Ph. D. Qualifying Exam Jan 2023

Modern and Statistical Physics

You can do all the problems—the best 3 count towards your total score

Problem 1: (40 points)

(a) The equation of state of a van der Waals gas is

$$\left(P + \frac{a}{v^2}\right)(v - b) = RT$$

where P is pressure, T is temperature, R is gas constant, v specific volume, a and b are characteristic constants for a given gas. Qualitatively explain the meaning of the terms $\frac{a}{v^2}$, and $-b$. [10 points]

(b) A van der Waals gas undergoes an isothermal expansion from specific volume v_1 to specific volume v_2. Calculate the change in the specific Helmholtz function. [15 points]

(c) Calculate the change in the specific internal energy in terms of v_1 and v_2. [15 points]

Problem 2: (40 points)

Boltzmann Gas

Here we consider a non-interacting gas of N classical particles in three dimensions. The gas is confined in a container of volume V. The energy of the i^{th} particle is given by $\varepsilon_i = \mathbf{p}_i^2 / (2m)$.

(a) Calculate the canonical partition function

$$Z^{(c)}(T,N) = \frac{c_N}{h^f} \int e^{-H\left(\{\mathbf{q}_i,\mathbf{p}_i\}\right)/(k_B T)} d\Gamma$$

where H is the Hamiltonian of the system and $d\Gamma$ is the volume element of the $2f$-dimensional phase-space spanned by the components of the coordinates $\mathbf{q}_i$ and momenta $\mathbf{p}_i$ $(i=1,\ldots,N)$. Express your result in terms of the thermal de Broglie wavelength, $\lambda_B = h/\sqrt{2\pi m k_B T}$, and the average particle distance, $a = (V/N)^{1/3}$.

What is the number of spatial or momentum degrees of freedom, f, of the gas? What is the value and meaning of the constant c_N? [15 points]

(b) Calculate the free energy F from $Z^{(c)}$. F is the thermodynamic potential associated with the canonical ensemble, i.e., [5 points]

$$F = -k_B T \ln Z^{(c)}(T,N)$$

(c) Now, the particle number is not fixed anymore and we go over to the grand canonical ensemble. Calculate the grand canonical partition function by Laplace transformation (use the canonical partition function without applied Stirling's formula)

$$Z^{(gc)}(T,\mu) = \sum_{N=0}^{\infty} e^{\frac{\mu}{k_B T}N} Z^{(c)}(T,N)$$

What is the meaning of μ? [10 points]

(d) From the grand canonical partition function, you can now obtain its thermodynamic potential $J(T,\mu) = -k_B T \ln Z^{(\mathrm{gc})}(T,\mu)$ (Planck-Massieu function) and average particle number $\bar{N} = \frac{\partial \ln Z^{(\mathrm{gc})}(T,\mu)}{\partial\left(\mu/(k_B T)\right)}$. In the thermodynamic limit (TDL), for large N and V: $\bar{N} \longrightarrow N$. Use $\bar{N}$ to obtain μ and verify that $J = F - \mu N$ in the TDL. [10 points]

Hint: Use Stirling's formula for large N: $N! \approx N^N e^{-N}$

Do not forget to answer the short questions in (a) and (c)!

Problem 3: (40 points)

A top quark decays to a bottom quark and a W boson. The resulting W boson then decays to a positron and a neutrino. Give your answers below in terms of the masses M_t, M_b, and M_W of the top quark, the bottom quark, and the W boson, respectively. You may treat the positron and the neutrino as massless. You are encouraged to set the speed of light as $c = 1$.

(a) What is the momentum of the bottom quark, in the rest frame of the top quark? [13 points]

(b) What is the maximum possible momentum of the positron, in the rest frame of the top quark? [13 points]

(c) Suppose the top quark has speed $0.5c$ in the lab frame, before it decays. What is the maximum possible momentum of the positron in the lab frame? [14 points]

Problem 4: (40 points)

Michelson Interferometer illuminated with an optical pulse:

A Michelson interferometer has two arms, the first of length ℓ, and the second of length $\ell + d$. The signals from the two arms are recombined and sent to a detector. The interferometer is illuminated by an optical source with a Gaussian spectral intensity distribution given by

$$I(f) = \frac{I_0}{\sqrt{2\pi\sigma^2}} e^{-\frac{(f-f_0)^2}{2\sigma^2}}$$

where f is the frequency.

(a) Draw a diagram of the Michelson-interferometer setup and label all its components. Explain the meaning of the parameters I_0, f_0 and σ, in the spectral-intensity distribution, and comment on the expected temporal properties of the optical signal. [10 points]

(b) Describe qualitatively how the interference pattern associated with the pulse described above differs from the case where the interferometer is illuminated with a monochromatic optical signal. [10 points]

(c) Calculate the total intensity of light observed as a function of the arm offset d. Does this result confirm your prediction from part (a)? Under what conditions does the result reduce to the monochromatic-light case? [20 points]

<u>Hint:</u> the following integral will be useful:

$$\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{2\sigma^2}} \cos(kx)\, dx = \sqrt{2\pi\sigma^2} \cos(ky)\, e^{-\frac{k^2\sigma^2}{2}}$$

Problem 5: (40 points)

Nuclear Energy Levels

Consider a simplified model of an atomic nucleus as a number of independent particles in an infinite potential well in one dimension. Take the potential to be

$$\begin{cases} U(x) = \infty & \text{for } x < 0 \\ U(x) = 0 & \text{for } 0 \le x \le a \\ U(x) = \infty & \text{for } x > a \end{cases}$$

(a) Solve the time-independent Schrodinger equation for a single particle of mass m and find the energy and wavefunctions in terms of the quantum number n. [15 points]

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$$

(b) Find the ground state energies for the following nuclei in terms of the ground state energy for hydrogen ${}^1_1H$, (call it E_0). Make sure to take into account that the proton and neutron are distinguishable and each have spin 1/2. [15 points]

i. ${}^4_2\mathrm{He}$

ii. ${}^6_3\mathrm{Li}$

iii. ${}^{12}_6\mathrm{C}$

(c) The true nuclear binding energy also includes the effects of the attractive strong interaction and the repulsive electromagnetic interaction. One might naively expect that neutron heavy nuclei would be more stable since there is only the attractive strong force and no repulsive electric force. Using the previous results, explain why, to first order, most light nuclei have an equal number of protons and neutrons. [10 points]

Ph. D. Qualifying Exam Aug 2022

Modern and Statistical Physics

You can do all the problems—the best 3 count towards your total score

Problem 1: (40 points)

Entropy of the van der Waals Gas

(a) Write the combined first and second law (including the entropy S and the configuration work) for a closed system of thermodynamics. Use intensive variables, the specific entropy s, and the specific volume v. [5 points]

