NIU Physics PhD Candidacy Exam - Fall 2018

Quantum Mechanics

YOU MAY SOLVE ALL FOUR PROBLEMS! THE **three best graded** COUNT TOWARDS YOUR TOTAL SCORE.
(40 POINTS EACH; TOTAL POSSIBLE SCORE: 120 POINTS)

I. ANGULAR MOMENTUM *[(10+12+10+8) PTS]*

We consider the motion of a particle with mass m in three dimensions. Thus in the following we have position $= (x, y, z)$, momentum $p = (p_x, p_y, p_z)$ and orbital angular momentum $L = r \times p$.

a) Evaluate the commutators $[L_z, r^2]$ and $[L_z, p^2]$.

b) Prove that for *any* potential $V(r)$, we have

$$\frac{d}{dt}\langle L\rangle = \langle N\rangle \qquad \text{where} \qquad N = r \times (-\nabla V)$$

Hint: Use part a).

c) Show $d\langle L\rangle/dt = 0$ for any spherically symmetric potential $V(r)$.

d) Interpret your results in (b) and (c).

II. PERTURBED HYDROGEN *[(16+10+10+4) PTS]*

Suppose that a Hydrogen atom is initially in an energy Eigenstate $|\Psi_0\rangle$ with energy E_0. Then at time $t = 0$, a light wave of (angular) frequency ω and electric field amplitude E_0 shines on the atom. According to first order perturbation theory, the probability amplitude for finding the electron in a new energy state $|\Psi_n\rangle$ (energy E_n) is given by

$$\langle\psi_n|\,\psi_0\rangle = \frac{e^{-\imath E_n t/\hbar}}{\imath h}\int_0^t e^{\imath(E_n-E_0)\tau/\hbar}\,\langle\Psi_n|\,V(x,\tau)\,|\Psi_0\rangle\,d\tau \tag{1}$$

If the potential $V(x, t)$ for the light wave is given by

$$V(x,t) = \frac{\imath E_0}{k}e^{\imath(px-\hbar\omega t)/\hbar} = V(x)e^{-\imath\omega t} \tag{2}$$

a) Show that the probability, $P_{0\to n}(t)$, that the electron makes a transition from E_0 to E_n is given by

$$P_{0\to n}(t) = C\frac{\sin^2(\Delta E t/(2\hbar))}{\Delta E^2}|\,\langle\Psi_n|\,V(x)\,|\Psi_0\rangle\,|^2\,, \tag{3}$$

with $\Delta E = E_n - E_0 - \hbar\omega$ and a numerical constant C (calculate its value).

b) For a fixed time, sketch the graph of $P_{0\to n}(t)$ as function of ΔE assuming the spatial integral, $\langle\Psi_n|\,V(x)\,|\Psi_0\rangle$ is constant with respect to E_n and ω and indicate the values of ΔE where $P_{0\to n}(t)$ goes to 0.

c) Explain how this plot, or its analytical expression, shows that energy conservation can and will be violated by an amount ΔE as long as $\Delta E \leq h/t$.

d) Explain how Eq. (3) can also be interpreted as a special case of the uncertainty principle, $(\Delta E)t \geq h$.

III. ENTANGLED QUBITS & QUANTUM INFORMATION *[(5+5+5+15+10) PTS]*

Here we consider two-level systems (e.g., Qubits) having eigenbasis states $|0\rangle$ and $|1\rangle$. An arbitrary state $|\psi\rangle$ in such a system can be written as superposition $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ with complex coefficients $\alpha_{0,1}$. Now, two of these systems — system A and system B — are brought into contact forming the composite system C. A general state in system C is given by

$$|\psi\rangle_C = \sum_{i,j\in\{0,1\}} c_{ij} |i\rangle_A |j\rangle_B \ .$$

If $|\psi\rangle_C$ can be written in form $|\psi\rangle_C = |\psi_A\rangle_A |\psi_B\rangle_B$ it is called a product state. However, not all states are product states and a state is called *entangled* if one c_{ij} exists, which cannot be written as product of the basis coefficients $\alpha_{0,1}^{(A,B)}$. Let us start with a particular state:

$$|\phi\rangle_C \equiv \gamma \left(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B\right)$$

(This is one of the four Bell states of a two-Qubit system.)

