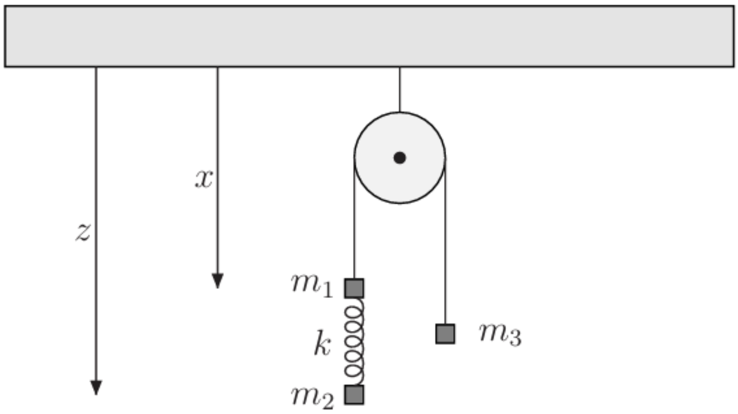
Classical Mechanics

You may solve all four problems, if you choose. Only the three best problem scores count towards your total score. Total points on each problem = 40. Total possible score = 120.

Problem 1 [40 points]:

Three masses m_1, m_2, and m_3 are suspended as shown in a uniform gravitational field with acceleration g. Masses m_1 and m_3 are connected by an unstretchable string which moves over a massless pulley whose center does not move. Mass m_2 is suspended from mass m_1 by a spring with constant k and unstretched length L. Call x and z the heights of masses m_1 and m_2, respectively, as shown.



(a) [**10 points**] Find the Lagrangian of the system, using x and $y = z - x - L$ as your configuration variables.

(b) [**10 points**] Find the equations of motion of the system.

(c) [**10 points**] Solve for the relative motion of the masses m_1 and m_2, and in particular find the angular frequency of oscillation.

(d) [**5 points**] What is the oscillation frequency if $m_1 \gg m_2, m_3$? Show using your answer above and/or physical arguments.

(e) [**5 points**] What is the oscillation frequency if $m_2 \gg m_1, m_3$? Show using your answer above and/or physical arguments.

Problem 2 [40 points]:

Consider the motion of a particle of mass m and initial angular momentum ℓ in the central 3-dimensional potential corresponding to a "spring" with a non-zero relaxed length a:

$$V(r) = \frac{1}{2}k(r-a)^2,$$

where r is the distance of the particle to the origin.

(a) **[16 points]** Find an algebraic relation between the radius r_0 of circular orbits and the given quantities k, ℓ, m, a. (It is not necessary to solve for r_0.)

(b) **[16 points]** For nearly circular orbits, find the angular frequency of small oscillations of r about r_0, with ℓ eliminated from your answer.

(c) **[8 points]** Use your results to find the period of time between successive radial maxima in nearly circular orbits for $a/r_0 \ll 1$, expressing your answer to first order in an expansion in a/r_0. Show that orbits are closed in the limit $a/r_0 \to 0$, by comparing to the orbital period for exactly circular orbits.

Problem 3 [40 points]:

(a) [**10 points**] Show that the moment of inertia of a thin, uniform rod about its center of mass is $ml^2/12$, where l is its total length and m is its total mass.

(b) [**15 points**] A long thin tube of negligible mass is pivoted so that it may rotate without friction in a horizontal plane. A thin rod of mass m and length l slides without friction inside the tube. Choose suitable coordinates and write down Lagrange's equations of motion for this system.

(c) [**15 points**] Initially the rod is centered over the pivot and the tube is rotating with angular velocity ω_0. Show that the rod is unstable in this position. Describe its subsequent motion if it is disturbed slightly. What are the radial and angular velocities of the rod after a long time? (Assume that the tube is long enough for the rod to remain inside.)

Problem 4 [40 points]:

A non-relativistic particle of mass m moves in one dimension x under the force

$$F(x) = ax^3 - bx,$$

where a and b are positive constants.

(a) [**6 points**] Why does a potential energy function exist for this force? Is the total energy conserved?

(b) [**6 points**] Find an expression for the potential energy $V(x)$, and sketch its graph. Choose the arbitrary constant so that the potential energy vanishes at $x = 0$.

(c) [**8 points**] Find all of the turning points of the motion if the total energy is E.