(b) Derive the following Tds equation under the isochoric (constant volume) condition,

$$Tds = c_v dT + T\left(\frac{\partial P}{\partial T}\right)_v dv \qquad (\, s = s(T, v))$$

Here c_v is the specific heat under the isochoric condition. [10 points]

(c) Show $Tds = c_v dT + \frac{T\beta}{\kappa} dv$ from the result above. Here $\beta \left(\equiv \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P\right)$ is the expansivity under the isobaric condition, and $\kappa \left(\equiv -\frac{1}{v}\left(\frac{\partial v}{\partial P}\right)_T\right)$ is the compressibility under the isothermal condition. [10 points]

(d) Calculate the entropy for a van der Waals gas using the above Tds equation. The equation of state of the van der Waals gas is

$$\left(P + \frac{a}{v^2}\right)(v - b) = RT$$

Here P is the pressure, v the specific volume, T is the absolute temperature, R is the gas constant, and the constant b is another characteristic constant of the gas. [15 points]

Problem 2: (40 points)

(a) Derive the expression of the number of quantum states whose energy lies in the energy range ε to $\varepsilon + d\varepsilon$:

$$g(\varepsilon)d\varepsilon = \gamma_s \frac{4\sqrt{2}\pi V}{h^3} m^{3/2} \varepsilon^{1/2} d\varepsilon = A\varepsilon^{1/2} d\varepsilon,$$

where $g(\varepsilon)$ is the density of states, γ_s is a spin factor, h the Planck's constant, and m the mass of the fermion. [10 points]

(b) Show that the average energy per fermion is $\frac{3}{5}\varepsilon_F$ at T = 0 K by making a direct calculation of $\frac{U(0)}{N}$, where ε_F is the Fermi energy, U(0) is the internal energy at 0K, and N is the number of fermions (fermion gas particles). [15 points]

(c) Similarly, calculate the average speed of a fermion gas particle at T = 0 K. Use the Fermi velocity v_F, defined by $\varepsilon_F = \frac{1}{2} m v_F^2$. [15 points]

Problem 3: (40 points)

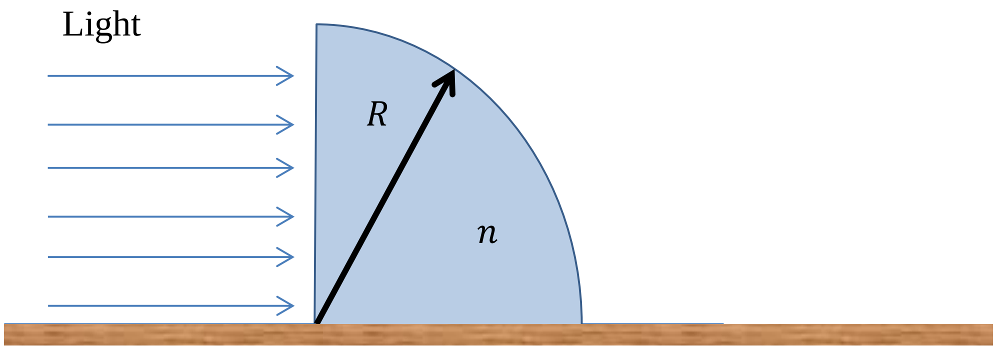
A particle with mass M and energy E collides with another particle of the same mass M which is initially at rest. The result of the collision is two identical particles, each of mass m.

(a) Find a Lorentz transformation to the center-of-momentum frame, and find the energies of each of the two initial particles in that frame. [20 points]

(b) Find the maximum and minimum possible energies E_{max} and E_{min} of either of the two final state particles in the lab frame. [10 points]

(c) Find the energies of the two final state particles, if one of them is emitted at a right angle to the initial direction of the incident particle in the lab frame. [10 points]

Problem 4: (40 points)



A glass prism in the shape of a quarter-cylinder lies on a horizontal table. A uniform, horizontal light beam falls on its vertical plane surface, as shown above. The radius of the cylinder is R and the refractive index of the glass is $n = 1.5$. A patch of light will form on the table.

(a) What distance from the lens does the patch of light start? (express your result in terms of R) [20 points]

(b) How far from the lens does the patch of light extend? (express your result in terms of R) [20 points]

The Lensmaker's equation might be helpful for this problem:

$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}\right]$$

Problem 5: (40 points)

Explain what was learned about quantization of radiation or of the mechanical system for <u>two</u> of the following experiments: [20 points each]

(a) Photoelectric effect

(b) Black body radiation spectrum

(c) Franck-Hertz experiment

(d) Davisson-Germer experiment

(e) Compton scattering

Describe your selected experiments in detail, indicate which of the measured effects were non-classical and why, and explain how they can be understood as quantum phenomena. Give equations if appropriate.

Ph. D. Qualifying Exam   January 2022

Modern and Statistical Physics

You can do all the problems—the best 3 count towards your total score

Problem 1: (40 points) ***van der Waals equation of state, the critical point***. The equation of state of a van der Waals gas is $\left(P + \frac{a}{v^2}\right)(v - b) = RT$, where P is pressure, T is temperature, R is gas constant, v specific volume, a and b are characteristic constants for a given gas. Here we modify the above van der Waals equation of state as $P = P(v, T)$ as a function of v and T.

a. What are the two other conditions at the critical point in the critical isotherm ($T = T_C$) on a P-v diagram in addition to the above equation of state? [15 points]

b. Express the critical specific volume, v_C, the critical temperature, T_C, and the critical pressure, P_C at the critical point, in terms of the constants a and b. [25 points]

Problem 2: (40 points) Fermi-Dirac statistics

a. What are the two conditions for Fermi-Dirac statistics that the fermions obey? [5x2 = 10 points]

b. There is an assembly of N non-interacting fermions under the conservation of particles and energy:

$$\sum_{j=1}^{n} N_j = N, \quad \sum_{j=1}^{n} N_j \varepsilon_j = U,$$

where N_j is the number of particles with single-particle energy ε_j; N and U are fixed. There are g_j quantum states at the jth energy level.

i. *Derive* the thermodynamic probability ω_j of this assembly of fermions for jth energy level ε_j for jth. [10 points]

ii. Using the above result, find the total number of microstates corresponding to an allowable configuration ω_{FD}. [5 points]

iii. *Express and Derive* the Fermi-Dirac distribution function for a discrete energy level ε_j. *Hint*: Find the occupation number of each energy level when the thermodynamic probability is a maximum, i.e., the equilibrium macrostate N_i. Apply the method of *Lagrange multipliers* for a function with two variables under the conservation of particles and energy. Set one of the multipliers to be $1/kT$ and the other to be μ/kT, where μ is the chemical potential. [3 + 12 points]

