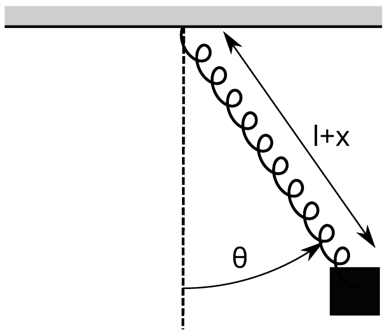
NIU Physics PhD Candidacy Exam – Spring 2024 – Classical Mechanics

You may solve all four problems, if you choose. Only the three best problem scores count towards your total score. Total points on each problem = 40. Total possible score = 120.

Problem 1. A spring pendulum consists of a mass m attached to one end of a massless spring with spring constant k. The other end of the spring is tied to a fixed support. When no weight is on the spring, its length is l. Assume that the motion of the system is confined to a vertical plane (that is, the plane of this page). The acceleration due to gravity is g. The spring can stretch but cannot bend. Let the displacements from equilibrium be denoted by the angle θ and the distance x, as shown.



(a) Obtain the Lagrangian for this system. [10 points]
(b) Obtain the equations of motion. [10 points]
(c) Solve the equations of motion for the approximation of a small stretching and small angular displacement. [10 points]
(d) Find the Hamiltonian of the system and Hamilton's equations. Is the system conservative? [10 points]

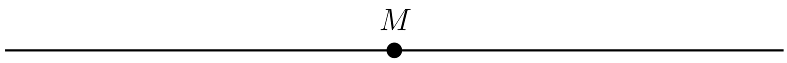
Problem 2. Express your answers for the questions below in terms of Newton's gravitational constant G, the masses, and the distance L. Assume that the stars move on circular orbits.

(a) What is the rotational period T of an equal-mass ($M_1 = M_2 = M$) double star of separation L? [10 points]

(b) What is the rotational period T of an unequal-mass ($M_1 \neq M_2$) double star of separation L? [15 points]

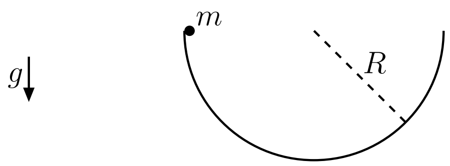
(c) What is the rotational period T of an equal-mass ($M_1 = M_2 = M_3 = M$) triple star, where the stars are located at the corners of an equilateral triangle (side L)? [15 points]

Problem 3. Consider two strings with the same mass density per unit length μ and the same tension T, joined at a point by a mass M, as shown. There is no gravity. Suppose that a transverse wave with angular frequency ω is incident from the left.



(a) Write down the form of the solutions on either side of the mass M, and find the boundary conditions that relate them. [14 points]
(b) Find the transmission and reflection coefficients. [20 points]
(c) What happens to the transmission coefficient in the three special cases $m = 0$, $\omega \to \infty$, and $\omega \to 0$? (You can get full credit for correct answers with short explanations here, even if your previous answers are incorrect.) [6 points]

Problem 4. A point particle of mass m starts from rest at the edge of a fixed hemispherical bowl of radius R, and slides down on the inside surface. The acceleration due to gravity is g. The coefficient of friction (the ratio of the tangential frictional force to the normal force) between the particle and the bowl surface is μ.



(a) Find an expression for the time needed for the particle to reach the bottom of the bowl. You may leave your answer for this part in terms of a definite integral. [25 points]

(b) Find the maximum height to which the particle rises on the other side of the bowl. How does your answer behave in the limits $\mu \to 0$ and $\mu \to 1$? [15 points]

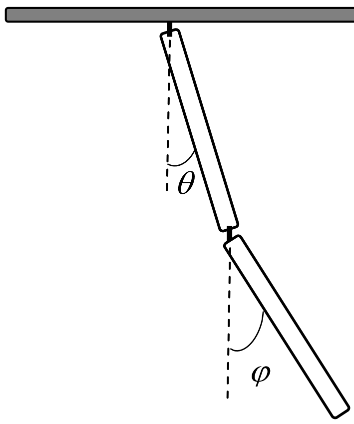
NIU Physics PhD Candidacy Exam – Fall 2023 – Classical Mechanics

You may solve all four problems, if you choose. Only the three best problem scores count towards your total score. Total points on each problem = 40. Total possible score = 120.

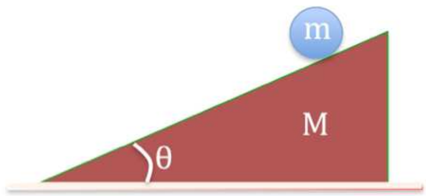
Problem 1. Two identical rods of mass m and length l are connected to the ceiling and together vertically by small flexible pieces of string. The acceleration due to gravity is g. The system then forms a physical double pendulum.

(a) Find the frequencies of the normal modes of this system for small oscillations around the equilibrium position. [25 points]

(b) Describe the motion of each of the normal modes. [15 points]



Problem 2. Consider a solid cylinder of mass m and radius r sliding without rolling down the smooth inclined face of a wedge of mass M and angle θ, as shown. The wedge is free to move on a horizontal plane without friction. The acceleration due to gravity is g.



(a) [16 points] How far has the wedge moved by the time the cylinder has descended from rest a vertical distance h?

(b) [16 points] Now suppose that the cylinder is free to roll down the wedge without slipping. How far does the wedge move in this case?

(c) [8 points] In which case does the cylinder reach the bottom faster? How does this depend on the radius of the cylinder?

Problem 3. Consider a point particle of mass m moving in three dimensions in a central potential

$$V(r) = -\frac{g}{r} - \frac{k}{r^2}$$

where g and k are positive constants and r is the distance from the origin. The particle approaches from very far away with speed v and impact parameter b, as shown. (The dashed line is what the path would be if there were no potential.)

(a) Find $r_{\rm min}$, the distance of closest approach of the particle to $r = 0$. Show that the particle will go through the origin if $k > k_{\rm crit}$, where $k_{\rm crit}$ is a critical value that you will determine in terms of the other given quantities. [14 points]

In the remainder of this problem, you should assume $k < k_{\rm crit}$, and you may leave your answers in terms of $r_{\rm min}$ and the other given quantities.

(b) What is the maximum speed reached by the particle on its trajectory? [8 points]
(c) What is the maximum acceleration reached by the particle on its trajectory? [8 points]
(d) When the particle is very far from the origin again, find the angle by which it has been scattered from its original direction. You may leave your answer in terms of a single definite integral. [10 points]

Problem 4. A smooth wire is bent into the form of a helix, whose equations in cylindrical coordinates (r, ϕ, z) are $r = a$ and $z = b\phi$, with a and b constants. A bead of mass m moves frictionlessly on the wire, and is attracted to the origin ($r = z = 0$) by a force whose magnitude is a constant k multiplied by the linear distance between the bead and the origin. There is no gravitational field.