a) Calculate the coefficient γ in $|\phi\rangle_C$.

b) Prove that $|\phi\rangle_C$ is entangled.

c) Write down another entangled state of system C with $c_{00} \neq 0$.

d) The degree to which a state $|\phi\rangle_C$ is entangled is measured by the von Neumann entropy, $S = -\mathrm{Tr}(\hat{\rho}_{A,B} \ln(\hat{\rho}_{A,B}))$ of either of the reduced density operators $\hat{\rho}_{A,B} = \mathrm{Tr}_{B,A}\hat{\rho}_C$. Calculate this entropy for $|\phi\rangle_C$. Justify that it is maximum by comparing it to pure states of one of the sub-systems.

e) Now we have two observers of Qubits A and B, Alice and Bob, respectively. Both measure in the eigenbasis states of their respective system, i.e., they apply operators $|i\rangle_{A,B} \langle i|_{A,B}$ to the composite state $|\phi\rangle_C$. Alice makes the measurement first and measures either state $|0\rangle_A$ or $|1\rangle_A$. What are the probabilities to measure these states and what are the resulting composite states of system C? After this measurement, what does Bob measure in either case (give the corresponding probabilities)? What does this mean for system B (even if systems A and B are spatially separated)? Does this mean that information can be transmitted instantaneously? (justify)

Hints: Read carefully and answer all sub-questions. The density operator related to $|\psi\rangle_C$ is given by $\hat{\rho}_C = |\psi\rangle_C \langle\psi|_C$. $\hat{\rho}_A = \mathrm{Tr}_B\hat{\rho}_C = \sum_{i\in\{0,1\}} \langle i|_B \, \hat{\rho}_C \, |i\rangle_B$. For an arbitrary state in a two-level system, the von Neumann entropy can be written as $S = -\mathrm{Tr}(\hat{\rho}\ln(\hat{\rho})) = -\sum_{i\in\{0,1\}} \eta_i \ln \eta_i$, if $\hat{\rho}$ is written in eigendecomposition $\hat{\rho} = \sum_{i\in\{0,1\}} \eta_i |i\rangle \langle i|$.

IV. ATTRACTIVE δ FUNCTION NEAR A WALL *[(15+7+10+8) PTS]*

We want to study the change in the bound-state solution of an attractive δ-function potential when it is near a wall. The potential then becomes

$$U(x) = \begin{cases} \infty & x < -d \\ -U_0\delta(x) & x \geq -d \end{cases} \ .$$

a) Find the solution for the wavefunction of the bound state and derive the transcendental equation that the complex wavenumber has to satisfy. There is no need to normalize the wavefunction.

b) Give a sketch of the wavefunction and compare it with that for an attractive δ-function in the absence of a wall.

c) Calculate the correction to the energy if the wall is far away. Explain the approximations made.

d) What is the condition for U_0 and d for the existence of at least one bound state?

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I. SPINS IN A CUBIC BOX *[(22+9+9) PTS]*

Two *identical* spin-$1/2$ fermions of mass M are confined in a cubic box of side L. The sides of the box have infinite potential. The identical fermions interact according to the attractive potential:

$$V(\vec{r}_1, \vec{r}_2) = -\epsilon L^3 \delta^{(3)}(\vec{r}_1 - \vec{r}_2),$$

where ϵ is small and positive, and should be treated as a perturbation. Do not neglect the effects of spin degrees of freedom in this problem. [Hint: you may or may not find $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and/or $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$ to be useful in doing an integral.]

a) Find the total spin, the degeneracy, and the energy of the ground state, working to first order in ϵ.

b) What is the total spin and the degeneracy of the first excited state?

c) If the potential is instead repulsive ($\epsilon < 0$), what is the total spin and the degeneracy of the first excited state?