Problem 3: (40 points) Charged pion decay

A charged pion can decay according to $\pi^- \to \mu^- \bar{\nu}_\mu$. In this problem, you should use units in which the speed of light is $c = 1$, and treat neutrinos and anti-neutrinos as massless. You can call the masses of the pion and muon as M and m, respectively, and you will not need their numerical values. Suppose that the total relativistic energy of the pion in the lab frame happens to be $E = \frac{5}{3}M$.

a) What is the expected mean distance traveled by the pion in the lab frame before it decays? Give your answer in terms of the mean lifetime τ_π of the pion in its rest frame. [6 points]

b) What are the maximum and minimum possible energies of the final state μ^- in the lab frame? (Hint: the maximum and minimum energies of the μ^- will occur when it goes in certain special directions that you can identify without calculation.) [16 points]

c) Now suppose that the muon that came from the pion decay then itself decays according to $\mu^- \to e^- \nu_\mu \bar{\nu}_e$. What is the maximum possible energy of the final state electron in the lab frame? (Here you should treat the electron as massless.) [11 points]

d) Give two other possible final states for π^- decay in which the number of final state particles is three or less. (You should assume that the laws of the Standard Model are in effect. These additional decays are very rare.) [7 points]

Problem 4: (40 points) Beta decay distribution

The density of states for a free particle of momentum p confined within a box is given by $n(p) = Cp^2$ where C is a constant.

a) The density of states can also be written as a function of energy, $n(E)$. Let $n(E) = C'E^\alpha$. Use the energy-momentum relationship for a non-relativistic electron to show that $\alpha = \frac{1}{2}$. [8 points]

b) Show that $\alpha = 2$ for a massless neutrino using the appropriate energy-momentum relationship. [8 points]

c) Suppose a nucleus undergoes beta decay and emits a non-relativistic electron and a massless neutrino. The total decay energy is E_{tot}. Let the amount of energy that goes to the electron be E, and the amount to the neutrino is $E_{tot} - E$. Write an expression for the joint density of states $n_J(E)$ using the expressions obtained in parts a) and b). [8 points]

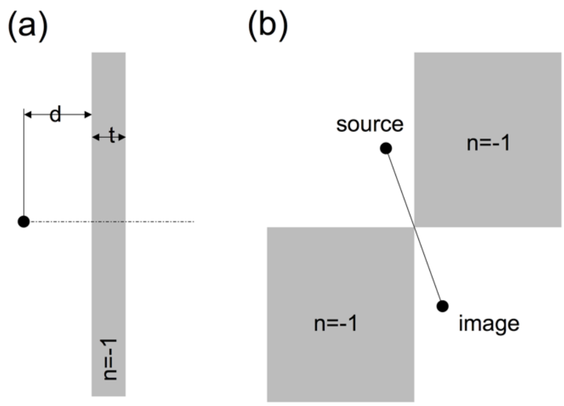
d) Find the energy corresponding to the maximum of $n_J(E)$. [8 points]

e) Sketch a graph of $n_J(E)$ over the range $0 < E < E_{tot}$. [8 points]

Problem 5: (40 points) Metamaterial lens

Artificial materials (or metamaterials) can be engineered to provide a negative index of refraction $n<0$. In this problem we explore the amazing properties of two arrangements of metamaterials described in the Figure below.

1. (**6 points**) Consider an incoming optical ray at the interface from vacuum to a metamaterial. Write down the expression for the refracted angle and draw the incoming and refracted ray for $n=-1$. Indicate the difference(s) with the result obtained for conventional materials.
2. (**17 points**) We now consider a slab of metamaterial with thickness t as drawn in the Figure (a) with $n=-1$.
   a. **(6 points)** Let a point-like source object be at a distance $d=t$ upstream of the slab, show that there is an image and give the location of this image.
   b. **(6 points)** Same question if the object is at a distance $d=1/2t$.
   c. **(5 points)** Comment on the properties of a simple slab with negative index of refraction compared to a standard (n>0) slab.
3. (**17 points**) Consider the configuration shown in Figure (b).
   a. **(6 points)** Via geometrical construction, show that the source is imaged on the lower-right quadrant.
   b. **(6 points)** Show that rays emitted by the source are also imaged on the source itself.
   c. **(5 points)** Discuss possible applications of such imaging/recirculating configurations.



Ph. D. Qualifying Exam Aug 2021

Modern and Statistical Physics

You can do all the problems—the best 3 count towards your total score

Problem 1: (40 points)

(a) The equation of state of a van der Waals gas is

$$\left(P + \frac{a}{v^2}\right)(v - b) = RT$$

where P is pressure, T is temperature, R is gas constant, v specific volume, a and b are characteristic constants for a given gas. Qualitatively explain the meaning of the terms $\frac{a}{v^2}$, and $-b$. [10 points]

(b) A van der Waals gas undergoes an isothermal expansion from specific volume v_1 to specific volume v_2. Calculate the change in the specific Helmholtz function f. [15 points]

(c) Calculate the change in the specific internal energy in terms of v_1 and v_2. [15 points]

Problem 2: (40 points)

(a) What is the core assumption of the Einstein's model of the heat capacity of solids? Is there anything missing to perfectly model the heat capacity of solid? If so, what is it? [10 points]

Using the partition function of an Einstein solid

$$Z = \frac{e^{-\theta_E/2T}}{1 - e^{-\theta_E/T}}$$

where θ_E is the Einstein temperature, answer the following four questions:

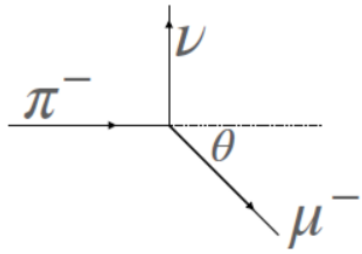
(b) Calculate the Helmholtz function *F*. [10 points]

(c) Calculate the Entropy *S*. [10 points]

(d) Show that the entropy approaches zero as the temperature goes to absolute zero. [5 points]

(e) Find the entropy at high temperatures. [5 points]

Problem 3: (40 points)



(a) A pion traveling at speed v decays into a muon and a neutrino, $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$. If the neutrino emerges at 90 degrees with respect to the original pion direction as in the image, at what angle θ does the muon come off at? Feel free to use natural units where $c = 1$ such that $\beta = v / c = v$ and $\gamma = (1 - v^2)^{-0.5}$. [20 points]

(b) Using basic knowledge of QCD and the Standard Model, find the leading order branching ratio for the $t\bar{t}$ system to decay with an electron (or anti-electron) and a tau (or anti-tau) in the final state. [10 points]

(c) Sketch the lowest order Feynman diagrams for Delbruck scattering (no need to label the particles running through the loops): $\gamma + \gamma \rightarrow \gamma + \gamma$ [5 points]

(d) Consider Delbruck scattering from the previous problem. Which particle/anti-particle pair dominates in the loop in the matrix element calculation? [5 points]

Problem 4: (40 points)

Typical dust particles in the Solar System have mass density ρ. They experience both gravitational and radiation pressure effects from the Sun (mass $M = 2 \times 10^{30}$ kg, emitted light power $P = 4 \times 10^{26}$ W).