(d) [**8 points**] For what values of the total energy E is the motion bounded?

(e) [**12 points**] For positive values of E less than that found in part (d), find the frequency for periodic oscillations. Do not assume that the oscillations are small. You may leave your answer in terms of a definite integral.

NIU Physics PhD Candidacy Exam – Fall 2025 – Classical Mechanics

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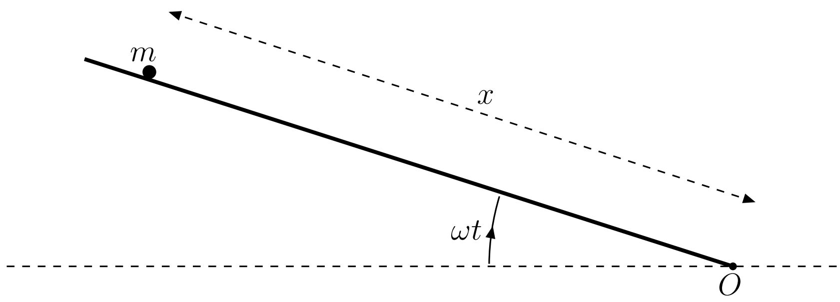
Problem 1. A small mass m is sitting on a frictionless flat board resting on a horizontal table. The board pivots about one fixed edge O, so that starting at time $t = 0$, the left edge of the board is raised. The angle that the board makes with the horizontal increases with constant angular velocity ω. The mass slides towards the fixed downhill end, under the influence of a uniform gravitational field of acceleration g.



(a) Using coordinate x, the distance of the mass from the fixed downhill edge, find the Lagrangian for the mass. [10 points]
(b) Find the equation of motion, and the general solution. [15 points]
(c) Find the particular solution of the equation of motion, assuming that the mass started from rest and was at position $x = x_0$ at time $t = 0$. [10 points]
(d) Assume that ω is small but non-zero. Find a simplified form for the $x(t)$ found in the previous part up to terms second-order in ω. [5 points]

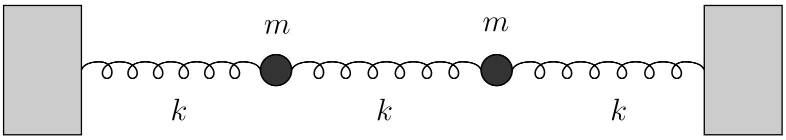
<u>Problem 2.</u> A very small comet with mass m is discovered approaching the Sun (with mass $M \gg m$) along an orbit that is a perfect parabola. At the present time the comet is noted to be moving with speed v, and has an impact parameter distance b with respect to the Sun. For parts (b) and (c) below, you may leave your answer in terms of definite integrals, if you choose.

(a) Find the distance of closest approach of the comet to the Sun, and find its speed when it reaches closest approach. [16 points]

(b) How long does it take the comet to again reach its present distance from the Sun? [14 points]

(c) When the comet again reaches its present distance from the Sun, what angle (as measured from the Sun) has it swept out on its orbit? [10 points]

Problem 3. Consider a system of two masses m and three identical springs with spring constant k between two stationary walls as shown. At equilibrium, the lengths of the springs are a, and if they were unstretched their lengths would be b. Consider only longitudinal motions (along the axis of the springs).

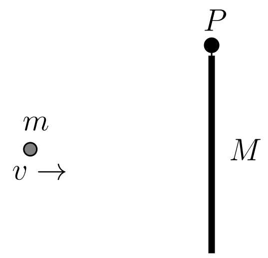


(a) Find the Lagrangian and Lagrange's equations for the system. [14 points]

(b) Find the normal-mode frequencies of vibration and the eigenvectors. [13 points]

(c) Suppose that at time $t = 0$ the mass on the left is displaced from equilibrium by a distance X to the right, while the mass on the right is not displaced, and both masses are at rest. Compute the motion of the left mass for $t > 0$. [13 points]

Problem 4. A thin rod of length L and mass M is suspended in a uniform gravitational field of acceleration g from a fixed frictionless pivot point P. The rod is struck midway along its length by a small lump of clay with mass m moving horizontally at a high speed v. The lump sticks to the middle of the rod after the collision.