II. MYSTERY POTENTIAL *[(10+10+6+14) PTS]*

The wave function for a particle of mass m moving in one dimension, subject to a potential $V(x)$ is given by

$$\psi(x,t) = \Theta(x) x e^{-Bx} e^{-iCt/\hbar},$$

where $\Theta(x)$ is the Heaviside step function (0 for $x < 0$ and 1 for $x > 0$) and B & C are real constants such that $\psi(x,t)$ is a properly normalized wave function that obeys the time-dependent Schrödinger equation for potential $V(x)$.

a) Sketch this wave function at time $t = 0$. Mark any significant features.

b) Using what you know about ψ, make a qualitative sketch of the potential $V(x)$ governing this system, indicating in particular any classically forbidden regions and classical turning points.

c) Is the particle in a state corresponding to a definite energy? If so, what is the energy (in terms of any or all of B and C); if not, why not?

d) Determine the potential $V(x)$ in terms of B, C, m, and $\hbar$. Does your result agree with your sketch?

III. ISOTROPIC HARMONIC OSCILLATOR *[(8+10+10+12 PTS]*

We consider a particle with mass m confined in a three-dimensional isotropic harmonic potential

$$V_0 = \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2)$$

a) Give a general expression for the energy eigenvalues. Also, give explicit expressions for the lowest five energy eigenvalues including their degeneracies.

b) We add a perturbation $V_1 = k\,x$ with a real constant k. Evaluate the energy eigenvalues in lowest nonvanishing order perturbation theory. Again give the lowest five energy eigenvalues including their degeneracies.

c) Next we consider the perturbation $V_2 = \frac{k}{\sqrt{2}}\,(x+y)$. Again evaluate the energy eigenvalues in lowest nonvanishing order perturbation theory? What are now the lowest five energy eigenvalues including their degeneracies?

d) Solve the Schrödinger equation for the potential $V_0 + V_2$ exactly. What are the energy eigenvalues? Compare with part a).

IV. P-ELECTRON *[(10+10+10+10) PTS]*

An electron in a p orbital ($l = 1$) feels an additional interaction

$$\mathrm{H}' = -\frac{\omega_0}{\hbar}(\mathrm{L}_x^2 - \mathrm{L}_y^2),$$

where ω_0 is a parameter and

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

a) Write H' in matrix form.

b) Determine the eigenvalues and eigenvectors of H'.

c) If the electron is initially in the state with $m = 1$, how does it oscillate between the different states as a function of time?

d) Calculate the probability of finding $m = 1$ as a function of time.

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I. RECTANGULAR POTENTIAL BARRIER *[(5+20+10+5) PTS]*

A particle with energy E encounters a rectangular potential barrier in one dimension, $V(x) = V_0$ for $0 \le x \le a$ and $V(x) = 0$ for $x < 0$ and $x > a$, coming from the left. Here $V_0 > 0$ and $a > 0$.

a) Write down the Schrödinger equation, make an Ansatz for the wave function including the wave numbers, and give the boundary conditions at $x = 0$ and $x = a$.

b) Calculate the transmission *coefficient*, t, as a function of the *two* wave numbers and a.

c) The related transmission *probability* $T = |t|^2$ is

$$\left(1 + \frac{\sin^2(k_{\mathrm{II}}a)}{4\epsilon(\epsilon - 1)}\right)^{-1},$$

with $\epsilon = E/V_0$ for $\epsilon > 1$ and k_{II} the wave numbers for $0 < x < a$. What is the lowest energy at which the transmission probability becomes 1? What would this energy be for a classical particle? What is the tunneling probability for $E < V_0$ (the same expression as above holds, but write T as a function of real quantities only!)?

d) Sketch T as function of ϵ such that it shows the correct physical behavior for $\epsilon < 1$, $\epsilon = 1$, and $\epsilon > 1$.

Hint for b: The boundary conditions give you four equations for the six unknown coefficients of the wave functions. However, the definition of the problem (particle comes from the left!) defines two of the coefficients (explain in your answer)! Then you only need to solve the linear equation system for the transmission coefficient.

II. PARTICLE IN A BOX *[(6+4+30) PTS]*

A spinless particle of charge e and mass m is confined to a cubic box of side L. A weak uniform electric field E_0 is applied, with direction parallel to one of the sides of the box, and the electrostatic potential for the field is taken to be zero at the center of the cubic box.

a) Write down the unperturbed energy eigenvalues and corresponding normalized wave functions.

b) Show explicitly that, to first order in the perturbation E_0, the ground state energy is unchanged.

c) Find the change in the ground state energy at second order in E_0. You should leave your answer in terms of an infinite sum of the form

$$\sum_{n=2,4,6,\ldots} \frac{n^p}{(n^2-1)^q}$$

where p and q are certain integers that you will find. How does the change in the ground state energy scale with the size of the box L?

