NIU Physics PhD Candidacy Exam - Fall 2019

Quantum Mechanics

You may solve ALL four problems! The three best graded count towards your total score.
(40 points each; total possible score: 120 points)

I. SPIN IN FIELD [(20+20) PTS]

A spin-1/2 particle has a magnetic moment $\vec{\mu} = \mu \vec{S}$, where $\mu$ is a constant and $\vec{S}$ is the spin operator. It is in a uniform time-varying magnetic field $\vec{B} = B(t) \hat{z}$. Suppose that at time $t = 0$, the spin of the particle is measured to be along the direction $\hat{n} = \hat{x} \sin(\theta) + \hat{z} \cos(\theta)$. You may express your answers to the following questions in terms of a definite integral involving $B(t)$.

a) Find the expectation value of the spin operator $\vec{S}$ as a function of $t$.

b) If the $\hat{y}$ component of the spin is measured at time $t$, what is the probability that one obtains the result $S_y = +\hbar/2$?

II. PARTICLE IN 1D [(8+8+10+14) PTS]

A particle of mass $m$ moves in one dimension under the influence of a potential $V(x)$. Suppose it is in an energy eigenstate $\psi(x) = (\gamma^2 / \pi)^{1/4} \exp(-\gamma^2 x^2 / 2)$ with energy $E = h^2 \gamma^2 / (2m)$.

a) Find the mean position of the particle.

b) Find the mean momentum of the particle.

c) Find the potential $V(x)$.

d) Find the probability $P(p) dp$ that the particle’s momentum is between $p$ and $p + dp$.

Hint: There are two ways to find the normalized wave function $\psi(p)$: Either rewrite the Schrödinger equation in momentum representation and use a Gaussian Ansatz or use the Fourier transform.

III. QUANTUM SPEED TRAP [(12+12+12+4) PTS]

A quantum mechanic is asked to construct a Speed Trap operator, $\hat{O}$, for wave riders with two eigenvalues: Ticket ($T$) and No-ticket ($N$). The eigenstate for Ticket, $|\psi_T\rangle$, shall be the (coherent) sum of the wave functions for Age>16 years riders, $|\psi_{A>16}\rangle$, and Speed>30 mph riders, $|\psi_{S>30}\rangle$.

a) Write the matrix for $\hat{O}$ in the $|\psi_T\rangle$, $|\psi_N\rangle$ basis representation.

b) Write the wave function for Ticket, $|\psi_T\rangle$, and No-ticket, $|\psi_N\rangle$, as a linear combination of $|\psi_{A>16}\rangle$ and $|\psi_{S>30}\rangle$. The wave functions should be correctly normalized.

c) Show why a wave rider in an eigenstate $|\psi_{S>30}\rangle$ cannot be in an eigenstate of Ticket.

d) How can you modify $|\psi_T\rangle$ and $\hat{O}$, such that $|\psi_T\rangle$ takes into account both, Speed and Age, and $\hat{O}$ catches all speeding wave riders regardless of age?
IV. CHARGED 2D HARMONIC OSCILLATOR [(20+10+10) PTS]

Let us consider a charged particle in a two-dimensional harmonic oscillator subject to a uniform electric field applied along the x-axis, i.e., with additional potential $U(\hat{x}) = -eE\hat{x}$.

a) Calculate the exact change in energy levels of the charged linear oscillator.

b) Calculate the exact change in the eigenfunctions of the charged oscillator. Express the wave functions for $E \neq 0$ in terms of the eigenfunctions of the two-dimensional harmonic oscillator.

c) Calculate the polarizability of the oscillator in these eigenstates. The polarizability, $\alpha$, determines the mean dipole moment, $P = \alpha E$, induced by the weak external electric field in an isotropic system. It can also be expressed as the negative second derivative of the energy with respect to the electric field. How does $\alpha$ depend on the quantum number?
Consider an electron confined to the one-dimensional double potential well shown in the figure above, along with the two lowest energy eigenfunctions, \( \psi_a \) (anti-symmetric) and \( \psi_s \) (symmetric). These eigenfunctions have energies, \( E_a \) and \( E_s \) respectively.

a) Using these two wavefunctions construct properly normalized wavefunctions, \( \psi_l \) and \( \psi_r \) which represent an electron localized in the left well (at \( x_l \)) and the right well (at \( x_r \)), respectively.

b) Find an expression for \( \psi_r(t) \).

c) Find the probability of finding the electron in the right well as a function of time assuming it starts at \( t = 0 \) in the left well.

II. CHARGED PARTICLE IN A MAGNETIC FIELD \([(5+15+20) \ PTS]\)

The Hamiltonian for a spinless charged particle in a magnet field \( \vec{B} \) is given by

\[
\hat{H} = \left( \hat{\vec{p}} - \frac{q}{\epsilon} \vec{A}(\vec{r}) \right)^2 ,
\]

where \( q \) is the charge of the particle, \( \hat{\vec{p}} = (p_x, p_y, p_z) \) its momentum conjugate to the particle’s position \( \vec{r} \), and \( \vec{A} \) the magnetic vector potential. Here we assume for the vector potential \( \vec{A} = -B_0 y \vec{e}_x \), where \( \vec{e}_x \) is the unit vector in \( x \)-direction.

a) Write down the Schrödinger equation for the particle. What is \( \vec{B} \)?

b) Prove that \( \hat{p}_x \) and \( \hat{\vec{p}}_z \) are constants of motion. What does this imply for the eigenfunctions of this problem?

\textit{Hint:} An observable \( \hat{A} \) is a constant of motion if its expectation value \( \langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle \) is time independent.

c) Calculate the energy eigenvalues of the system.

\textit{Hints:} Use b)! Show that the energy eigenvalues do not depend on \( p_x \).
III. VARIATIONAL TREATMENT OF THE HARMONIC OSCILLATOR [(20+10+10) PTS]

By considering the one-dimensional harmonic oscillator potential \( V = \frac{m\omega^2}{2} x^2 \) comparable to a particle in a box, one can obtain an approximation of the ground state wavefunction

\[
\psi(x) = \frac{1}{\sqrt{L}} \cos \frac{\pi}{2L} x \quad \text{for} \quad |x| \leq L
\]

where \( L \) is the size of the box. The wavefunction is zero outside of the box. The effective size of the box is considered a free parameter.

a) Calculate the expectation value of the energy for this wavefunction. You can use the integral

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y^2 \cos^2 y \, dy = \frac{\pi}{24}(\pi^2 - 6).
\]

b) Find the value of \( L \) that minimizes the energy.

c) What is the minimum value of the energy?

IV. TIME-DEPENDENT PERTURBATION THEORY [(24+8+8) PTS]

A one-dimensional harmonic oscillator is in its ground state for \( t < 0 \). At \( t = 0 \) a perturbation of the form

\[
V = Ax^2 e^{-t/\tau}
\]

is switched on.

a) Using time-dependent perturbation theory to lowest nonvanishing order, calculate the probability that the system has made a transition to a given excited state. Which final states can be reached that way?

b) Show that for \( t \gg \tau \) your expression is independent of time.

c) Specify any requirements on the parameters of the problem necessary for the validity of the approximations made in the application of time-dependent perturbation theory. Compare the value with the zero-point energy \( \frac{1}{2} \hbar \omega \) of the harmonic oscillator.