I. CONFINED ELECTRON \([10+20+10] \text{PTS}\)

Consider an electron confined to the one-dimensional double potential well shown in the figure above, along with the two lowest energy eigenfunctions, \(\psi_a\) (anti-symmetric) and \(\psi_s\) (symmetric). These eigenfunctions have energies, \(E_a\) and \(E_s\) respectively.

a) Using these two wavefunctions construct properly normalized wavefunctions, \(\psi_l\) and \(\psi_r\) which represent an electron localized in the left well (at \(x_l\)) and the right well (at \(x_r\)), respectively.

b) Find an expression for \(\psi_r(t)\).

c) Find the probability of finding the electron in the right well as a function of time assuming it starts at \(t = 0\) in the left well.

II. CHARGED PARTICLE IN A MAGNETIC FIELD \([5+15+20] \text{PTS}\)

The Hamiltonian for a spinless charged particle in a magnet field \(\vec{B}\) is given by

\[
\hat{H} = \left( \hat{\vec{p}} - \frac{q}{c} \vec{A}(\vec{r}) \right)^2,
\]

where \(q\) is the charge of the particle, \(\hat{\vec{p}} = (p_x, p_y, p_z)\) its momentum conjugate to the particle’s position \(\vec{r}\), and \(\vec{A}\) the magnetic vector potential. Here we assume for the vector potential \(\vec{A} = -B_0 y \hat{e}_x\), where \(\hat{e}_x\) is the unit vector in \(x\)-direction.

a) Write down the Schrödinger equation for the particle. What is \(\vec{B}\)?

b) Prove that \(\hat{p}_x\) and \(\hat{p}_z\) are constants of motion. What does this imply for the eigenfunctions of this problem?

\textit{Hint:} An observable \(\hat{A}\) is a constant of motion if its expectation value \(\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle\) is time independent.

c) Calculate the energy eigenvalues of the system.

\textit{Hints:} Use b)! Show that the energy eigenvalues do not depend on \(p_x\).
III. VARIATIONAL TREATMENT OF THE HARMONIC OSCILLATOR [(20+10+10) PTS]

By considering the one-dimensional harmonic oscillator potential \( V = \frac{m\omega^2}{2}x^2 \) comparable to a particle in a box, one can obtain an approximation of the ground state wavefunction

\[
\psi(x) = \frac{1}{\sqrt{L}} \cos \frac{\pi}{2L} x \quad \text{for} \quad |x| \leq L
\]

where \( L \) is the size of the box. The wavefunction is zero outside of the box. The effective size of the box is considered a free parameter.

a) Calculate the expectation value of the energy for this wavefunction. You can use the integral

\[
\int_{-\pi/2}^{\pi/2} y^2 \cos^2 y \, dy = \frac{\pi}{24} (\pi^2 - 6).
\]

b) Find the value of \( L \) that minimizes the energy.

c) What is the minimum value of the energy?

IV. TIME-DEPENDENT PERTURBATION THEORY [(24+8+8) PTS]

A one-dimensional harmonic oscillator is in its ground state for \( t < 0 \). At \( t = 0 \) a perturbation of the form

\[
V = Ax^2 e^{-t/\tau}
\]

is switched on.

a) Using time-dependent perturbation theory to lowest nonvanishing order, calculate the probability that the system has made a transition to a given excited state. Which final states can be reached that way?

b) Show that for \( t \gg \tau \) your expression is independent of time.

c) Specify any requirements on the parameters of the problem necessary for the validity of the approximations made in the application of time-dependent perturbation theory. Compare the value with the zero-point energy \( 1/2 \hbar \omega \) of the harmonic oscillator.