Hint: you may find the following definite integrals useful:

$$\int_{-L/2}^{L/2} du\, \sin\left(\frac{n\pi u}{L} + \frac{n\pi}{2}\right) \sin\left(\frac{n'\pi u}{L} + \frac{n'\pi}{2}\right) = \begin{cases} \frac{L}{2} & (n = n') \\ 0 & (n \ne n'). \end{cases}$$

and

$$\int_{-L/2}^{L/2} du\, u \sin\left(\frac{n\pi u}{L} + \frac{n\pi}{2}\right) \sin\left(\frac{n'\pi u}{L} + \frac{n'\pi}{2}\right) = \begin{cases} -\dfrac{4L^2}{\pi^2} \dfrac{nn'}{(n^2 - n'^2)^2} & (n + n' = \text{odd}) \\ 0 & (n + n' = \text{even}). \end{cases}$$

III. SPIN *[(20+20) PTS]*

A particle of spin $1/2$ is described by the eigenspinor

$$\frac{1}{\sqrt{5}} \begin{pmatrix} i \\ 2 \end{pmatrix} .$$

a) What are the values of $\langle S_z \rangle$, $\langle S_y \rangle$, $\langle S_y^2 \rangle$, and $\Delta(S_y) = \langle (S_y - \langle S_y \rangle)^2 \rangle$?

b) What is the probability that a measurement of S_y gives the value $\hbar/2$?

Hint: The Pauli matrices are on the formula sheet.

IV. TWO SPRINGS *[(10+10+10+10) PTS]*

Let us consider the Hamiltonians

$$H = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2 \qquad \text{and} \qquad H_1 = \frac{1}{2} m\omega_1^2 x^2 .$$

a) Determine the eigenvalues of the Hamiltonian $H_{\text{tot}} = H + H_1$.

b) Let us take the solutions of H to be $|n\rangle$ (note that these are not the solutions of H_1 or H_{tot}). Calculate the matrix elements $\langle n'|H_1|n\rangle$. Show that for the following matrix elements we can write $\langle 0|H_1|0\rangle = \frac{1}{2}\alpha$, $\langle 2|H_1|2\rangle = \frac{5}{2}\alpha$, and $\langle 0|H_1|2\rangle = \langle 2|H_1|0\rangle = \alpha/\sqrt{2}$. Determine α.

c) Let us now consider the situation where $\omega_1 \ll \omega$. In this limit we can take as a trial solution for the change in the lowest eigenstate of H due to the presence of H_1,

$$|\psi\rangle = \cos\theta |0\rangle + \sin\theta |2\rangle,$$

where $\cos\theta$ and $\sin\theta$ are coefficients. Explain why this is a reasonable trial wave function and determine the θ value that minimizes $\langle \psi | H_{\text{tot}} | \psi \rangle$.

d) Show that the exact eigenfunction for $H_{\text{tot}} = H + H_1$ with $n_{\text{tot}} = 0$, is indeed a combination of the $n = 0$ and $n = 2$ eigenfunctions of H in the limit $\omega_1 \ll \omega$. (There is no need to find the exact combinations, only show that it contains a combination of the right Hermite polynomials).
The solutions of H for $n = 0$ and 2 are

$$\varphi_0(x) = \frac{1}{\sqrt{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\varphi_2(x) = \frac{1}{2\sqrt{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2} \left(\frac{4m\omega}{\hbar}x^2 - 2\right)$$

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You may solve ALL four problems! The **three best graded** count towards your total score.
(40 points each; total possible score: 120 points)

I. SPHERICAL POTENTIAL WELL *[(10+10+10+10) PTS]*

Consider a spherical potential well with

$$U(r) = 0 \qquad r < a \qquad ; \qquad U(r) = \infty \qquad r > a$$

a) Give the radial part of the Hamiltonian for $r < a$.

b) We are now only interested in the wavefunction for $l = 0$. Give the general solution of the differential equation by trying a solution $R = P/r$. (Normalization is not necessary).

c) Determine the behavior of the wavefunction for $r \to 0$. What does the condition that the wavefunction should not diverge at $r = 0$ imply?

d) Determine the eigenvalues.