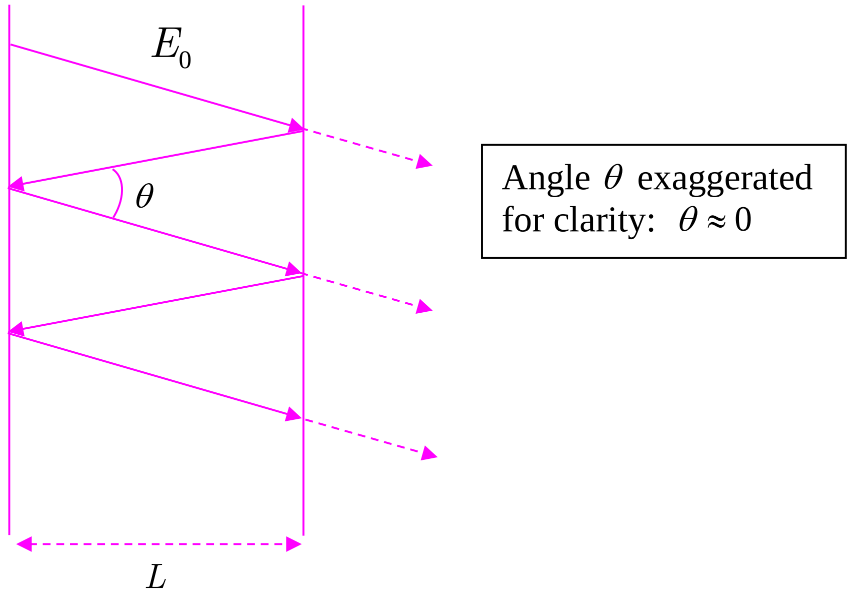
(a) Explain why smaller dust particles are more likely to be ejected from the Solar System in terms of the forces on them. (Assume the dust particles are spherical) [10 points]

(b) Find the critical radius R of a particle that would not be ejected from the Solar System. (find R in terms of P, M, ρ, the gravitational constant G, and the speed of light c) [25 points]

(c) Provide a rough numerical estimate to part (b). (Use: $G \approx 10^{-10}$ m^3/kg s^2, $\rho \approx 10^3$ kg/m^3 [ice]). [5 points]

Problem 5: (40 points)

Consider a laser cavity with mirrors of amplitude reflectivity r and reflection coefficient $r^2 = R \approx 0.99$ and a length L between the mirrors. The laser beam continuously reflects between the mirror ends. The high reflectivity permits the beam in opposite directions to be considered of approximately equal amplitudes so that standing waves are generated which constitute the longitudinal modes of the cavity.



(a) What is the phase change, ϕ, during one round trip of the laser cavity? Express your answer in terms of L and the radiation frequency, ν. Assume $\theta \approx 0$. [5 points]

(b) For what mode frequencies ν can standing waves be maintained in the cavity? (that is, find all the possible frequencies ν that can exist inside the cavity). Express your answer in terms of L. [10 points]

(c) Find an expression for the intensity of radiation transmitted out the far end of the laser cavity in terms of ϕ, E_0, R, and t, where t is the amplitude transmission. [20 points]

(d) As the frequency changes from the standing wave value, the intensity out of the end of the cavity decreases. Use your results from (a) and (c) to determine the shift in frequency over which the intensity drops to half its maximum value. [5 points]

Ph. D. Qualifying Exam March 2021

Modern and Statistical Physics

You can do all the problems—the best 3 count towards your total score

Problem 1: (40 points)

Thermodynamics, Applications of the First Law

The equation of state for radiant energy in equilibrium with the temperature of the walls of a cavity of volume V is $P = aT^4/3$, where a is a constant. The energy equation is $U = aT^4V$.

(a) Show that the heat supplied in an isothermal doubling of the volume of the cavity is $(4/3)aT^4V$. [20 points]

(b) Show that in an adiabatic process, VT^3 is constant. [20 points]

Problem 2: (40 points)

In a pin-hole camera the distance of the pin hole from the photographic plate is $\ell = 10$ cm. You want to take a picture of the sun in the visible spectrum ($\lambda \approx 5000\,\text{Å}$).

(a) In terms of ℓ and λ, what diameter of the pin hole should you use to obtain the sharpest resolution? [35 points]

(b) Estimate this pin hole diameter given the values for ℓ and λ above. [5 points]

Problem 3: (40 points)

Ising Spins in a Magnetic Field

Here we consider N non-interacting, localized (Ising) spins in a magnetic field h. The Hamiltonian is given by

$$\mathcal{H} = -h\sum_{i=1}^{N} s_i$$

with $s_i = \pm 1$. Calculate as a function of temperature, T, and h:

(a) the (canonical) partition function Z. [15 points]

(b) using (a), the free energy, F, and internal energy, E. [7 points]

(c) using (b), the entropy, S. Calculate the limiting values $S(T \to 0, h \neq 0)$ and $S(T, h \to 0)$. [8 points]

(d) the magnetization (average magnetic moment): $m = \langle s_i \rangle$. [10 points]

Hint: For (d), rewrite the expression for the average as a suitable derivative of the logarithmic partition function.

Problem 4: (40 points)

An ambitious new experiment is being proposed to produce Higgs bosons, by colliding a beam of positrons with energy E against a fixed target of electrons at rest. The energy of the positron beam is to be fixed to be the minimum value at which a single Higgs boson can be produced. The produced Higgs boson will sometimes decay into two photons. Give your answers in simplest form in terms of the Higgs boson mass m_H and the electron mass m_e, and feel free to set the speed of light to $c = 1$. (Don't worry about the fact that this experiment isn't very practical. . .)