(a) Find the angular velocity ω_0 of the rod immediately after the collision. [Hint: at the moment of the collision, the total energy of the rod and the clay lump is not conserved, but there is another conserved quantity.] [14 points]

(b) Find an expression for the minimum speed $v = v_{\text{crit}}$ that will result in the rod making a complete circle around the pivot point. [13 points]

(c) For a given $v > v_{\text{crit}}$, find an expression for the time needed for for the rod to make the complete circle around the pivot point. You should leave your answer in terms of a definite integral. [13 points]

NIU Physics PhD Candidacy Exam – Spring 2025 – Classical Mechanics

You may solve all four problems, if you choose. Only the three best problem scores count towards your total score. Total points on each problem = 40. Total possible score = 120.

Problem 1. A small planet of mass m orbits a star. The attractive force on the planet is in the radial direction, and is given by

$$F_r = -\frac{k}{r^2} - ar,$$

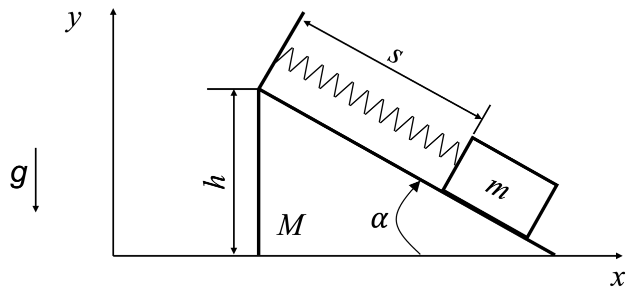
where k and a are positive constants and $r(t)$ is the radial distance between the planet and the star. (It is not directly relevant to the rest of the problem, but this is the force law that would result if the star and planet are immersed in a uniformly dense spherical cloud of dark matter particles that only interact gravitationally.) Assume that the extra force is very small (so that $a \ll k/r^3$). Consider nearly circular orbits with average radius r_0. All of your answers should be expressed in terms of only the quantities r_0, m, k, and a, and you should work to first order in a.

(a) Find the angular frequency of small radial oscillations of r about the mean r_0. [20 points]

(b) Find the average angular velocity of the planet's orbit about the star. [12 points]

(c) From the results of the previous two parts, find the rate of precession of the perihelion of the orbit. [8 points]

Problem 2. A block of mass m is attached to a wedge of mass M by a spring with spring constant k. The inclined frictionless surface of the wedge makes an angle α to the horizontal. The wedge is free to slide on a horizontal frictionless surface, as shown in the figure. There is a uniform downward gravitational field with acceleration g.

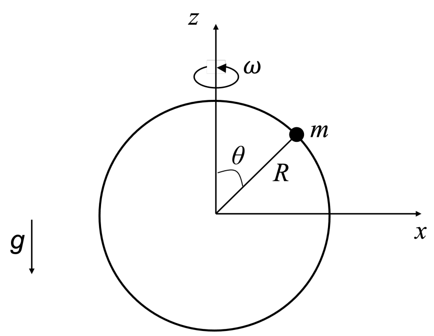


(a) Given that the relaxed length of the spring alone (if the block were not attached) is d, find the value s_0 when both the block and the wedge are at rest. [10 points]

(b) Find the Lagrangian for the system as a function of the x coordinate of the wedge and the length of the spring s. Write the equations of motion. [15 points]

(c) What is the natural frequency of vibration? [15 points]

Problem 3. A bead of mass m is constrained to move on a circular hoop of radius R. The hoop is forced to rotate with constant angular velocity ω around a vertical axis, which coincides with a diameter of the hoop. There is a uniform downward gravitational field with acceleration g.



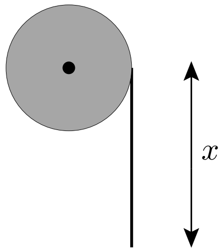
(a) Set up the Lagrangian and obtain the equations of motion for the bead. [14 points]

(b) Find the critical angular velocity Ω, such that for $\omega < \Omega$ the bottom of the hoop provides a stable equilibrium position for the bead. [14 points]

(c) Find the stable equilibrium position for $\omega > \Omega$. [12 points]

In parts (b) and (c), you need to prove the stability of the respective positions.