II. SPINS IN THE BOX *[(22+9+9) PTS]*

Two *identical* spin-$1/2$ fermions of mass M are confined in a cubic box of side L. The sides of the box have infinite potential. The identical fermions interact according to the attractive potential:

$$V(\vec{r}_1, \vec{r}_2) = -\epsilon L^3 \delta^{(3)}(\vec{r}_1 - \vec{r}_2),$$

where ϵ is small and positive, and should be treated as a perturbation. Do not neglect the effects of spin degrees of freedom in this problem. [Hint: you may or may not find $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and/or $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$ to be useful in doing an integral.]

a) Find the total spin, the degeneracy, and the energy of the ground state, working to first order in ϵ.

b) What is the total spin and the degeneracy of the first excited state?

c) If the potential is instead repulsive ($\epsilon < 0$), what is the total spin and the degeneracy of the first excited state?

III. PERTURBED BOUND STATES *[(5+5+12+18) PTS]*

Consider a particle of mass m in a two-dimensional infinite square well of width a

$$V_0(x, y) = \begin{cases} 0 & 0 \le x \le a,\ 0 \le y \le a \\ \infty & \text{otherwise} \end{cases}$$

a) Write down the time-independent Schrödinger equation for this problem.

b) Write down the energy eigenfunctions and corresponding energy eigenvalues for the ground and first excited states. (You need not derive the answer if you know it.)

We now add a time-independent perturbation

$$V_1(x, y) = \begin{cases} \lambda xy & 0 \le x \le a,\ 0 \le y \le a \\ 0 & \text{otherwise} \end{cases}$$

c) Obtain the first-order energy shift for the ground state.

d) Obtain the zeroth-order energy eigenfunctions and the first-order energy shifts for the first excited states.

IV. CHARGED 1D LINEAR OSCILLATOR *[(20+10+10) PTS]*

Let us consider an one-dimensional harmonic oscillator in a uniform electric field applied along the oscillation axis, i.e., with additional potential $U(\hat{x}) = -e\mathcal{E}\hat{x}$.

a) Find the change in energy levels of the charged linear oscillator.

b) Find the change in the eigenfunctions of the charged linear oscillator. Express the wavefunctions for $\mathcal{E} \neq 0$ in terms of the eigenfuncions of the one-dimensional harmonic oscillator.

c) Find the *polarizability* of the oscillator in these eigenstates. The polarizability, α, determines the mean dipole moment, $p = \alpha\mathcal{E}$, induced by the weak external electric field in an isotropic system. It can also be expressed as the negative second derivative of the energy with respect to the electric field. How does α depend on the quantum number?

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I. ATOMIC ORBITALS *[(10+10+10+10) PTS]*

We want to study the change in energy of of atomic orbitals feeling a potential energy $\hat{H}_1$ due to a constant electric field in the x direction, where $\hat{H}_1$ is given by

$$\hat{H}_1 = eE_x x.$$

We will consider the coupling of the $n, l = 0$ orbital to $n', l = 1$ orbitals, where the unperturbed orbitals are given by

$$\psi_{n00}(r, \theta, \varphi) = R_{n0}(r)Y_{00}(\theta, \varphi) \qquad \text{and} \qquad \psi_{n'1m}(r, \theta, \varphi) = R_{n'1}(r)Y_{1m}(\theta, \varphi)$$

a) Express x in terms of r and spherical harmonics.

b) Evaluate the matrix element $\langle n'1m|\mathrm{H}_1|n00\rangle$. Express the radial part of this matrix element in terms of integrals involving R_{n0} and $R_{n'1}$. For the angular part, make use of the fact that Y_{00} is a constant and the normalization condition of the spherical harmonics. Show that the result can be written as

$$\langle n'1m|\mathrm{H}_1|n00\rangle = P\left(\delta_{m,-1} - \delta_{m,1}\right),$$

where P is the factor containing the radial parts of the matrix element of H_1.

c) Using the result from b), set up the 4×4 Hamiltonian in matrix form for the basis set $|n00\rangle, |n'11\rangle, |n'1, -1\rangle, |n'10\rangle$. Include also an energy difference between the states with different n, i.e. $\Delta = \langle n'1m|\mathrm{H}|n'1m\rangle - \langle n00|\mathrm{H}|n00\rangle$, where H is the unperturbed Hamiltonian of the hydrogen atom.

d) Find the eigenenergies for the Hamiltonian obtained in c).