(a) Find the Lagrangian equation of motion for z, in terms of only m, a, b, and k. [16 points]

(b) Find the Hamiltonian of the system in terms of z and its conjugate momentum, and Hamilton's equations of motion. [12 points]

(c) If the bead is released at rest from its position on the helix with $z = h$, what is its velocity when it reaches $z = 0$? [12 points]

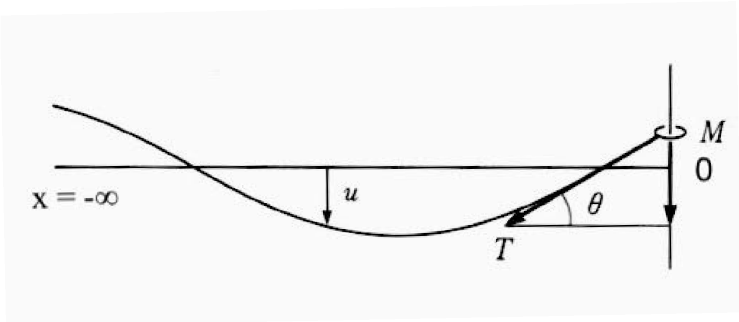
NIU Physics PhD Candidacy Exam – Spring 2023 – Classical Mechanics

You may solve all four problems, if you choose. Only the three best problem scores count towards your total score. Total points on each problem = 40. Total possible score = 120.

Problem 1. An infinitely long string with linear mass density σ and tension T is attached at one end to a ring with mass M. The ring slides frictionlessly on a vertical pole at $x = 0$ as shown. The diameters of the ring and the pole can be ignored. Gravity can also be ignored. Suppose a sinusoidal wave with transverse (vertical) displacement

$$u_I(x, t) = A\cos(kx - \omega t)$$

is incident from the far left ($x = -\infty$). Note that the constant phase angle of the incoming wave has been set to 0 for convenience. The displacement is defined so that $u = 0$ at equilibrium, and the angle θ as defined in the figure is assumed to remain small.



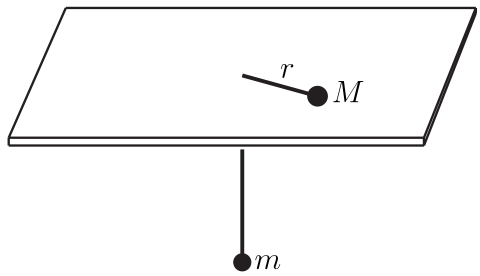
(a) Derive the relationship between the angular frequency ω, tension T, linear mass density σ, and wavevector k using the equation of motion. [10 points]
(b) Derive the reflected wave transverse displacement $u_R(x, t)$. In particular, find the relationship between the constant phase angle ϕ of the reflected wave, the mass of the ring M, and the angular frequency ω. [24 points]
(c) Discuss the dependences of $\cos\phi$ from part (b) on M for constant ω, including $M = 0$ and ∞, and its dependences on the ω of the incoming wave, including $\omega = 0$ and ∞. [6 points]

Problem 2. A particle of mass M is constrained to move on a smooth horizontal plane. A second particle of mass m is attached to it by hanging from a string passing through a hole in the plane as shown, and is constrained to move in a vertical line in a uniform gravitational field of acceleration g. All motion is frictionless and the string is massless.

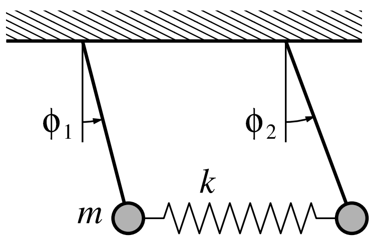
(a) Find the Lagrangian for the system and derive the equations of motion. [15 points]

(b) Consider solutions in which M moves in a circle with a constant speed v_0. Find the radius of the circle r_0 in terms of the other quantities. [12 points]

(c) Show that the solution in part (b) is stable and find the angular frequency of small oscillations about the stable circular orbit. [13 points]



Problem 3. Consider two identical pendulums, each of length l and mass m. The pendulums are constrained to move in a common plane, with positions specified by the angles ϕ_1 and ϕ_2. The pendulums are joined by a massless spring with force constant k. The natural length of the spring is equal to the distance between the two supports, so the equilibrium position is at $\phi_1 = \phi_2 = 0$ with the two pendulums vertical.



(a) Write down the total kinetic energy, the gravitational potential energy, and the spring potential energy. (Assume that both angles remain small at all times.) Write down the Lagrangian for this problem. [10 points]

(b) Write down the Lagrangian equations of motion. [10 points]

(c) Find the frequencies of the normal modes for these two coupled pendulums. [10 points]

(d) Find the normal coordinates for the normal modes. Briefly discuss the type of motion they describe. [10 points]

Problem 4. A small satellite with mass m is in a circular orbit of radius r around a planet of mass M. The planet's thin atmosphere results in a frictional force $F = Av^{\alpha}$ on the satellite, where v is the speed of the satellite and A and α are constants. It is observed that with the passage of time, the orbit of the satellite remains circular, but with a radius that decreases very slowly with time according to $dr/dt = -C$, where C is a constant independent of the orbit's radius and the speed. Assume that $m \ll M$.

(a) Show that α can only be a certain integer, and find that integer. [20 points]

(b) Solve for the constant A. Express your answer in terms of C, the masses m and M, and Newton's gravitational constant G. [20 points]

NIU Physics PhD Candidacy Exam – Fall 2022 – Classical Mechanics

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Problem 1. Suppose that the law of gravitational interaction between two objects with masses M and m is modified by replacing the potential term in the Lagrangian according to:

$$\frac{GMm}{r} \rightarrow \frac{GMm}{r}\left(1 + \frac{\dot{r}^2}{C^2}\right)$$

where $r = |\vec{r}|$ with $\vec{r}$ the relative position vector, a dot means a time derivative, and $C =$ a new constant. In the following, assume that $m \ll M$ and treat the heavier object as stationary.