<u>Problem 4.</u> A very long, perfectly flexible material is rolled up on a fixed, horizontal, massless, very thin axle. The rolled-up portion rotates freely on the axle. The material has a (small) thickness s, width w, and, when completely unrolled, length ℓ. The mass density per unit volume of the material is ρ. At time $t = 0$, a length x_0 hangs from the roll and the system is at rest. For $t > 0$, the material unrolls under the influence of a constant gravitational field g (downward). [Useful information: the moment of inertia of a solid cylinder of mass M and radius R about its axis of symmetry is $I = MR^2/2$.]



(a) Using the length x of the hanging part of the material as your configuration variable, find the Lagrangian of the system. [13 points]
(b) Find the canonical momentum conjugate to x, and the Hamiltonian. [9 points]
(c) What is the velocity of the free end of the material at the time when half of it is off the roll ($x = \ell/2$)? [18 points]

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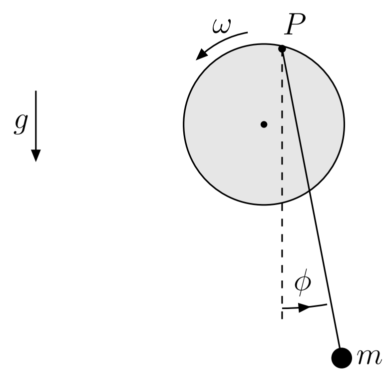
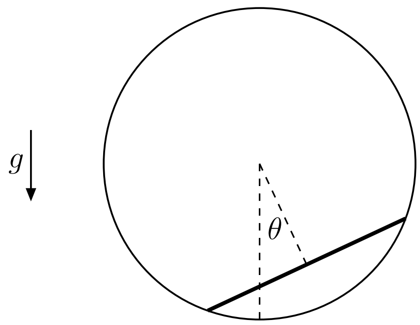
Problem 1. Consider a circular disc of radius R as shown above. The disc is forced to rotate counterclockwise about its center at a constant angular velocity ω. A pendulum with mass m and length L hangs from a point P on the edge of the disc. There is a uniform gravitation field with acceleration g pointing down. At time $t = 0$, the point P is at its highest point. The following identities might be useful: $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and $\cos(A + B) = \cos A \cos B - \sin A \sin B$.



(a) Find the position and velocity of the point P as a function of time. [4 points]
(b) Find the Lagrangian for the system in terms of the configuration variable ϕ, which is the angle made by the pendulum with the vertical. Simplify the result so that each term has at most one trigonometric function. [14 points]
(c) Find the equation of motion for ϕ. [12 points]
(d) Find the total (net) force vector on the mass at time $t = 0$, in terms of the angle (ϕ_0) and its time derivative ($\dot{\phi}_0$) at that moment. [10 points]

Problem 2. A thin straight rod of length 2ℓ and mass m is constrained to move with its ends on a fixed circle of radius R. The circle is in a vertical plane, as shown, in a uniform gravitational field with acceleration g pointing down. The contacts between the circle and the rod are frictionless. The longer dashed line is vertical and has length R, and the shorter dashed line goes between the center of the circle and the midpoint of the sliding rod. Together they define the angle θ.



(a) Find the Lagrangian and derive the equation of motion for the rod. Use as your configuration variable the angle θ shown. [16 points]

(b) Find the angular frequency of small oscillations. [9 points]

(c) Suppose the rod is at rest with its center at the highest possible point ($\theta = \pi$), and then is very slightly perturbed. Find the velocity of its center when it reaches its lowest possible point ($\theta = 0$). [15 points]

Problem 3. A point particle of mass m moves subject to a 3-dimensional central potential:

$$V(r) = -\frac{k}{r^n}$$

where k and n are positive constants.

(a) If the particle has angular momentum L, what is the radius R for which the orbit is circular? [10 points]

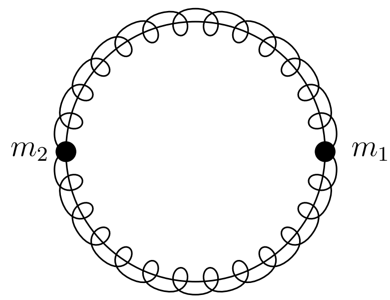
(b) Suppose the motion is close to the circular orbit mentioned in the previous part. Writing $r(t) = R + \delta r(t)$ and assuming that $\delta r(t)$ is small, find an equation of motion for $\delta r(t)$. Write your equation in a form that does not involve the angular momentum L. [15 points]

(c) Solve this equation for δr. For what values of n are the circular orbits stable? [15 points]

Problem 4. Two beads, with unequal masses m_1 and m_2, are constrained to slide frictionlessly on a stationary circular hoop of radius R. The beads are connected as shown by two identical springs with spring constant k and unstretched length d. There is no gravity.