II. POSITRONIUM *[(20+20) PTS]*

In the ground orbital state of positronium (an electron-positron bound state), the Hamiltonian in the presence of an external magnetic field $B\hat{z}$ is given to a good approximation by

$$H = a\vec{S}_e \cdot \vec{S}_p + b(S_e^z - S_p^z)$$

where a is a constant and b is proportional to B, and $\vec{S}_e$ and $\vec{S}_p$ are the spin operators for the electron and positron respectively.

a) Find the energy eigenvalues and eigenstates of H.

b) Now suppose the magnetic field is off, so $b = 0$. At time $t = 0$, the electron and the positron spin components in the z direction are measured to be up and down, respectively. What is the probability that the electron spin will be down when measured again at a later time t?

III. ATTRACTIVE POTENTIAL *[(30+10) PTS]*

Let us consider a particle with mass m in a one-dimensional square potential well with an attractive delta-potential at its center, i.e.,

$$U(x) = \begin{cases} \infty & x \leq -a \\ -U_0\delta(x) & -a < x < a \\ \infty & x \geq a \end{cases},$$

where $2a$ is the width of the square well and U_0 the strength of the delta potential.

a) Derive the equation determining the eigenenergy ϵ of the bound eigenstate of the delta potential.
Hint: This equation has the form $\tanh(ka) = \alpha \frac{k}{U_0}$ with a constant α you should find and wave number $\hbar k = \sqrt{2m|\epsilon|}$.

b) What is the minimum U_0 for which a state with $\epsilon < 0$ exists?

IV. 1D HARMONIC OSCILLATOR *[(10+10+20) PTS]*

Given is a 1D harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

and a wave function which is a mixture of the $n = 0$ and $n = 1$ states

$$\psi(x) = 1/\sqrt{(}5)(u_0(x) - 2u_1(x)) \,.$$

a) Draw $\psi(x)$

b) What is $\langle E \rangle$ in terms of m and ω?

c) What are $\langle \hat{x} \rangle$, $\langle \hat{x}^2 \rangle$, and δx?

NIU Physics PhD Candidacy Exam - Spring 2016

Quantum Mechanics

Do Only Three Out Of Four Problems

I. INTERACTING SPINS *[(16+8+16) PTS]*

We consider two spin-1/2 particles interacting via the operator

$$f = a + b\,\boldsymbol{s}_1 \cdot \boldsymbol{s}_2$$

where a and b are constants and $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are the spin operators for particles 1 and 2. The total spin angular momentum is $\boldsymbol{j} = \boldsymbol{s}_1 + \boldsymbol{s}_2$.

a) Show that f, $\boldsymbol{j}^2$ and j_z can be measured simultaneously.

b) Derive the matrix representation for f in the $|j, m, s_1, s_2\rangle$ basis. (Label rows and columns of your matrix.)

c) Derive the matrix representation for f in the $|s_1, s_2, m_1, m_2\rangle$ basis. (Again, label rows and columns of your matrix.)

II. POTENTIAL WELL *[(14+13+13) PTS]*

Let the potential $V = 0$ for $r < a_0$ (the Bohr radius) and $V = \infty$ for $r > a_0$. V is a function of r only.

a) What is the energy of an electron in the lowest energy state of this potential?

b) How does that compare to the energy of the 1S state of Hydrogen?

c) What is the approximate energy of the lowest energy state with angular momentum greater than 0? Do not try to solve the equation but instead base your approximation on the shape of the well.

III. HARMONIC OSCILLATOR IN ELECTRIC FIELD *[(6+14+20) PTS]*

Consider a simple harmonic oscillator in one dimension with the usual Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2\,.$$

The eigenfunction of the ground state can be written as $\psi_0(x) = \mathcal{N} e^{-\alpha^2 x^2/2}$.

a) Determine the constant $\mathcal{N}$.

b) Calculate both the constant α and the energy eigenvalue of the ground state.

c) At time $t = 0$, an electric field $|E|$ is switched on, adding a perturbation of the form $\hat{H}_1 = e|E|\hat{x}$ to the Hamiltonian. What is the new ground state energy?