(a) Find the equations of motion for orbital motion in appropriate variables. [12 points]

(b) Does the radial vector of the small orbiting object sweep out equal areas in equal times? (Explain briefly.) [4 points]

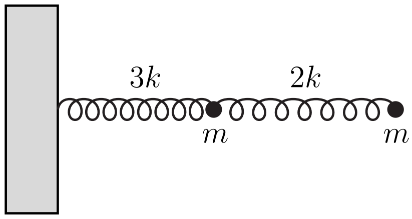
(c) Find the angular frequency of small oscillations about a circular orbit of radius R. Write your answer in terms of only G, M, m, R, and C. (Do not assume that C is small.) [12 points]

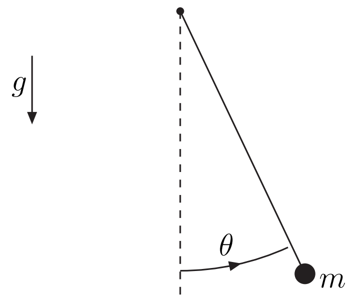
(d) Let M_E and R_E be the mass and the radius of the Earth (assumed to be a non-rotating perfect sphere). Compute the escape velocity for an object initially on the surface of the Earth, as a function of G, M_E, R_E, C, and $\alpha =$ the angle between the initial velocity at the Earth's surface and the vertical. [12 points]

Problem 2. Two identical point masses m are connected by springs as shown below. The spring constants are $3k$ and $2k$, and the spring on the left is attached to a fixed wall. All motion is in one dimension, along the axes of the springs.

(a) Write the Lagrangian for the system in terms of the displacements of the masses from their equilibrium positions, and derive the equations of motion. [16 points]

(b) Find the normal modes of oscillation and their frequencies. [24 points]





<u>Problem 3</u> A simple pendulum (string length L and mass m) is freely swinging in a plane under the gravitational force, mg, directed downward, as shown above.

(a) Find the equation of motion of this pendulum. Do not assume that θ is small. [4 points]

(b) Show that this non-linear differential equation can be approximated by a linear differential equation (simple harmonic oscillator) under a certain condition. What is the condition, and what is the period T_0 of this "approximated" simple harmonic oscillator? [4 points]

(c) For the non-linear differential equation in part (a), suppose that the motion is such that the maximum angular displacement is $\theta = \theta_0$. Find the angular velocity $d\theta/dt$ as a function of θ. Write your answer in terms of g, L, θ_0, and θ. [16 points]

(d) Now suppose that $\theta = 0$ at time $t = 0$. Note that $\theta = \theta_0$ (the maximum angular deviation) at time $t = T/4$, where T is the period of the *non-linear* oscillation. Find an expression for T in terms of θ_0 and T_0 [the period found in part (b)], in the form of an expansion in $\sin(\theta_0/2)$. Keep terms up to and including second order in that expansion. [16 points]

[Hints for part (d): You may find it useful to replace cosines using the formula $\cos(\theta) = 1 - 2\sin^2(\theta/2)$. You may then find it useful to write $\sin(\theta/2) = \sin(\theta_0/2)\sin\phi$, which defines a new variable ϕ. Finally, the expansion $1/\sqrt{1-x^2} = 1 + x^2/2 + \ldots$ may be useful.]

Problem 4 Consider a one-dimensional system with a coordinate q and its conjugate momentum p, and Hamiltonian

$$H = \frac{p^2}{2m} - \frac{k}{2q^2}, \tag{1}$$

where m and k are constants.

(a) Show that $\alpha pq - tH$ is a constant of the motion for a particular value of the constant α, which you should determine. [20 points]

(b) Suppose that at time $t = 0$, $q = q_0$ and $p = 0$. Find $p(t)$ and $q(t)$ for all $t > 0$. [20 points]

NIU Physics PhD Candidacy Exam – Spring 2022 – Classical Mechanics

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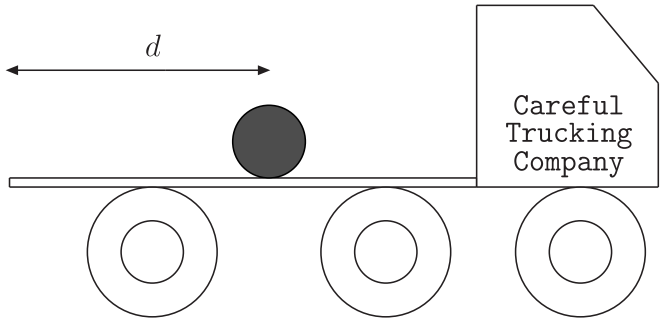
Problem 1. A very heavy flat-bed truck starts at rest on a level surface. It has a ball which is a solid sphere of radius R and mass M on the bed as shown, with its center a distance d from the back end. When the truck starts moving, the truck accelerates uniformly with acceleration a_T, and the ball rolls without slipping on the truck bed.

(a) Suppose the ball has uniform mass density. How much time does it take for the ball to roll off of the truck? (Note: moment of inertia of a uniform sphere about the center is $I = \frac{2}{5}MR^2$.) [28 points]

(b) If you could choose a different density distribution in the ball (but constrained to have the same total mass M and radius R), what choice would minimize the time to fall off? What is this minimum time? [12 points]



Problem 2. Consider a particle of mass m scattering from a central potential $V = -k/r^n$, where k and n are positive constants. The particle approaches from very far away with a non-zero impact parameter distance b and initial velocity v_0.

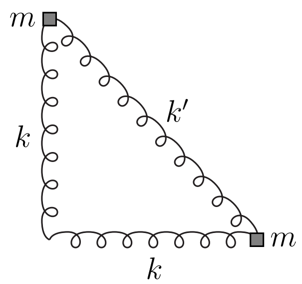
(a) Show that for the particle to have a chance at hitting the origin ($r = 0$), it is necessary that n is greater than or equal to a certain number that you will determine. [15 points]

(b) Now taking $n = 4$, show that a necessary and sufficient condition for the particle to hit the origin is that $b < b_{\text{crit}}$, where b_{crit} is a quantity that you will determine in terms of k, v_0, and m. [20 points]

(c) Still taking $n = 4$, what is the cross section for particles to hit the origin? [5 points]

Problem 3. A spherical star of a definite radius is constructed from an incompressible fluid of constant mass density ρ. The star is held together by its own gravitational attraction. The total mass of the star is M. Find the pressure $P(r)$ within the star as a function of the distance from the center, assuming that the star is not rotating. [40 points]

Problem 4. Two equal masses (m) are constrained to move without friction, the first on the positive x axis and the second on the positive y axis. They are attached to two identical springs (force constant k and natural length L) whose other ends are attached to the origin. In addition, the two masses are connected to each other by a third spring of force constant k'. The length of the third spring is chosen so that the system is in equilibrium with all three springs at their relaxed natural lengths. Use coordinate x as the displacement of the first mass from equilibrium and coordinate y as the displacement of the second mass from equilibrium.