(a) Find the normal modes of the system and their angular frequencies. [25 points]

(b) Now suppose that at time $t = 0$ the beads are on opposite sides of the hoop, with bead m_1 having an instantaneous speed v and bead m_2 at rest. Solve for the subsequent motion of the first bead. [15 points]



NIU Physics PhD Candidacy Exam – Spring 2024 – Classical Mechanics

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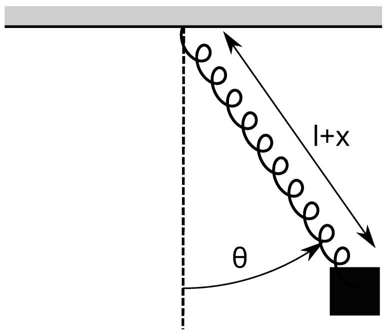
Problem 1. A spring pendulum consists of a mass m attached to one end of a massless spring with spring constant k. The other end of the spring is tied to a fixed support. When no weight is on the spring, its length is l. Assume that the motion of the system is confined to a vertical plane (that is, the plane of this page). The acceleration due to gravity is g. The spring can stretch but cannot bend. Let the displacements from equilibrium be denoted by the angle θ and the distance x, as shown.



(a) Obtain the Lagrangian for this system. [10 points]
(b) Obtain the equations of motion. [10 points]
(c) Solve the equations of motion for the approximation of a small stretching and small angular displacement. [10 points]
(d) Find the Hamiltonian of the system and Hamilton's equations. Is the system conservative? [10 points]

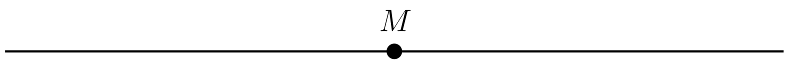
Problem 2. Express your answers for the questions below in terms of Newton's gravitational constant G, the masses, and the distance L. Assume that the stars move on circular orbits.

(a) What is the rotational period T of an equal-mass ($M_1 = M_2 = M$) double star of separation L? [10 points]

(b) What is the rotational period T of an unequal-mass ($M_1 \neq M_2$) double star of separation L? [15 points]

(c) What is the rotational period T of an equal-mass ($M_1 = M_2 = M_3 = M$) triple star, where the stars are located at the corners of an equilateral triangle (side L)? [15 points]

Problem 3. Consider two strings with the same mass density per unit length μ and the same tension T, joined at a point by a mass M, as shown. There is no gravity. Suppose that a transverse wave with angular frequency ω is incident from the left.

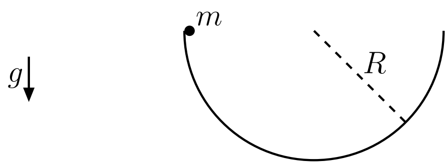


(a) Write down the form of the solutions on either side of the mass M, and find the boundary conditions that relate them. [14 points]

(b) Find the transmission and reflection coefficients. [20 points]

(c) What happens to the transmission coefficient in the three special cases $m = 0$, $\omega \to \infty$, and $\omega \to 0$? (You can get full credit for correct answers with short explanations here, even if your previous answers are incorrect.) [6 points]

Problem 4. A point particle of mass m starts from rest at the edge of a fixed hemispherical bowl of radius R, and slides down on the inside surface. The acceleration due to gravity is g. The coefficient of friction (the ratio of the tangential frictional force to the normal force) between the particle and the bowl surface is μ.