IV. EXPANDING POTENTIAL WELL *[(12+16+12) PTS]*

Let us take am infinite potential well in one dimension of width L with a particle of mass m moving inside it ($-\frac{L}{2} < x < \frac{L}{2}$). The particle is initially in the lowest eigenstate.

a) Calculate the energy and wavefunction.

b) At $t = 0$, the walls of the potential well are suddenly moved to $-L$ and L. Calculate the probability of finding the particle in the eigenstates of the new system.

c) What is the expectation value of the energy in the new eigenstates?
(You can make use of the series $\sum\limits_{m=0}^{\infty} \frac{(2m+1)^2}{[(2m+1)^2-4]^2} = \frac{\pi^2}{16}$).

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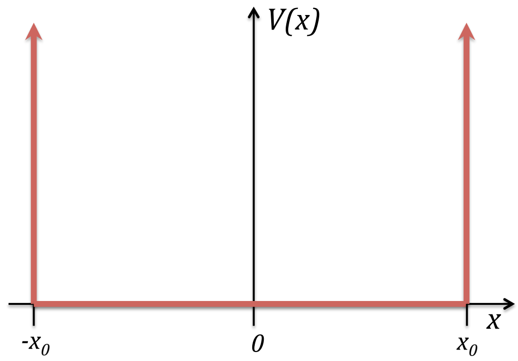
Do Only Three Out Of Four Problems

I. POTENTIAL WELL *[(8+10+10+12) PTS]*

Let us consider a particle of mass m in an infinite square potential well $V(x)$ [0 for $|x| < x_0$, ∞ otherwise]. The potential well is depicted in the figure.



a) Redraw the potential well and sketch the lowest three eigenfunctions (for $n = 1, 2, 3$ with n being the quantum number).

b) Calculate the eigenenergies E_n of the particle. (*Hint:* $E_n = \kappa \frac{n^2\hbar^2}{mx_0^2}$ with a numerical prefactor κ, which you need to calculate.)

Next, we consider the wave function

$$\psi(x) = \frac{\gamma}{\sqrt{x_0}} \left[\cos\left(\pi x/(2x_0)\right) + 2\sin\left(\pi x/x_0\right)\right] ,$$

which is a linear combination of the lowest two eigenfunctions.

c) Find the value of γ, such that $\psi(x)$ is normalized.

d) Calculate the expectation value of the kinetic energy

II. HARMONIC OSCILLATOR *[(20+20) PTS]*

Given is a 2D harmonic oscillator with Hamiltonian

$$\hat{H} = \hat{\mathbf{p}}^2/(2m) + \frac{m\omega^2}{2}(\hat{x}^2 + \hat{y}^2) + km\hat{x}\hat{y} ,$$

with $\hat{\mathbf{p}} = (\hat{p}_x, \hat{p}_y)$.

a) For $k = 0$, what are the energies of the ground state and first and second excited states? What are the degeneracies of each state?

b) For $k > 0$, using first order perturbation theory, what are the energy shifts of the ground state and the first excited states?

III. SPIN IN A MAGNETIC FIELD *[(20+20) PTS]*

The Hamiltonian for a spin $\mathbf{s}$ of a particle with charge e in an applied magnetic field $\mathbf{B}$ is given by

$$\hat{H} = -\frac{ge}{2m}\hat{\mathbf{s}} \cdot \mathbf{B}.$$

where g is the gyromagnetic ratio.

a) Calculate $d\hat{\mathbf{s}}/dt$.

b) Describe the motion if the magnetic field is in the y direction. Express the results in terms of the initial spin components.

IV. TWO FERMIONS *[(10+10+20) PTS]*

Suppose that two **identical** spin-1/2 fermions, each of mass m, interact only via the potential

$$V(r) = \frac{4}{3}\frac{a^2}{\hbar^2}\frac{\vec{S}_1 \cdot \vec{S}_2}{r}$$

where r is the distance between the particles, and $\vec{S}_1$ and $\vec{S}_2$ are the spins of particles 1 and 2 respectively, and a is a constant.

a) What is the value of the total spin for the bound states of the system?

b) What values of the orbital angular momentum are allowed for bound states?

c) Find the energies and degeneracies of the ground state and the first two excited states of the system.