(a) Find the Lagrangian of the system. [12 points]

(b) Find the equations of motion of the system, assuming small displacements. [10 points]

(c) Solve for the normal frequencies, and find and describe the corresponding normal modes. [18 points]

NIU Physics PhD Candidacy Exam – Fall 2021 – Classical Mechanics

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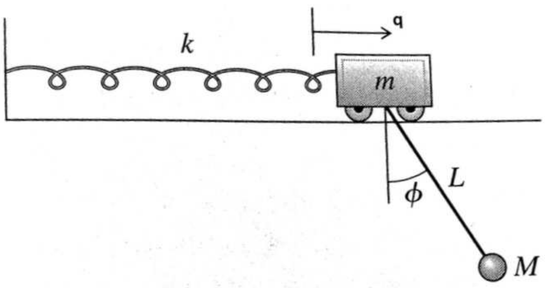
Problem 1. Consider the system below of a cart of mass m on a horizontal track that is attached to a spring with spring constant k. There is a bob of mass M attached to the cart by a massless rod of length L free to rotate in the two-dimensional plane of the page. The acceleration of gravity is g. The horizontal displacement from equilibrium of the cart is q, and the angle of the rod with respect to the vertical is ϕ, as shown.



(a) Write down the Lagrangian for the system, making no approximations on the size of ϕ. [15 points]

(b) Make the small angle approximation (note, you probably want to keep an extra term in your expansion of the potential energy) and write down the Euler-Lagrange equations for the system. [10 points]

(c) Solve the Euler-Lagrange equations for the case where $m = M = L = g = 1$ and $k = 2$, finding the eigenfrequencies of the system. [10 points]

(d) Describe the motion for the two modes of the system given above, indicating which motion corresponds to each of the two eigenfrequencies. [5 points]

Problem 2. Consider a particle of mass m moving in a three-dimensional central potential

$$V(r) = -k/(r - b),$$

where k and b are fixed positive constants. Let the angular momentum of the particle be ℓ.

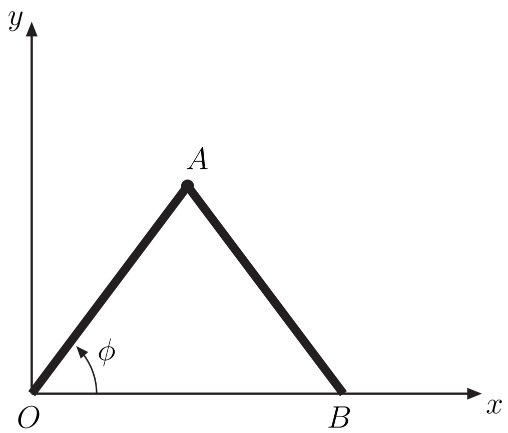
(a) Find the differential equation of motion for r, the radial coordinate of the particle. [10 points]

(b) Derive an algebraic equation for the possible values of the radius r_c of a circular orbit. (You do not need to solve the equation.) [12 points]

(c) Consider a small perturbation of a circular orbit, $r = r_c + \epsilon$. Find a linearized differential equation for ϵ, written in terms of r_c, b, and k (with ℓ eliminated). Use it to show that circular orbits are stable if either $r_c < b$ <u>or</u> $r_c > nb$, where n is a certain integer that you will find. [10 points]

(d) Using graphical methods, show that the equation for a circular orbit that you found in part (b) has either one or three solutions, depending on whether ℓ^2/mbk is smaller or larger than a certain rational number that you will find. [8 points]

Problem 3 Two identical thin uniform rods, each of length d and mass m, are hinged together at the point A. The rod on the left has one end hinged at the fixed point O, while the end B of the other rod slides freely along the horizontal x axis. The system is in a uniform vertical gravitational field with acceleration g. All motion is frictionless.



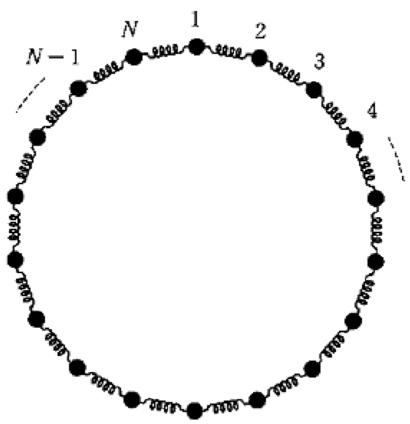
(a) Find the total kinetic energy of the system. You should find the result:

$$T = md^2\dot{\phi}^2(a + b\sin\phi + c\sin^2\phi)$$

where a, b and c are constant numbers that you will determine. (Exactly one of a, b, and c is 0.) [25 points]

(b) Suppose the system is at rest at time $t = 0$ with $\phi = \phi_0$. What is the velocity of the hinge A when it hits the horizontal x axis? [15 points]

Problem 4 Consider N identical masses m, coupled with N identical springs with spring constant k, constrained to move freely on a circular ring as shown. (There is no gravity in this problem.) Let the displacement along the ring for the mass n be $x_n(t)$ for $n = 1, \ldots, N$. This can be extended to define $x_0 = x_N$ and $x_{N+1} = x_1$, and more generally to the boundary condition $x_n = x_{n+N}$ for any integer n.



(a) Derive the equations of motion for the $x_n(t)$. [12 points]
(b) Look for normal mode solutions of the system with the amplitude for the mass labeled n given by $a_n = A\sin(n\alpha+\beta)$, where A, α, and β are constants independent of n. What are the allowed values of α, and how many distinct normal modes are there? Find the corresponding solutions for ω. (Note that your ω cannot depend on n.) [28 points]

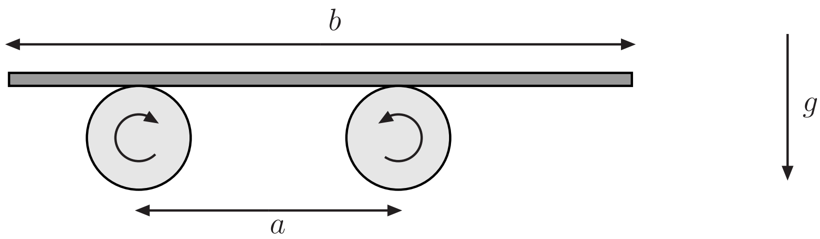
The following trigonometric identities may or may not be useful to you:

$$\begin{aligned}\sin(u+v) &= \sin u\cos v + \cos u \sin v\\ \cos(u+v) &= \cos u\cos v - \sin u\sin v\end{aligned}$$

NIU Physics PhD Candidacy Exam – March 2021 – Classical Mechanics

You may solve all four problems, if you choose. Only the three best problem scores count towards your total score. Total points on each problem = 40. Total possible score = 120.