(a) What is the required positron beam energy E? [10 points]

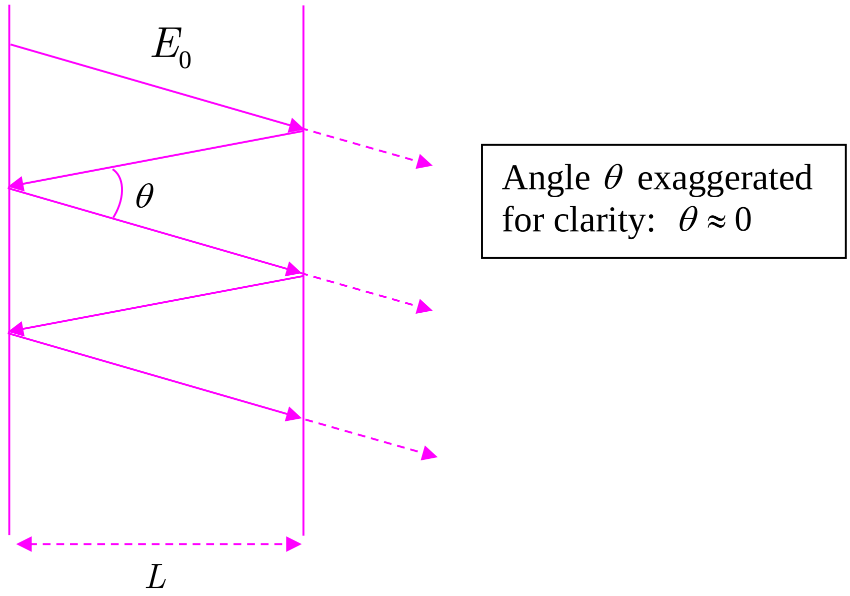
(b) What is the velocity of the produced Higgs bosons? [10 points]

(c) What are the maximum and minimum energies of the photons emitted from the Higgs boson decay? [10 points]

(d) If a photon is emitted in a direction perpendicular to the beam in the Higgs boson rest frame, what angle will it make with the beam in the target frame? [10 points]

Problem 5: (40 points)

Consider a laser cavity with mirrors of amplitude reflectivity r and reflection coefficient $r^2 = R \approx 0.99$ and a length L between the mirrors. The laser beam continuously reflects between the mirror ends. The high reflectivity permits the beam in opposite directions to be considered of approximately equal amplitudes so that standing waves are generated which constitute the longitudinal modes of the cavity.



(a) What is the phase change, ϕ, during one round trip of the laser cavity? Express your answer in terms of L and the radiation frequency, ν. Assume $\theta \approx 0$. [5 points]

(b) For what mode frequencies ν can standing waves be maintained in the cavity? (that is, find all the possible frequencies ν that can exist inside the cavity). Express your answer in terms of L. [10 points]

(c) Find an expression for the intensity of radiation transmitted out the far end of the laser cavity in terms of ϕ, E_0, R, and t, where t is the amplitude transmission. [20 points]

(d) As the frequency changes from the standing wave value, the intensity out of the end of the cavity decreases. Use your results from (a) and (c) to determine the shift in frequency over which the intensity drops to half its maximum value. [5 points]

Ph. D. Qualifying Exam Jan. 2021

Modern and Statistical Physics

You can do all the problems—the best 3 count towards your total score

Problem 1: (40 points)

Classical Thermodynamics

(a) Applying the first law of thermodynamics, show that the total sum of the Carnot ratio of the Carnot engine is

$$\frac{Q_1}{T_1} + \frac{Q_2}{T_2} = 0$$

where $T_1 < T_2$, Q_2 is the heat absorbed during isothermal expansion process, and Q_1 the heat given off during isothermal compression process of the Carnot cycle. [20 points]

(b) Using the above result, derive the Clausius inequality

$$\oint \frac{\bar{d}Q}{T} \leq 0$$

for an arbitrary Carnot cycle. Here $\bar{d}$ represents an inexact differential. [20 points]

Problem 2: (40 points)

Heat capacity of a solid

(a) State the difference between Einstein's theory and Debye's theory of the heat capacity of a solid. [20 points]

(b) Derive Debye's T^3 law:

$$C_V = \frac{12\pi^4}{5} Nk \left(\frac{T}{\theta_D} \right)^3$$

where C_V is the heat capacity under constant volume, N is the total number of phonons, θ_D is the Debye temperature, $\theta_D \equiv h\nu_m / k$, and ν_m is the cutoff frequency. Hint: $\int_0^\infty \frac{x^4 e^x}{(e^x - 1)^2} dx = \frac{4\pi^4}{15}$ [20 points]

Problem 3: (40 points)

A muon-antimuon collider has equal beam energies that are tuned to produce Higgs bosons at rest in the lab frame, and then look for the rare decay to a Z boson and a photon, $H \to Z\gamma$. In your answers below, you can take the masses of the Higgs boson and the Z boson to be m_H and m_Z. The mass of the muon is numerically 105.7 MeV/c^2, and its mean lifetime at rest is 2.2 microseconds

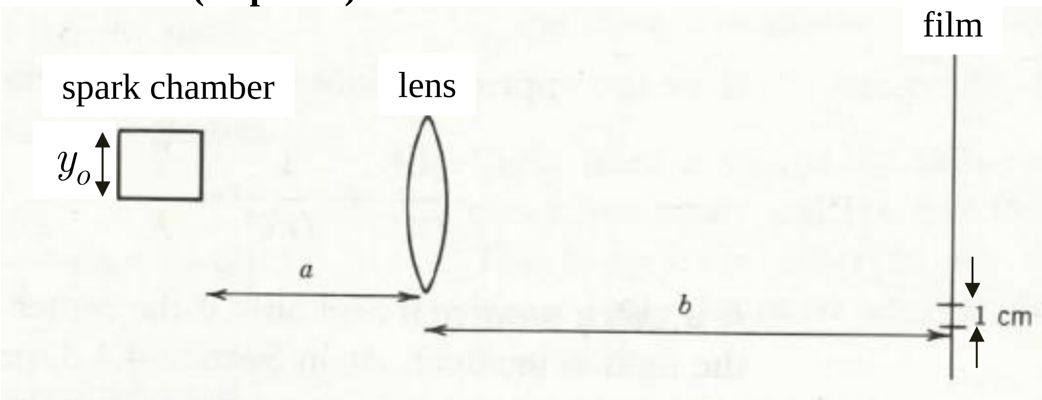
(a) What is the mean length in meters (to one significant digit) that muons in the beam travel before they decay? (Take $m_H = 125$ GeV/c^2 in this part.) [6 points]

(b) Find the energies and momenta of the Z boson and photon produced in the Higgs decay. Give your answers in terms of the symbols m_H and m_Z, rather than numerically. [14 points]

(c) If the Z boson decays to a lepton-antilepton pair, find the maximum and minimum possible momentum of the lepton in the lab frame. In this part, take the mass of the lepton to be 0. Give your answers in terms of the symbols m_H and m_Z, rather than numerically. [14 points]

(d) By what numerical factor (to one significant digit) would the rate for this process be different if the muon and antimuon beams were replaced by electron and positron beams? [6 points]

Problem 4: (40 points)



A lens must be selected to photograph a cubic spark chamber y_o cm on a side. The image size is to be 1 cm. The distance between the spark chamber and the film is L cm.

