch of the potential $U(x)$ governing this system, indicating in particular any classically forbidden regions and classical turning points.

c) Is the particle in a state corresponding to a definite energy? If so, what is the energy (in terms of any or all of B and C); if not, why not?

d) Derive an expression for the potential $U(x)$ in terms of B, C, m, and $\hbar$. Does your result agree with your sketch?

IV. RELATIVISTIC OSCILLATOR *[40 PTS]*

A particle of mass m_0 moves one-dimensionally in the oscillator potential $\frac{1}{2}m_0\omega^2x^2$. In the non-relativistic limit, where the kinetic energy T and momentum p are related by $T = \frac{p^2}{2m_0}$, the ground state and its energy are well known. The latter is $E_0 = \frac{1}{2}\hbar\omega$.
Now we allow for relativistic corrections in the relation between T and p.

- Calculate the ground state level shift ΔE to order c^{-2} (c =speed of light).

Hint: The relativistic kinetic energy is $T_r = mc^2 - m_0c^2$ with the relativistic mass $m = m_0(1 - v^2/c^2)^{-1/2}$. You need to express T_r in terms of p, m_0, and c, then expand in small $p/(m_0c)$.

NIU Physics PhD Candidacy Exam - Spring 2020

Quantum Mechanics

You may solve ALL four problems! The **three best graded** count towards your total score.
(40 points each; total possible score: 120 points)

I. HARMONIC OSCILLATOR AND UNCERTAINTY *[(15+15+10) PTS]*

a) For a simple harmonic oscillator with $\hat{H} = (\hat{p}^2/m - kx^2)/2$, show that the energy of the ground state has the lowest value compatible with the uncertainty principle.

b) The wave function of the state where the uncertainty principle minimum is realized is a Gaussian function $\exp(-\alpha x^2)$. Making use of this fact, but without solving any differential equation, find the value of α.

c) Making use of the creation and annihilation operators, but again without solving any differential equation, calculate the (non-normalized) wave function of the first excited state of the harmonic oscillator.

II. ATTRACTIVE δ FUNCTION NEAR A WALL *[(15+5+10+10) PTS]*

We want to study the change in the bound-state solution of an attractive δ function potential when it is near a wall. The potential then becomes

$$U(x) = \begin{cases} \infty & x < -d \\ -U_0\delta(x) & x \geq -d \end{cases} .$$

a) Find the solution for the wavefunction of the bound state and derive the transcendental equation that the complex wavenumber has to satisfy. There is no need to normalize the wavefunction.

b) Give a sketch of the wavefunction and compare it with that for an attractive δ-function in the absence of a wall.

c) Calculate the correction to the energy if the wall is far away. Explain the approximations made.

d) What is the condition for U_0 and d for the existence of at least one bound state.

III. BOUND MUON *[(15+15+10) PTS]*

Consider a muon bound to an atom of atomic number Z in the $1s$ orbit with the wavefunction given by the form:

$$\psi(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}} ,$$

where

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{Zme^2} .$$

Assume that muon lifetime is determined by its interaction with the nucleus and is thus proportional to the probability of finding the muon inside the nucleus.

a) Approximate the atomic weight by $A \approx 2Z$ and assume that the nuclear radius $r_N \sim A^{1/3}$. Find the Z dependence of the muon lifetime.

b) For a neutral atom, the inner orbitals will be filled by electrons. Discuss whether you would expect the muon lifetime to be longer or shorter for a neutral atom relative to a fully ionized atom.

c) Discuss whether you think the effect in part b will make a small fractional change in the lifetime or a large fractional change in the lifetime. Give your reasoning, not just an answer.

IV. HYDROGEN APPROXIMATION *[40 PTS]*

The scientists on the faraway planet Xxzxy don't know how to solve differential equations, but they do know how to differentiate and integrate, and they are also experts in the variational principle. They study the ground state of the Hydrogen atom (electron mass μ, potential $V(r) = -e^2/r$, neglect spin) by using a Gaussian trial wavefunction of the form $\psi(r) = \mathcal{N} e^{-k^2 r^2}$, where $\mathcal{N}$ is a normalization constant. What are their best estimates for k and for the ground state energy? How does their answer for the ground state energy compare to the correct value?