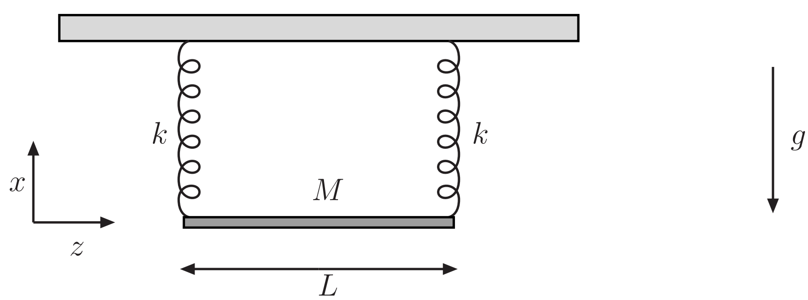
Problem 1. Two cylinders with equal radii R are forced to rotate with the same constant angular velocity Ω, but in *opposite* directions. The axes of the cylinders are parallel, at the same height and a distance a apart. The center of the cylinder rotating clockwise is at $x = -a/2$, and the center of the cylinder rotating counterclockwise is at $x = a/2$. A thin board with mass m (distributed uniformly) and length b is placed on the cylinders, slightly off-center so that its center is initially at rest at the position $x = x_0$. The acceleration of gravity is g. The coefficient of friction (ratio of tangential to normal force) between the cylinders and the board is μ. Solve for the motion of the board. [40 points.]



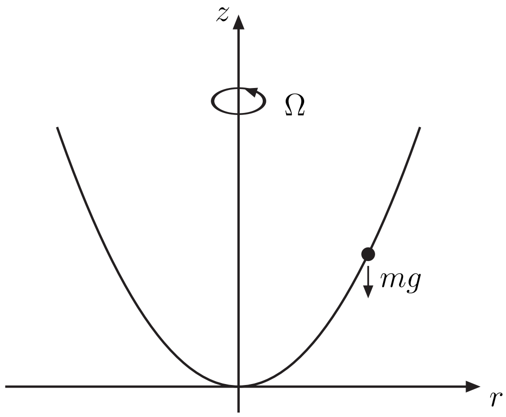
Problem 2. A thin rigid uniform bar of mass M and length L is supported in equilibrium in a horizontal position by two massless springs attached to each end, as shown. The springs have the same force constant k, and are suspended from a horizontal ceiling in a uniform gravitational field. The motion of the bar is constrained to the x, z plane, and the center of gravity of the bar is further constrained to move parallel to the vertical x-axis.

(a) Write a Lagrangian describing the dynamics of this system for small deviations from equilibrium, using as configuration variables the heights of the endpoints of the bar.
[15 points]

(b) Find the normal modes and angular frequencies of small vibrations of the system.
[25 points]



Problem 3. A wire in the shape of a parabola $z = \frac{r^2}{2a}$ is forced to rotate about the vertical z axis with constant angular frequency Ω. Here r is the distance from the z axis, and a is a constant. A constant gravitational acceleration g is directed in the negative z direction. A small bead of mass m slides on the wire without friction.



(a) Find the Lagrangian and obtain Lagrange's equations of motion for the bead, using r as the coordinate. [12 points]
(b) There exists a solution of Lagrange's equations for which dr/dt = constant. What is this constant? From where does the energy come to permit such unbounded motion? [9 points]
(c) Obtain the equations of motion for small deviations of the bead from rest at the bottom of the parabola. Give a condition for stable oscillation about this position. [9 points]
(d) Find the canonical momentum, the Hamiltonian, and Hamilton's equations of motion for the bead. Is the Hamiltonian conserved? [10 points]

Problem 4. The force of attraction between a very heavy star and a planet of mass m is:

$$F = \frac{a}{r^2} + \frac{3b\ell^2}{r^4}$$

where ℓ is the angular momentum of the planet and a, b are both positive constants. [Note: this does approximate the force of attraction between a planet and a black hole, in the non-relativistic limit, with $a = GMm$ where M is the mass of the star.]
(a) [15 points] Under what conditions is a stable circular orbit possible? Give the radius of the stable circular orbit in terms of the given parameters (m, a, b, ℓ).
(b) [15 points] What is the smallest radius possible for any circular orbit as a function of a and b, allowing for arbitrary ℓ? (Hint: this occurs in the limit of very large ℓ.) Is this circular orbit stable or unstable?
(c) [10 points] If the planet travels in a slightly non-circular orbit about a stable radius, find an expression for the angular frequency of small radial oscillations.

NIU Physics PhD Candidacy Exam – Spring 2021 – Classical Mechanics

You may solve all four problems, if you choose. Only the three best problem scores count towards your total score. Total points on each problem = 40. Total possible score = 120.

<u>Problem 1.</u> Consider a galaxy consisting of a very large number N of identical stars, each of mass m, distributed uniformly as a sphere with radius R. At the center of the galaxy is a very dense, very small core of mass M. The stars only interact gravitationally with each other and with the core.

(a) If the star orbits are perfectly circular, find the speed v and the angular momentum ℓ (about the center of the galaxy) for each star, as a function of the star's distance from the center r and N, R, m, M and G (Newton's constant). [14 points]

(b) Now consider star orbits that can be non-circular. The equation of motion of r for each star with angular momentum ℓ is the same as that of an equivalent 1-dimensional problem. For that problem, find the effective potential as a function of r. Your answer may depend on N, R, m, M, G, and ℓ. [13 points]

(c) Consider a star with a nearly circular orbit $r(t) = r_0 + \epsilon(t)$, where r_0 is constant and $\epsilon(t)$ is very small. Find an expression for the frequency of oscillations for ϵ. Put your answer in a form that does not depend on ℓ. [13 points]

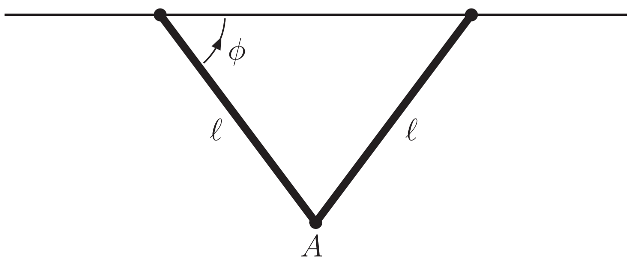
<u>Problem 2.</u> Two identical thin uniform rods, each of fixed length ℓ and mass m, are hinged together at the point A, as shown. The other ends of the rods both slide freely on a horizontal rail. The system is in a uniform vertical gravitational field with acceleration g. All motion is frictionless. At time $t = 0$, the angle ϕ is equal to ϕ_0, and the rods are released from rest.