(a) What focal length lens (in cm) should be chosen? (express in terms of L and y_o) [20 points]

(b) The F-stop number of a lens is the ratio f / D, where f is the focal length and D is the diameter of the lens. If the F-stop number of the lens for this problem is F, find the diffraction limited angular resolution of the lens in terms of λ, the wavelength of light, and F, L, and y_o (assume Fraunhofer diffraction occurs resulting in simple Airy disk patterns). [15 points]

(c) What is the smallest sized spot that can be imaged on the film? [5 points]

Problem 5: (40 points)

The acceleration due to gravity on the surface of Mercury is g_M (3.5 m/sec^2). The radius of Mercury is R_M. Suppose that the atmosphere of Mercury was pure H_2 gas.

(a) What would the temperature be so that the rms speed of the H_2 molecules of mass m matched the escape speed? (express this in terms of the parameters given in this problem and any other universal physical constants). [25 points]

(b) Qualitatively, what is the effect on the temperature of the remaining gas? Explain your reasoning. [5 points]

(c) Would there be a similar effect if the actual temperature was less than the result in (a)? Explain your reasoning. [5 points]

(d) If Mercury's atmosphere had two or more components of differing molecular mass, what would happen to the composition of the atmosphere as a function of time? Explain your reasoning. [5 points]

Ph. D. Qualifying Exam Aug. 2020

Modern and Statistical Physics

You can do all the problems—the best 3 count towards your total score

Problem 1: (40 points)

Classical Thermodynamics

Two equal quantities of water, each of mass m and at temperatures T_1 and T_2 are adiabatically mixed together, the pressure remaining constant. The specific heat capacity of the water at constant pressure is c_P.

(a) Calculate the entropy changes of the masses of water initially at T_1 and T_2, respectively. [15 points]

(b) Calculate the entropy change of the thermodynamic universe. [10 points]

(c) Show that $\Delta S > 0$ for any finite temperatures T_1 and T_2. [15 points]

Problem 2: (40 points)

Statistical mechanics

(a) Calculate the entropy S and the Helmholtz function F for an assembly of *distinguishable* particles. [25 points]

(b) Show that the total energy U and the pressure P are the same for distinguishable particles as for molecules of an ideal gas (indistinguishable, dilute gas) while S is different. Explain why this makes sense. [15 points]

Problem 3: (40 points)

Mandelstam Variables

Consider the relativistic two-body scattering $A+B \rightarrow C+D$ with corresponding 4-vectors p_A, p_B, p_C, and p_D. The masses of the corresponding objects are m_A, m_B, m_C, and m_D. You can set the speed of light as $c=1$ if you would like. The Mandelstam variables are useful Lorentz invariants:

$$s = (p_A + p_B)^2$$
$$t = (p_A - p_C)^2$$
$$u = (p_A - p_D)^2$$

(a) Show that $s+t+u = m_A^2 + m_B^2 + m_C^2 + m_D^2$. [10 points]

(b) Show that the total center of mass energy is given by $\sqrt{s}$. [10 points]

(c) The LHC collides protons in the lab frame with an energy of 6.5 TeV against other protons with an energy of 6.5 TeV (so that $\sqrt{s} = 13$ TeV). One of the things looked for is the production of new Z' bosons. What is the most massive Z' that the LHC could directly observe? [10 points]

(d) Why can the photon not decay? [10 points]
Hint 1: Prove that the mass of a decaying particle must exceed the masses of the particles produced in its decay. In other words, if A decays into $B+C$, show that $m_A > m_B + m_C$.
Hint 2: Evaluate in the center of mass frame.

Problem 4: (40 points)

The Drude model for a metal assumes that the conduction electrons can be approximated by a gas of free electrons where the only important parameters for the gas are n, the number density of electrons, and τ_c, the time between collisions.

(a) Show that in this model the electrical resistivity of a metal can be expressed as [20 points]

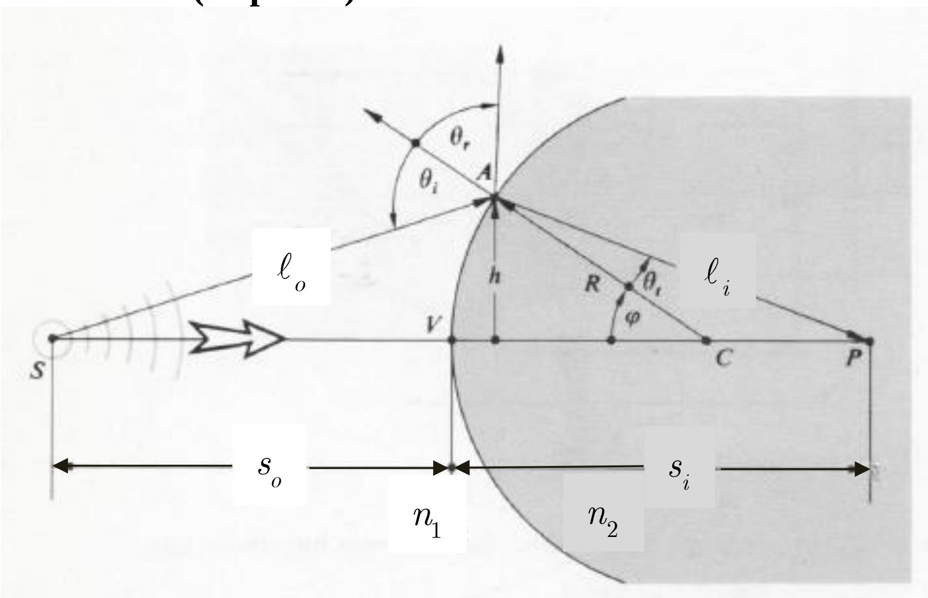
$$\rho = \frac{m_e}{ne^2\tau_c}$$

where m_e is the mass of an electron, and e is the charge of an electron.

(b) A trivalent metal having a mass density ρ_M is formed into a wire of length L and cross-sectional area A. Find the current flowing through the wire when a potential of V_0 volts is applied to its ends (the atomic mass is M for each metal atom).

Write your solution for the current in terms of the parameters given above as well as with any other important physical constants. Finally, put in the numbers in your formula for aluminum (but do not calculate) so that one can calculate the current (when given a calculator). For aluminum: $M = 27\,\mathrm{amu}$, density of aluminum is 2.7 g/cm^3, $\tau_c = 4\times10^{-14}\,\mathrm{s}$, $A = 2\,\mathrm{mm}^2$, $L = 20\,\mathrm{m}$, and $V_0 = 3\,\mathrm{V}$. [20 points]

Problem 5: (40 points)



In the figure above, a homogeneous region of refractive index n_1 is to the left of a spherical surface of radius R, and a homogeneous region of refractive index n_2 is to the right of this surface. Suppose that point P is the focus of the light emitted from point S.