(a) Find an expression for the time needed for the particle to reach the bottom of the bowl. You may leave your answer for this part in terms of a definite integral. [25 points]

(b) Find the maximum height to which the particle rises on the other side of the bowl. How does your answer behave in the limits $\mu \to 0$ and $\mu \to 1$? [15 points]

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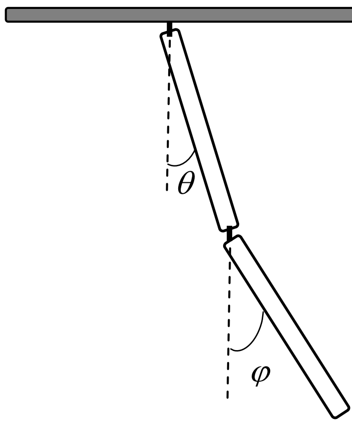
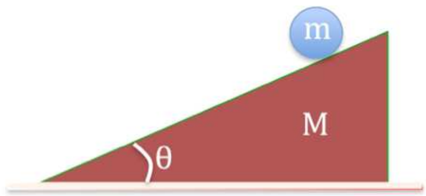
Problem 1. Two identical rods of mass m and length l are connected to the ceiling and together vertically by small flexible pieces of string. The acceleration due to gravity is g. The system then forms a physical double pendulum.

(a) Find the frequencies of the normal modes of this system for small oscillations around the equilibrium position. [25 points]

(b) Describe the motion of each of the normal modes. [15 points]



Problem 2. Consider a solid cylinder of mass m and radius r sliding without rolling down the smooth inclined face of a wedge of mass M and angle θ, as shown. The wedge is free to move on a horizontal plane without friction. The acceleration due to gravity is g.



(a) [16 points] How far has the wedge moved by the time the cylinder has descended from rest a vertical distance h?

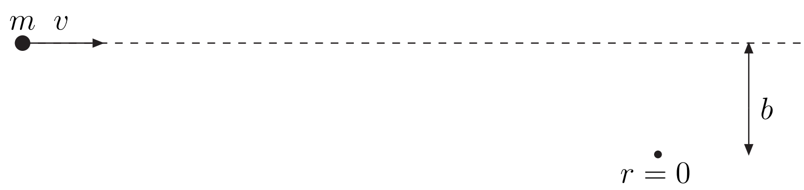
(b) [16 points] Now suppose that the cylinder is free to roll down the wedge without slipping. How far does the wedge move in this case?

(c) [8 points] In which case does the cylinder reach the bottom faster? How does this depend on the radius of the cylinder?

Problem 3. Consider a point particle of mass m moving in three dimensions in a central potential

$$V(r) = -\frac{g}{r} - \frac{k}{r^2}$$

where g and k are positive constants and r is the distance from the origin. The particle approaches from very far away with speed v and impact parameter b, as shown. (The dashed line is what the path would be if there were no potential.)



(a) Find r_{min}, the distance of closest approach of the particle to $r = 0$. Show that the particle will go through the origin if $k > k_{\text{crit}}$, where k_{crit} is a critical value that you will determine in terms of the other given quantities. [14 points]

In the remainder of this problem, you should assume $k < k_{\text{crit}}$, and you may leave your answers in terms of r_{min} and the other given quantities.

(b) What is the maximum speed reached by the particle on its trajectory? [8 points]
(c) What is the maximum acceleration reached by the particle on its trajectory? [8 points]
(d) When the particle is very far from the origin again, find the angle by which it has been scattered from its original direction. You may leave your answer in terms of a single definite integral. [10 points]

Problem 4. A smooth wire is bent into the form of a helix, whose equations in cylindrical coordinates (r, ϕ, z) are $r = a$ and $z = b\phi$, with a and b constants. A bead of mass m moves frictionlessly on the wire, and is attracted to the origin ($r = z = 0$) by a force whose magnitude is a constant k multiplied by the linear distance between the bead and the origin. There is no gravitational field.

(a) Find the Lagrangian equation of motion for z, in terms of only m, a, b, and k. [16 points]

(b) Find the Hamiltonian of the system in terms of z and its conjugate momentum, and Hamilton's equations of motion. [12 points]

(c) If the bead is released at rest from its position on the helix with $z = h$, what is its velocity when it reaches $z = 0$? [12 points]