NIU Physics PhD Candidacy Exam - Fall 2019

Quantum Mechanics

YOU MAY SOLVE ALL FOUR PROBLEMS! THE **three best graded** COUNT TOWARDS YOUR TOTAL SCORE.
(40 POINTS EACH; TOTAL POSSIBLE SCORE: 120 POINTS)

I. SPIN IN FIELD *[(20+20) PTS]*

A spin-1/2 particle has a magnetic moment $\vec{\mu} = \mu\vec{S}$, where μ is a constant and $\vec{S}$ is the spin operator. It is in a uniform time-varying magnetic field $\vec{B} = B(t)\hat{z}$. Suppose that at time $t = 0$, the spin of the particle is measured to be along the direction $\hat{n} = \hat{x}\sin(\theta) + \hat{z}\cos(\theta)$. You may express your answers to the following questions in terms of a definite integral involving $B(t)$.

a) Find the expectation value of the spin operator $\vec{S}$ as a function of t.

b) If the $\hat{y}$ component of the spin is measured at time t, what is the probability that one obtains the result $S_y = +\hbar/2$?

II. PARTICLE IN 1D *[(8+8+10+14) PTS]*

A particle of mass m moves in one dimension under the influence of a potential $V(x)$. Suppose it is in an energy eigenstate $\psi(x) = (\gamma^2/\pi)^{1/4}\exp(-\gamma^2 x^2/2)$ with energy $E = \hbar^2\gamma^2/(2m)$.

a) Find the mean position of the particle.

b) Find the mean momentum of the particle.

c) Find the potential $V(x)$.

d) Find the probability $P(p)dp$ that the particle's momentum is between p and $p + dp$.
Hint: There a two ways to find the normalized wave function $\psi(p)$: Either rewrite the Schrödinger equation in momentum representation and use a Gaussian Ansatz or use the Fourier transform.

III. QUANTUM SPEED TRAP *[(12+12+12+4) PTS]*

A *quantum mechanic* is asked to construct a *Speed Trap* operator, $\hat{O}$, for wave riders with two eigenvalues: Ticket (T) and No-ticket (N). The eigenstate for Ticket, $|\psi_T\rangle$, shall be the (coherent) sum of the wave functions for *Age*$>$*16* years riders, $|\psi_{A>16}\rangle$, and *Speed*$>$*30* mph riders, $|\psi_{S>30}\rangle$.

a) Write the matrix for $\hat{O}$ in the $|\psi_T\rangle$, $|\psi_N\rangle$ basis representation.

b) Write the wave function for Ticket, $|\psi_T\rangle$, and No-ticket, $|\psi_N\rangle$, as a linear combination of $|\psi_{A>16}\rangle$ and $|\psi_{S>30}\rangle$. The wave functions should be correctly normalized.

c) Show why a wave rider in an eigenstate $|\psi_{S>30}\rangle$ cannot be in an eigenstate of Ticket.

d) How can you modify $|\psi_T\rangle$ and $\hat{O}$, such that $|\psi_T\rangle$ takes into account both, Speed and Age, and $\hat{O}$ catches all speeding wave riders regardless of age?

IV. CHARGED 2D HARMONIC OSCILLATOR *[(20+10+10) PTS]*

Let us consider a charged particle in a two-dimensional harmonic oscillator subject to a uniform electric field applied along the x-axis, i.e., with additional potential $U(\hat{x}) = -e\mathcal{E}\hat{x}$.

a) Calculate the **exact** change in energy levels of the charged linear oscillator.

b) Calculate the **exact** change in the eigenfunctions of the charged oscillator. Express the wave functions for $\mathcal{E} \neq 0$ in terms of the eigenfunctions of the two-dimensional harmonic oscillator.

c) Calculate the *polarizability* of the oscillator in these eigenstates. The polarizability, α, determines the mean dipole moment, $P = \alpha\mathcal{E}$, induced by the weak external electric field in an isotropic system. It can also be expressed as the negative second derivative of the energy with respect to the electric field. How does α depend on the quantum number?