(a) Find the Lagrangian of the system and the equation of motion in terms of the single configuration variable ϕ. [12 points]

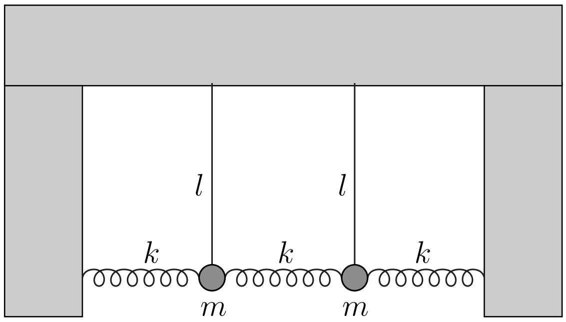
(b) For small times t, $\phi(t) = \phi_0 + \alpha t^2 + \beta t^4 + \ldots$. Find the constants α and β. [8 points]

(c) Find the amount of time needed for the hinge A to reach its lowest point. You may leave your answer in terms of a single definite integral. Do not assume the time is small. [10 points]

(d) Find the velocity at which each of the free ends of the rods are moving when the hinge A reaches its lowest point. [10 points]



Problem 3. Consider the system of coupled pendula in the figure. Both pendula have mass m, are suspended by massless strings of length l, and are coupled by identical massless springs with constant k. Let the deviation of the left mass from equilibrium be x_1, and the deviation of the right mass from equilibrium is x_2. There is a uniform gravitation field of acceleration g.



(a) Derive the set of coupled differential equations that describe the system. [5 points]

(b) Derive the eigenvalue equation for this system. Show all of your steps explicitly. [10 points]

(c) Find the angular frequencies of the normal modes for this system. [10 points]

(d) Find the corresponding normalized eigenvectors. [10 points]

(e) Describe the motion of each normal mode using words and pictures. [5 points]

Problem 4. A homogeneous solid cube with mass M and sides of length a is initially in a position of unstable equilibrium with one edge in contact with a horizontal plane. The cube is then given a tiny displacement and allowed to fall. (This is done in a uniform gravitational field with acceleration g. The moment of inertia of a cube about an axis through its center and parallel to an edge is $I = Ma^2/6$.)

(a) Find the angular velocity of the cube when one face strikes the plane, assuming that the edge cannot slide due to friction. [15 points]

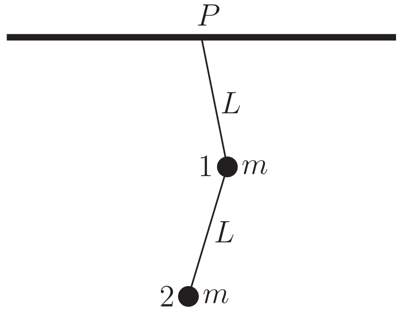
(b) Same question as (a), but now assuming that the edge slides without friction on the plane. [15 points]

(c) For the frictionless case in part (b), what is the force exerted by the surface on the cube just before the face strikes the plane? [10 points]

NIU Physics PhD Candidacy Exam – Fall 2020 – Classical Mechanics

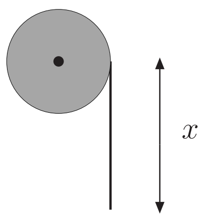
You may solve all four problems, if you choose. Only the three best problem scores count towards your total score. Total points on each problem = 40. Total possible score = 120.

<u>Problem 1.</u> Consider the double pendulum system shown below, suspended in a uniform gravitational field of acceleration g. Each of the sections of the double pendulum has length L, and both bobs have mass m, and x_1 and x_2 are the horizontal displacements of the upper and lower bobs relative to the support point P, respectively. Consider only small oscillations.



(a) Derive the eigenvalue equation (in matrix form) for the system. [18 points]
(b) Find the characteristic equation, and use it to find the normal mode angular frequencies. [12 points]
(c) Find the eigenvector solutions for the motion of x_1 and x_2. [10 points]

<u>Problem 2.</u> A very long, perfectly flexible material is rolled up on a fixed, horizontal, massless, very thin axle. The rolled-up portion rotates freely on the axle. The material has a (small) thickness s, width w, and, when completely unrolled, length ℓ. The mass density per unit volume of the material is ρ. At time $t = 0$, a length x_0 hangs from the roll and the system is at rest. For $t > 0$, the material unrolls under the influence of a constant gravitational field g (downward). [Useful information: the moment of inertia of a solid cylinder of mass M and radius R about its axis of symmetry is $I = MR^2/2$.]



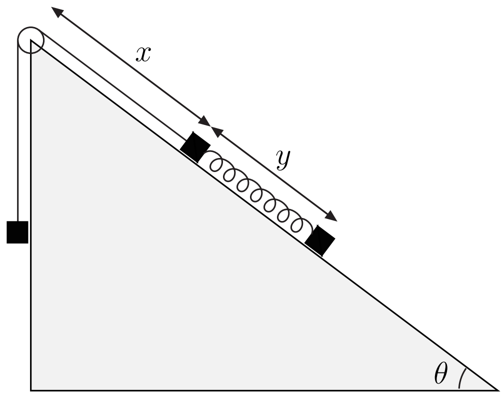
(a) Using the length x of the hanging part of the material as your configuration variable, find the Lagrangian of the system. [15 points]
(b) Find the canonical momentum conjugate to x, and the Hamiltonian. [9 points]
(c) What is the velocity of the free end of the material at the time when half of it is off the roll ($x = \ell/2$)? [16 points]

Problem 3. Consider a point particle which moves in a central potential subject to a radial force

$$F_r = -\frac{a}{r^3} - \frac{b}{r^4},$$

with a, b both positive. At the origin there is a particle eater, which swallows any particle that reaches $r = 0$. The point particle starts out at $r = R$ with angular momentum L and initial radial velocity $dr/dt = v_r$. Enumerate the conditions on R, v_r, and L that will result in the particle being eaten. [40 points]

Problem 4. Consider a stationary inclined plane ramp with a fixed angle θ, with three blocks with equal masses M attached by a spring of constant k and a string over a fixed pulley at the top of the ramp, as shown below. The positions of the blocks are described by the distances $x(t)$ (the distance from the pulley to the higher block on the ramp) and $y(t)$ (the extension of the spring), as shown. The system is in a uniform graviational field of acceleration g. The unstretched length of the spring is 0, and the pulley is massless.