(a) Why is the following statement true? [10 points]

$$\ell_o n_1 + \ell_i n_2 = s_o n_1 + s_i n_2$$

(b) Using the approximation for paraxial rays: $h << s_o, s_i, R$, show that [30 points]

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}.$$

Ph. D. Qualifying Exam Jan. 2020

Modern and Statistical Physics

You can do all the problems—the best 3 count towards your total score

Problem 1: (40 points)

Classical Thermodynamics

During a first-order phase transition:

(a) Show that the change of entropy of the system undergoing the transition is a linear function of the volume change. [20 points]

(b) Show that the change of internal energy is given by $\Delta U = L\left(1 - \frac{d(lnT)}{d(lnP)}\right)$, where L is the latent heat of transformation. T and P are temperature and pressure, respectively. [20 points]

Problem 2: (40 points)

Statistical mechanics

(a) Verify that the average energy per fermion is $(3/5)\varepsilon_F$ at absolute zero by making a direct calculation of $U(0)/N$. [20 points]

(b) Prove that the average speed of a fermion gas particle at $T = 0$ is $(3/4)v_F$, where the Fermi velocity v_F is defined by $\varepsilon_F = (1/2)mv_F^2$. [20 points]

Problem 3: (40 points)

A Compton scattering $(\gamma e^- \to \gamma e^-)$ experiment is performed in such a way that only those events are recorded in which the scattered photon and electron emerge perpendicular to each other. (This condition applies to all three parts below.) The initial photon energy is E and the initial electron is stationary.

(a) What is the momentum of the final-state photon? Give both its magnitude and its angle with respect to the initial photon direction. [20 points]

(b) For events recorded in the experiment, what is the threshold energy E_{min} for the initial state photon? [8 points]

(c) What is E for events recorded in the experiment, if in the center-of-momentum frame the final photon momentum is perpendicular to the initial photon momentum? [12 points]

Problem 4: (40 points)

(a) A *laser beam* having a diameter D in air strikes a piece of glass of index of refraction n at an angle θ. What is the diameter of the beam in the glass? [20 points]

(b) An exceedingly narrow *beam of white light* is incident at an angle θ on a sheet of glass of thickness L in air. The index of refraction for red light is n_r and for violet light it is n_v. Determine the approximate diameter of the emerging beam. [20 points]

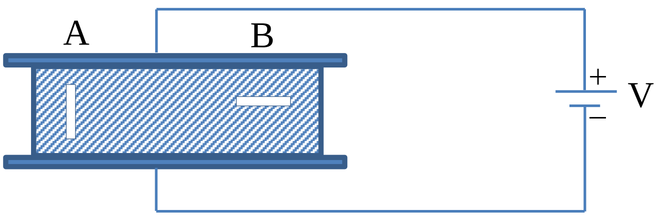
Problem 5: (40 points)

The following questions are to be answered by a single number (order of magnitude) or by brief explanations. [5 points each]

(a) What is the ionization potential of the hydrogen atom (in eV)?

(b) What is the electrostatic energy (in eV) of two electronic charges spaced 1Å apart?

(c) A capacitor has two cracks (A & B) in the dielectric as shown below. At which of the two cracks will breakdown occur first as the voltage is increased? Explain.



(d) Why is the sky blue and the sunset red?

(e) What is the ratio of the magnetic moment of the electron and the proton?

(f) Are most substances (e.g. water, benzene) paramagnetic or diamagnetic? Explain.

(g) Briefly describe how a laser works.

(h) Briefly describe how a bubble chamber works.

Ph. D. Qualifying Exam Aug. 2019

Modern and Statistical Physics

You can do all the problems—the best 3 count towards your total score

Problem 1: (40 points)

Ideal Gas of Relativistic Particles

We consider and ideal gas of N non-interacting relativistic particles in a volume $V = L^3$ in 3 dimensions. These particles follow Boltzmann statistics. The energy of a particle i is proportional to its momentum, i.e., $\varepsilon_i = c|\boldsymbol{p}_i|$.

(a) Calculate the canonical partition function $Z(T,V)$. [18 points]

(b) Calculate the internal energy E and the heat capacity C of the gas. [10 points]

(c) Calculate the Boltzmann entropy $S_B(E,N)$. [12 points]

Hint: $\int_0^\infty x^2 e^{-x} dx = 2$. Use the definition for the relativistic de Broglie wavelength, $\tilde{\lambda}_\beta \equiv \dfrac{hc}{k_B T}$.

Problem 2: (40 points)

Classical Thermodynamics

(a) In the big bang theory of the universe, the radiation energy initially confined in a small region adiabatically expands in a spherically symmetric manner. The expansion can be treated as a quasi-static process. The radiation cools down as it expands. Derive a relation between the temperature T and the radius R of the spherical volume of radiation, based purely on thermodynamic considerations. Here the radiation pressure is expressed as $p = U/3V$, and the black body radiation energy density is $u = U/V = aT^4$. [20 points]

(b) Find the total entropy of a photon gas as a function of its temperature T, volume V, and the constants k_B, h, c. [20 points]

Problem 3: (40 points)

Consider the following reaction from the collision of a proton and an antiproton in empty space: $p + \overline{p} \rightarrow p + p + \overline{p} + \overline{p}$. Find the threshold (minimum) speed of the initial state antiproton for the reaction to proceed, if:

(a) the initial state proton has the same energy with velocity in the opposite direction, [15 points]

(b) the initial state proton is at rest, [13 points]

(c) the initial state proton has the same speed but the collision occurs at a right angle. [12 points]

Problem 4: (40 points)

The acceleration due to gravity on the surface of Mercury is g_M (3.5 m/sec^2). The radius of Mercury is R_M. Suppose that the atmosphere of Mercury were pure H_2 gas.

(a) What would the temperature be so that the rms speed of the H_2 molecules of mass m matched the escape speed? (express this in terms of the parameters given in this problem and any other universal physical constants). [25 points]

(b) Qualitatively, what is the effect on the temperature of the remaining gas? Explain your reasoning. [5 points]

(c) Would there be a similar effect if the actual temperature was less than the result in (a)? Explain your reasoning. [5 points]

(d) If Mercury's atmosphere had two or more components of differing molecular mass, what would happen to the composition of the atmosphere as a function of time? Explain your reasoning. [5 points]

Problem 5: (40 points)

Commencement ceremonies are displayed on huge LED-based television displays. Estimate how close you can get to these displays and still see an image and not the individual LEDs. You may assume Fraunhofer diffraction occurs resulting in simple Airy disk patterns. Use the following information:

Aperture range for human eye (or pupil):	$D = 2 - 8 \times 10^{-3}$ m
Wavelength range for visible light:	$\lambda = 400 - 700 \times 10^{-9}$ m
Half pitch of LED displays:	$h = 3 - 8 \times 10^{-3}$ m
Minimum distance for diffraction limited seeing:	$L = ???$ m

Find the lower and upper bound values for L (make a rough estimate).