NIU Physics PhD Candidacy Exam - Spring 2015

Quantum Mechanics

DO ONLY THREE OUT OF FOUR PROBLEMS

I. FLUX QUANTUM *[(5+10+20+5) PTS]*

A charged particle (charge $-e$) moves in a (three dimensional) space having an infinite, impenetrable cylinder of radius a along the z-axis in its center. ψ_0 shall be the solution of the stationary Schrödinger equation outside the cylinder without magnetic field.

a) Now we apply a magnetic field $\mathbf{B}$ to the system, which is determined by the vector potential, $\mathbf{A}$. Write down the corresponding Schrödinger equation.

b) Here the vector potential will be

$$\mathbf{A}(r,\varphi,z) = \begin{cases} \frac{1}{2}Br\hat{\varphi}, & r < a \\ \frac{a^2 B}{2r}\hat{\varphi}, & r > a \end{cases}$$

where $\hat{\varphi}$ is the angular unit vector. Calculate the magnetic field distribution, $\mathbf{B}$, from this vector potential.

c) Use the functional form $\psi = e^{-i\gamma\chi}\psi_0$ with

$$\chi(\mathbf{x}) = \int_{\mathbf{x_0}}^{\mathbf{x}} \mathbf{A}\cdot d\mathbf{s}$$

for the wave function and solve the Schrödinger equation corresponding to the above vector potential. Find the constant γ.

d) For which flux Ba^2 is the wave function ψ unique?

Hint: $\nabla = (\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial z})$, $\nabla\times\mathbf{v} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\varphi} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ v_r & rv_\varphi & v_z \end{vmatrix}$.

II. NEUTRONS IN THE BOX *[(10+10+10+10) PTS]*

Eight non-interacting neutrons are confined to a 3D square well of size $D = 5$fm ($5\cdot 10^{-15}$m) such that $V = -50M$eV for $0 < x < D$, $0 < y < D$, $0 < z < D$ and $V = 0$ everywhere else.

a) How many energy levels are there in this well?

b) What is the degeneracy of each energy level?

c) What is the approximate Fermi energy for this system?

d) What is the relative probability to be in the lowest energy state to the fourth lowest energy state at $k_B T = 10MeV$? Just write down the ratio (do not calculate the value)

Useful constants: mass of neutron = 940 MeV/c^2, $c\hbar$ = 197 MeV·fm, hc = 1240 MeV·fm

III. ATTRACTIVE WALL *[(12+12+8+8) PTS]*

Let us consider a step potential with an attractive δ-function potential at the edge

$$U(x) = U\theta(x) - \frac{\hbar^2 g}{2m}\delta(x).$$

a) Calculate the wave function for $E > V$.

b) Calculate the reflection coefficient $|R|^2$ and discuss the limit $E \gg U \gg \hbar^2 g/2m$.

c) Determine the wavefunction for the bound state.

d) What is the energy of the bound state?

IV. K-CAPTURE *[(13+13+14) PTS]*

The K-capture process involves the absorption of an inner orbital electron by the nucleus, resulting in the reduction of the nuclear charge Z by one unit. This process is due to the non-zero probability that an electron can be found within the volume of the nucleus. Suppose that an electron is in the $1s$ state of a Hydrogen-like potential, with wavefunction given by:

$$\psi(r) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

where a_0 is the Bohr radius.

a) Calculate the probability that a $1s$ electron will be found within the nucleus. Take the nuclear radius to be R, which you may assume is much smaller than a_0.

b) Assume that an electron is initially in the ground state with $Z = 2$, and a nuclear reaction abruptly changes the nuclear charge to $Z = 1$. What is the probability that the electron will be found in the ground state of the new potential after the change in nuclear charge?

c) Now assume instead that the nuclear reaction leaves the electron in a state given by the wavefunction

$$\Psi(r,\theta,\phi) = A(\sin\theta\,\sin\phi + \sin\theta\,\cos\phi + \cos\theta)re^{-r/a_0},$$

where A is an appropriate normalization constant. What are the possible values that can be obtained in measurements of L^2 and L_z, and with what probabilities will these values be measured?