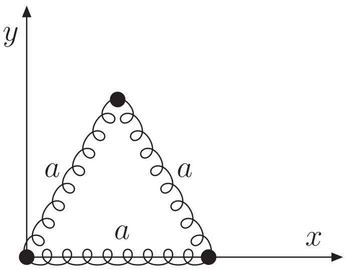


(a) [12 points] Find the Lagrangian and the equations of motion of the system.
(b) [20 points] The two blocks on the ramp are released from rest at the top with $x = y = 0$ at time $t = 0$. Find the motions of the blocks as a function of time.
(c) [8 points] The choice $\theta = 30$ degrees has a special significance, which should be apparent in the results of part (b). What is it?

NIU Physics PhD Candidacy Exam – Spring 2020 – Classical Mechanics

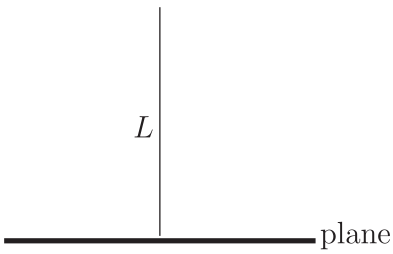
You may solve all four problems, if you choose. Only the three best problem scores count towards your total score.

<u>Problem 1.</u> Consider three identical masses m that are connected to each other by three identical springs, each of spring constant k. At equilibrium, the three masses form an equilateral triangle with sides a, as shown.



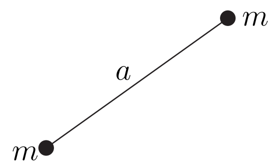
All three masses are allowed to move freely in the xy plane; ignore gravity and friction.
(a) Find the Lagrangian for the system, using generalized coordinates that are the Cartesian displacements relative to equilibrium. You should assume that the displacements are small with respect to a, and expand and simplify the potential energy accordingly. [20 points]
(b) Find the Lagrange equations of motion for the system. [10 points]
(c) There are 6 normal modes for the system. Without solving the Lagrangian equations of motion, how many of them have 0 frequency? Clearly describe the motions that they correspond to. [10 points]

<u>Problem 2.</u> A very thin rope of total mass M and length L is suspended vertically in a uniform downward gravitational field with acceleration g, so that one end of it is barely touching a horizontal plane, as shown.



At time $t = 0$, the rope is released from rest, and allowed to fall on to the plane (without bouncing or piling up significantly).
(a) Find the force $F(t)$ on the plane as a function of time $0 < t < \infty$. [25 points]
(b) Show that $F(t)$ reaches a maximum value equal to nMg, where n is a certain number (not equal to 1) that you will compute. When is the maximum reached? [15 points]

<u>Problem 3.</u> Two small asteroids of mass m in deep space are rotating each other at angular frequency Ω, under the influence of Newtonian gravity. The asteroids are also connected by a very strong, very light, unstretchable wire of length a.



(a) How large must Ω be for the wire to be under tension? [8 points]

(b) The wire suddenly disintegrates under the tension. If the asteroids then completely escape from each other and move off to infinity, what was the minimum possible value of Ω for this to occur? [8 points]

(c) Suppose that Ω was larger than the value found in part (a) but smaller than the value found in part (b). Find an algebraic equation whose solutions will yield the maximum and minimum distances between the asteroids that occur during their motion after the disintegration of the wire. (You do not need to solve the equation.) [12 points]

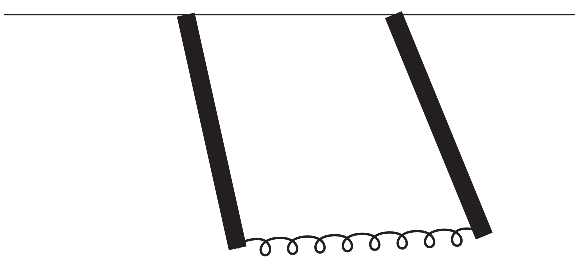
(d) Suppose instead that Ω was larger than the value found in part (b). A very long time after the disintegration of the wire, the masses are very far away from each other and moving on curves that approach straight parallel lines. Find the distance between the parallel lines. [12 points]

<u>Problem 4</u> Two rigid thin rods of uniform mass density each have mass M and length L. They are pivoted freely at their top ends, and joined by a spring at their bottom ends, as shown in the figure below. The rods are constrained to move in the plane of the page. The spring has spring constant k and length b when unstretched, and the distance between the pivot points is also b. The acceleration due to gravity is g.

(a) [16 points] Find the Lagrangian for this system.

(b) [12 points] Find the equations of motion for *small oscillations* of this system.

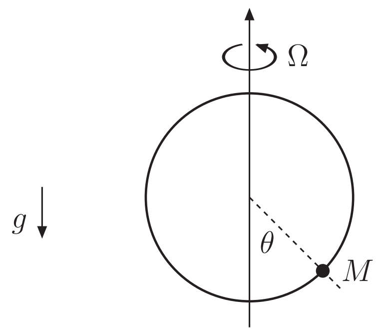
(c) [12 points] Sketch the motions corresponding to the normal modes. Find the corresponding normal frequencies for small oscillations.



NIU Physics PhD Candidacy Exam – Fall 2019 – Classical Mechanics

You may solve all four problems, if you choose. Only the three best problem scores count towards your total score. Total points on each problem = 40. Total possible score = 120.

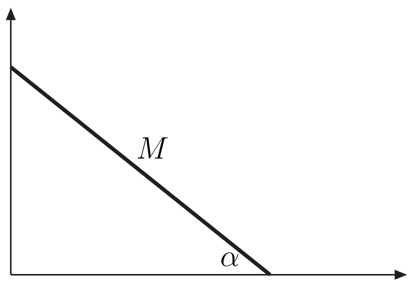
Problem 1. A small bead of mass M moves without friction on a circular wire hoop of radius R. The hoop is forced to rotate about its vertical diameter with a constant angular frequency Ω. There is a uniform downward gravitational field with acceleration g.