Ph. D. Qualifying Exam Jan. 2019

Modern and Statistical Physics

You can do all the problems—the best 3 count towards your total score

Problem 1: (40 points)

Non-Relativistic Fermi-Dirac Gases

(a) What is the definition of the Fermi energy ε_F? Express the Fermi-Dirac distribution function in the continuum approximation. (*note*: this problem concerns a *non-relativistic* Fermi gas) [5 points]

(b) What is the relationship between the internal energy, the number of fermions, $N(\varepsilon)d\varepsilon$, in the single particle energy range ε to $\varepsilon + d\varepsilon$, the Fermi distribution function $f(\varepsilon)$, and the degeneracy function $g(\varepsilon)$? [5 points]

(c) Verify that the average energy per fermion is $\frac{3}{5}\varepsilon_F$ at absolute zero $(T = 0\text{K})$. [15 points]
Hint: Make a direct calculation of *U*(0)/N, where *U*(0) is the internal energy at $T = 0\text{K}$.

(d) Prove that the average speed of a fermion gas particle at $T = 0\text{K}$ is $\frac{3}{4}v_F$. The Fermi velocity v_F is defined by $\varepsilon_F = \frac{1}{2}mv_F^2$. [15 points]

Problem 2: (40 points)

Entropy of the van der Waals Gas

(a) Write the combined first and second law (including the entropy S and the configuration work) for a closed system of thermodynamics. Use intensive variables, the specific entropy s, and the specific volume v. [5 points]

(b) Derive the following Tds equation under the isochoric (constant volume) condition,

$$Tds = c_v dT + T\left(\frac{\partial P}{\partial T}\right)_v dv \qquad s = s(T, v)$$

Here c_v is the specific heat under the isochoric condition. [10 points]

(c) Show $Tds = c_v dT + \frac{T\beta}{\kappa} dv$ from the result above. Here β is the expansivity under the isobaric condition, and κ is the compressibility under the isothermal condition. [10 points]

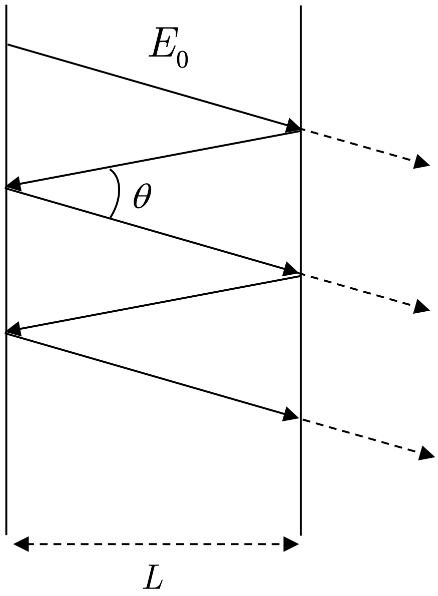
(d) Calculate the entropy for a van der Waals gas using the above Tds equation. The equation of state of the van der Waals gas is

$$\left(P + \frac{a}{v^2}\right)(v - b) = RT$$

Here P is the pressure, v the specific volume, T is the absolute temperature, R is the gas constant, and the constant b is another characteristic constant of the gas. [15 points]

Problem 3: (40 points)

Consider a laser cavity with mirrors of amplitude reflectivity r and reflection coefficient $r^2 = R \sim 0.99$ and a length L between the mirrors. The laser beam continuously reflects between the mirror ends. The high reflectivity permits the beam in opposite directions to be considered of approximately equal amplitudes so that standing waves are generated which constitute the longitudinal modes of the cavity.



Angle θ exaggerated for clarity: $\theta \sim 0$

(a) What is the phase change, ϕ, during one round trip of the laser cavity? Express your answer in terms of L and ν, the radiation frequency. [5 points]

(b) For what mode frequencies ν can standing waves be maintained in the cavity? (that is, find all the possible frequencies ν that can exist inside the cavity). Express your answer in terms of L. [10 points]

(c) Find an expression for the intensity of radiation transmitted out the far end of the laser cavity in terms of ϕ, E_0, R, and t, where t is the amplitude transmission. [20 points]

(d) As the frequency changes from the standing wave value, the intensity out of the end of the cavity decreases. Use your results from (a) and (c) to determine the shift in frequency over which the intensity drops to half its maximum value. [5 points]

Problem 4: (40 points)

(a) A negative muon can stop in Aluminum and be captured in a muonic atom. What is the binding energy of the muon in the 1*s* state? [15 points]

Hint: use the Bohr model of the atom with
E(1*s*) for Hydrogen = -13.6 eV
Z(Al)=13
$m_e = 0.511$ MeV/c^2
$m_\mu = 105.7$ MeV/ c^2
$\frac{1}{2} m_e c^2 \alpha^2 = 13.6$ eV

The muon can then interact with the Al nucleus and, with possible new physics mechanisms, produce the following reaction which violates lepton number. But if this reaction occurs, it may help to understand why neutrinos have mass.

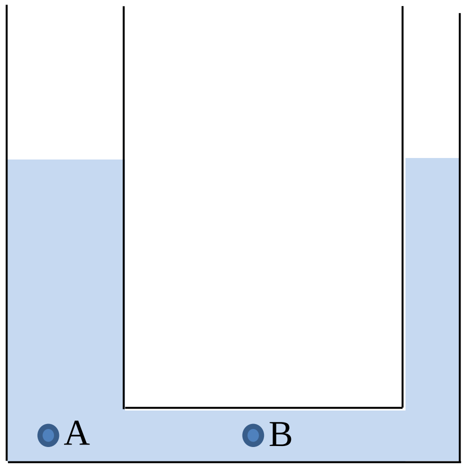
$$\mu^- + \mathrm{Al}^{27} \rightarrow e^+ + \mathrm{Na}^{27} \qquad \mathrm{Z(Na)} = 11$$

(b) What is the total kinetic energy of the electron+sodium? [10 points]

(c) What is the total energy of the emitted electron? [15 points]

Note:
mass $\mathrm{Al}^{27} = 26.9815$ amu
mass $\mathrm{Na}^{27} = 26.9941$ amu
mass electron = 0.00055 amu
1 amu = 931.5 MeV/ c^2

Hint: do not forget to keep track of the number of electrons and the muon binding energy.



Problem 5: (40 points)

A freshman comes to you with a puzzling problem regarding the pressure in a U-tube containing water. He heard that the pressure at a point in a fluid is due to the weight of the fluid above it. The student argues that the pressure at point A and point B must be different because they have different amounts of fluid above them. Provide a simple argument that explains why the student's reasoning is correct or incorrect.