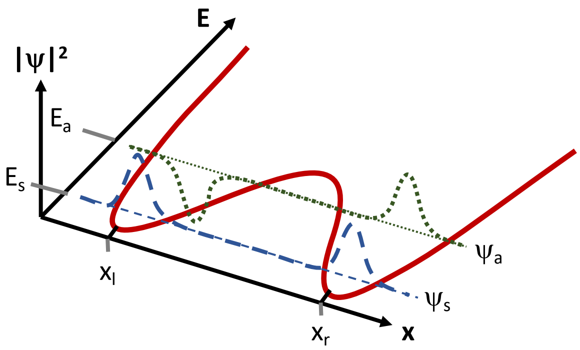
NIU Physics PhD Candidacy Exam - Spring 2019

Quantum Mechanics

YOU MAY SOLVE ALL FOUR PROBLEMS! THE **three best graded** COUNT TOWARDS YOUR TOTAL SCORE.
(40 POINTS EACH; TOTAL POSSIBLE SCORE: 120 POINTS)

I. CONFINED ELECTRON *[(10+20+10) PTS]*

Consider an electron confined to the one-dimensional double potential well shown in the figure above, along with the two lowest energy eigenfunctions, ψ_a (anti-symmetric) and ψ_s (symmetric). These eigenfunctions have energies, E_a and E_s respectively.



a) Using these two wavefunctions construct properly normalized wavefunctions, ψ_l and ψ_r which represent an electron localized in the left well (at x_l) and the right well (at x_r), respectively.

b) Find an expression for $\psi_r(t)$.

c) Find the probability of finding the electron in the right well as a function of time assuming it starts at $t = 0$ in the left well.

II. CHARGED PARTICLE IN A MAGNETIC FIELD *[(5+15+20) PTS]*

The Hamiltonian for a spinless charged particle in a magnet field $\vec{B}$ is given by

$$\hat{\mathcal{H}} = \left(\hat{\vec{p}} - \frac{q}{c}\vec{A}(\hat{\vec{r}})\right)^2 ,$$

where q is the charge of the particle, $\vec{p} = (p_x, p_y, p_z)$ its momentum conjugate to the particle's position $\vec{r}$, and $\vec{A}$ the magnetic vector potential. Here we assume for the vector potential $\vec{A} = -B_0 y \vec{e}_x$, where $\vec{e}_x$ is the unit vector in x-direction.

a) Write down the Schrödinger equation for the particle. What is $\vec{B}$?

b) Prove that $\hat{p}_x$ and $\hat{p}_z$ are constants of motion. What does this imply for the eigenfunctions of this problem?
Hint: An observable $\hat{A}$ is a constant of motion if its expectation value $\left\langle \hat{A} \right\rangle = \langle\psi|\,\hat{A}\,|\psi\rangle$ is time independent.

c) Calculate the energy eigenvalues of the system.
Hints: Use b)! Show that the energy eigenvalues do not depend on p_x.

III. VARIATIONAL TREATMENT OF THE HARMONIC OSCILLATOR *[(20+10+10) PTS]*

By considering the one-dimensional harmonic oscillator potential ($V = \frac{m\omega^2}{2}x^2$) comparable to a particle in a box, one can obtain an approximation of the ground state wavefunction

$$\psi(x) = \frac{1}{\sqrt{L}} \cos \frac{\pi}{2L} x \quad \text{for} \quad |x| \leq L$$

where L is the size of the box. The wavefunction is zero outside of the box. The effective size of the box is considered a free parameter.

a) Calculate the expectation value of the energy for this wavefunction. You can use the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y^2 \cos^2 y \, dy = \frac{\pi}{24}(\pi^2 - 6).$$

b) Find the value of L that minimizes the energy.

c) What is the minimum value of the energy?

IV. TIME-DEPENDENT PERTURBATION THEORY *[(24+8+8) PTS]*

A one-dimensional harmonic oscillator is in its ground state for $t < 0$. At $t = 0$ a perturbation of the form

$$V = Ax^2 e^{-t/\tau}$$

is switched on.

a) Using time-dependent perturbation theory to lowest nonvanishing order, calculate the probability that the system has made a transition to a given excited state. Which final states can be reached that way?

b) Show that for $t \gg \tau$ your expression is independent of time.

c) Specify any requirements on the parameters of the problem necessary for the validity of the approximations made in the application of time-dependent perturbation theory. Compare the value with the zero-point energy $1/2\hbar\omega$ of the harmonic oscillator.