(a) Find a Lagrangian that describes the motion of the bead in terms of the angle θ shown, and derive the equation of motion. [14 points]
(b) Find all of the equilibrium positions, and find the conditions under which each of these is stable. [12 points]
(c) For each possible stable equilibrium position, find the angular frequency of small oscillations. [14 points]

Problem 2. A very small comet with mass m is discovered approaching the Sun (with mass $M \gg m$) along an orbit that is a perfect parabola. At the present time the comet is noted to be moving with speed v, and has an impact parameter b with respect to the Sun. For parts (b) and (c) below, you may leave your answer in terms of definite integrals, if you choose.
(a) Find the distance of closest approach of the comet to the Sun, and find its speed when it reaches closest approach. [16 points]
(b) How long does it take the comet to again reach its present distance from the Sun? [14 points]
(c) When the comet again reaches its present distance from the Sun, what angle (as measured from the Sun) has it swept out on its orbit? [10 points]

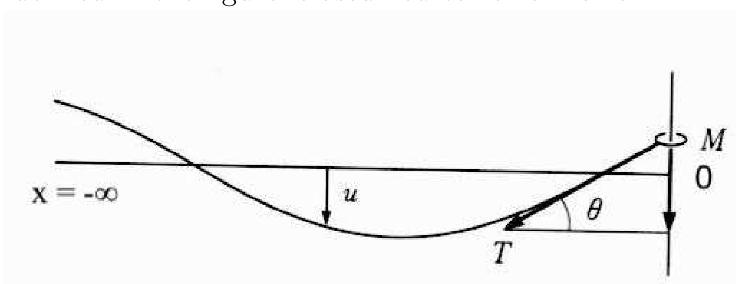
Problem 3. A thin stick of mass M (with uniform mass density) and length L is placed with one end on a wall and the other end on the floor. Both ends slide frictionlessly without losing contact, and the motion remains within the vertical plane. The angle of the stick with the floor is α. There is a uniform gravitational field with acceleration g downwards.



(a) Find a Lagrangian describing the system in terms of α and derive its equation of motion. [20 points]

(b) Suppose that at time $t = 0$ the stick is released from rest with $\alpha = \alpha_0$. At what time does the top end of the stick hit the floor? You may leave your answer in terms of a single definite integral over a dimensionless variable. [20 points]

Problem 4. An infinitely long string with linear mass density σ and tension T is attached at one end to a ring with mass M. The ring slides frictionlessly on a vertical pole at $x = 0$ as shown. The diameters of the ring and the pole can be ignored. Gravity can also be ignored. Suppose a sinusoidal wave with transverse (vertical) displacement $u_I(x,t) = A\cos(kx - \omega t)$ is incident from the far left ($x = -\infty$). Note that the constant phase angle of the incoming wave has been set to 0 for convenience. The displacement is defined so that $u = 0$ at equilibrium, and the angle θ as defined in the figure is assumed to remain small.



(a) Derive the relationship between the angular frequency ω, tension T, linear mass density σ, and wavevector k using the equation of motion. [10 points]

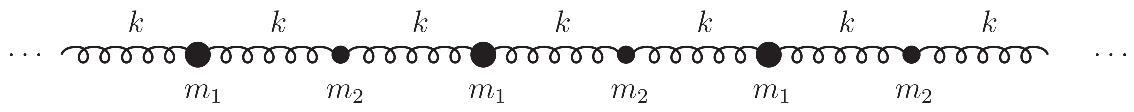
(b) Derive the reflected wave transverse displacement $u_R(x,t)$. In particular, find the relationship between the constant phase angle ϕ of the reflected wave, the mass of the ring M, and the angular frequency ω. [24 points]

(c) Discuss the dependences of $\cos\phi$ from part (b) on M for constant ω, including $M = 0$ and ∞, and its dependences on the ω of the incoming wave, including $\omega = 0$ and ∞. [6 points]

NIU Physics PhD Candidacy Exam – Spring 2019 – Classical Mechanics

You may solve all four problems, if you choose. Only the three best problem scores count towards your total score. Total points on each problem = 40. Total possible score = 120.

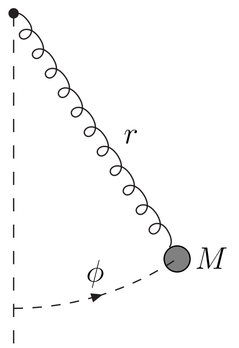
Problem 1. Consider an infinitely long 1-dimensional chain of alternating masses m_1 and m_2 and identical ideal springs with spring constant k, as shown below. The equilibrium distance d between neighboring masses is also equal to the unstretched spring lengths. Consider only longitudinal oscillations (with motion along the direction of the chain) with a fixed wavenumber $a = 2\pi/\lambda$, where λ is the wavelength. You may want to use a complex wave notation, with the positions of masses m_1 given by $Ae^{i[2nad-\omega t]}$ and the positions of masses m_2 given by $Be^{i[(2n+1)ad-\omega t]}$, where n are integers.



(a) Show that there are two collective modes with a given wavenumber a, and find their angular frequencies ω. [28 points]
(b) In the special case $m_1 = m_2 = m$ and the long wavelength limit $ad \ll 1$, characterize each of the two modes you found by identifying the motion of neighboring masses as either in phase or out of phase. [12 points]

Problem 2. Consider a binary system, consisting of two stars with masses m_1 and m_2 in circular orbits at a distance R from each other, in the presence of Newtonian gravity. Ignore all relativistic effects, and take the stars to be point masses.
(a) Derive the expression for the orbital period, T. [8 points]
(b) Suppose that $m_1 = 6M_{\text{Sun}}$ and $m_2 = 10M_{\text{Sun}}$ and that $R = 4$ a.u., where 1 a.u. is the distance between the Earth and the Sun. Find the orbital period in years. [4 points]
(c) Suppose that for a brief moment, the laws of physics are temporarily suspended and both stars are brought instantaneously to rest, following which Newtonian physics is restored. Find the amount of time needed for the stars to collide. If you wish, you may leave your answer in terms of a definite integral over a dimensionless variable. [16 points]
(d) Suppose that as in part (c) there is a very brief suspension of the laws of physics, but that in the center-of-mass frame it instantaneously brings m_2 to rest while increasing the speed of m_1 to v without affecting its direction. How large must v be in order for the stars to completely escape each other, leading to infinite separation? [12 points]

Problem 3. The pendulum shown below consists of a mass M attached to the end of a massless ideal spring with spring constant k and unstretched length a, in a uniform downward gravitational field with acceleration g. The other end of the spring is fixed, and the spring stretches and contracts but does not curve.



(a) Considering only motion confined to a vertical plane, find the Lagrangian for the system in terms of configuration variables r and ϕ, and derive the equations of motion. [10 points]
(b) Find the general solution for small oscillations about the equilibrium position, considering only motion confined to a vertical plane as in part (a). [18 points]
(c) Find the Hamiltonian for the system in terms of appropriate phase space variables, this time allowing motions of M that are not confined to the vertical plane. [12 points]

Problem 4. A yo-yo of total mass M and uniform mass density consists of 2 large cylinders of radius R and thickness d, joined by an axle of radius r and length also given by d. A very thin string is wound around the axle, with one end fixed to the ceiling in a uniform gravitational field with acceleration g. The axis of symmetry of the yo-yo remains parallel to the ground and does not rotate.
(a) Find the moment of inertia of the yo-yo about its axis of symmetry. [10 points]
(b) If the yo-yo starts from rest, find the time needed to descend a distance h. [15 points]
(c) Find the tension in the string as the yo-yo descends. [15 points]