NIU Physics PhD Candidacy Exam – Spring 2023 – Classical Mechanics

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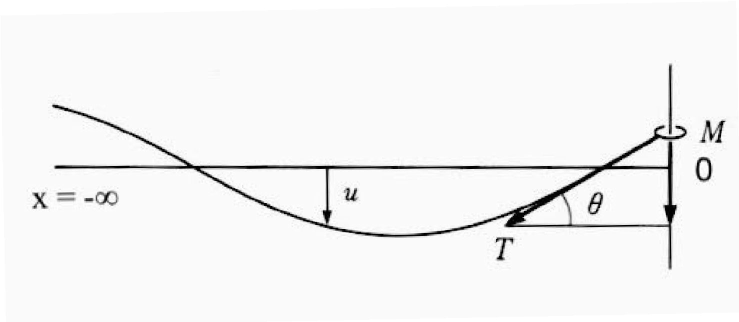
Problem 1. An infinitely long string with linear mass density σ and tension T is attached at one end to a ring with mass M. The ring slides frictionlessly on a vertical pole at $x = 0$ as shown. The diameters of the ring and the pole can be ignored. Gravity can also be ignored. Suppose a sinusoidal wave with transverse (vertical) displacement

$$u_I(x,t) = A\cos(kx - \omega t)$$

is incident from the far left ($x = -\infty$). Note that the constant phase angle of the incoming wave has been set to 0 for convenience. The displacement is defined so that $u = 0$ at equilibrium, and the angle θ as defined in the figure is assumed to remain small.



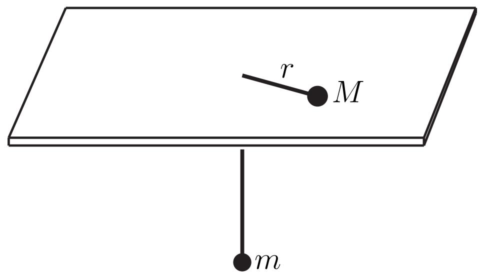
(a) Derive the relationship between the angular frequency ω, tension T, linear mass density σ, and wavevector k using the equation of motion. [10 points]
(b) Derive the reflected wave transverse displacement $u_R(x,t)$. In particular, find the relationship between the constant phase angle ϕ of the reflected wave, the mass of the ring M, and the angular frequency ω. [24 points]
(c) Discuss the dependences of $\cos\phi$ from part (b) on M for constant ω, including $M = 0$ and ∞, and its dependences on the ω of the incoming wave, including $\omega = 0$ and ∞. [6 points]

Problem 2. A particle of mass M is constrained to move on a smooth horizontal plane. A second particle of mass m is attached to it by hanging from a string passing through a hole in the plane as shown, and is constrained to move in a vertical line in a uniform gravitational field of acceleration g. All motion is frictionless and the string is massless.

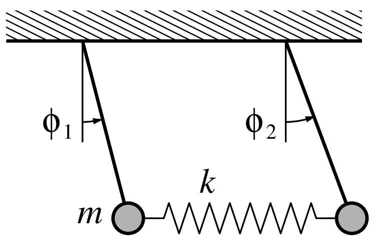
(a) Find the Lagrangian for the system and derive the equations of motion. [15 points]

(b) Consider solutions in which M moves in a circle with a constant speed v_0. Find the radius of the circle r_0 in terms of the other quantities. [12 points]

(c) Show that the solution in part (b) is stable and find the angular frequency of small oscillations about the stable circular orbit. [13 points]



Problem 3. Consider two identical pendulums, each of length l and mass m. The pendulums are constrained to move in a common plane, with positions specified by the angles ϕ_1 and ϕ_2. The pendulums are joined by a massless spring with force constant k. The natural length of the spring is equal to the distance between the two supports, so the equilibrium position is at $\phi_1 = \phi_2 = 0$ with the two pendulums vertical.



(a) Write down the total kinetic energy, the gravitational potential energy, and the spring potential energy. (Assume that both angles remain small at all times.) Write down the Lagrangian for this problem. [10 points]

(b) Write down the Lagrangian equations of motion. [10 points]

(c) Find the frequencies of the normal modes for these two coupled pendulums. [10 points]

(d) Find the normal coordinates for the normal modes. Briefly discuss the type of motion they describe. [10 points]

Problem 4. A small satellite with mass m is in a circular orbit of radius r around a planet of mass M. The planet's thin atmosphere results in a frictional force $F = Av^{\alpha}$ on the satellite, where v is the speed of the satellite and A and α are constants. It is observed that with the passage of time, the orbit of the satellite remains circular, but with a radius that decreases very slowly with time according to $dr/dt = -C$, where C is a constant independent of the orbit's radius and the speed. Assume that $m \ll M$.

(a) Show that α can only be a certain integer, and find that integer. [20 points]

(b) Solve for the constant A. Express your answer in terms of C, the masses m and M, and Newton's gravitational constant G. [20 points]