NIU Physics PhD Candidacy Exam – Fall 2010 – Quantum Mechanics
DO ONLY THREE OUT OF FOUR QUESTIONS

Problem 1.

We consider a spinless particle with mass m and charge q that is confined to move on a circle of radius R centered around the origin in the x-y plane.

(a) Write down the Schrödinger equation for this particle and solve it to find the eigenenergies and corresponding normalized eigenfunctions. Are there degeneracies? [10 points]
(b) This system is perturbed by an electric field $\boldsymbol{E}$ pointing along the x axis. To lowest nonvanishing order in perturbation theory, find the corrections to the eigenenergies of the system. [10 points]
(c) What are the corrections to the eigenfunctions due to the field $\boldsymbol{E}$ in lowest nonvanishing order? [10 points]
(d) Next we consider instead of the electric field $\boldsymbol{E}$ the effect of a magnetic field $\boldsymbol{B}$ pointing along the z axis. Evaluate to lowest nonvanishing order in perturbation theory the corrections to the eigenenergies of the system. [10 points]

Problem 2.

Let us consider two spins S and S' with $S = S' = \frac{1}{2}$. The z components of the spin are $S_z = \pm\frac{1}{2}$ and $S'_z = \pm\frac{1}{2}$. We can define a basis set as $|SS_z, S'S'_z\rangle$ (or simplified $|S_z, S'_z\rangle$). The spins interact with each other via the interaction

$$H = T\mathbf{S} \cdot \mathbf{S}', \tag{1}$$

where T is a coupling constant. $\mathbf{S}$ and $\mathbf{S}'$ work on the spins S and S', respectively.

(a) Rewrite the interaction in terms of S_z, S'_z and step up and down operators $S_\pm$ and $S'_\pm$. [10 points]
(b) Find the eigenvalues of H when the spins are parallel. [10 points]
(c) Find the eigenvalues of H for $S_z + S'_z = 0$. [13 points]
(d) Give a physical interpretation of the eigenenergies and eigenstates of H.[7 points]

Problem 3.

Given a one-dimensional harmonic oscillator with Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2 \tag{2}$$

and a wavefunction which is a mixture of the $n = 0$ and $n = 1$ states

$$\psi(x) = \frac{1}{\sqrt{5}}(u_0(x) - 2u_1(x)), \tag{3}$$

where u_0 and u_1 are the normalized eigenfunctions of the lowest two energy states. Note that $a^{\pm} = (\mp i p_x + m\omega x)/\sqrt{2\hbar m\omega}$.

(a) draw $\psi(x)$. [5 points]

(b) what is $\langle E\rangle$ in terms of m and ω? [8 points]

(c) what are $\langle x\rangle$, $\langle x^2\rangle$ and Δx? [17 points]

(d) what is $\langle p\rangle$? [10 points]

Problem 4.

The (unnormalized) eigenfunctions for the lowest energy eigenvalues of a one-dimensional simple harmonic oscillator (SHO) are

$$\psi_0(x) = e^{-x^2/a^2}, \qquad \psi_1(x) = \tfrac{x}{a}\, e^{-x^2/a^2}, \qquad \psi_2(x) = \left(1 - \tfrac{4x^2}{a^2}\right) e^{-x^2/a^2},$$

$$\psi_3(x) = \left(\tfrac{3x}{a} - \tfrac{4x^3}{a^3}\right) e^{-x^2/a^2}, \qquad \psi_4(x) = \left(3 - \tfrac{24x^2}{a^2} + \tfrac{16x^4}{a^4}\right) e^{-x^2/a^2}.$$

Now consider an electron in "half" a one-dimensional SHO potential (as sketched below)

$$V(x) = \begin{cases} K\,x^2 & x > 0 \\ \infty & x \le 0 \end{cases} \tag{4}$$

(a) Sketch the ground and first excited state for this new potential. [6 points]

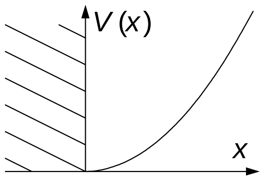
(b) Write the normalized wave function for the ground state in terms of the electron mass m and the oscillator frequency ω (corresponding to the spring constant K). [6 points]

(c) What are the energy eigenvalues for the potential $V(x)$? [6 points]

(d) Now we add a constant electric field $\mathcal{E}$ in x direction. Use first-order perturbation theory to estimate the new ground state energy.[8 points]

(e) We go back to $\mathcal{E} = 0$. Now we add a second electron. Ignoring the Coulomb interaction between the electrons, write the total energy and the new two-particle wave function, assuming that the electrons are in a singlet spin state with the lowest possible energy. (You can ignore wave function normalization now.) [7 points]

(f) Repeat part (e) assuming that the electrons are instead in a triplet spin state with the lowest possible energy. [7 points]



NIU Physics PhD Candidacy Exam – Spring 2010 – Quantum Mechanics
DO ONLY THREE OUT OF FOUR QUESTIONS

Problem 1. In many systems, the Hamiltonian is invariant under rotations. An example is the hydrogen atom where the potential $V(r)$ in the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(r),$$

depends only on the distance to the origin.
An infinitesimally small rotation along the z-axis of the wavefunction is given by

$$R_{z,d\varphi}\psi(x, y, z) = \psi(x - yd\varphi, y + xd\varphi, z),$$

(a) Show that this rotation can be expressed in terms of the angular momentum component L_z. [10 points]
(b) Starting from the expression of L_z in Cartesian coordinates, show that L_z can be related to the derivative with respect to φ in spherical coordinates. Derive the φ-dependent part of the wavefunction corresponding to an eigenstate of L_z. [10 points]
(c) Show that if $R_{z,d\varphi}$ commutes with the Hamiltonian, then there exist eigenfunctions of H that are also eigenfunctions of $R_{z,d\varphi}$. [10 points]
(d) Using the fact that L_i with $i = x, y, z$ commute with the Hamiltonian, show that $\mathbf{L}^2$ commutes with the Hamiltonian. [10 points]

Problem 2. To an harmonic oscillator Hamiltonian

$$H = \hbar\omega a^\dagger a,$$

we add a term

$$H' = \lambda(a^\dagger + a).$$

This problem is known as the displaced harmonic oscillator. It can be diagonalized exactly by adding a constant (let us call it Δ) to the step operators.
(a) Express the constant Δ in terms of $\hbar\omega$ and λ. [8 points]
(b) The energies are shifted by a constant energy. Express that energy in terms of $\hbar\omega$ and λ. [8 points]
(c) Express the new eigenstates $|\tilde{n}\rangle$ in terms of the displaced oscillator operator $\tilde{a}^\dagger$. [8 points]
(d) Calculate the matrix elements $\langle\tilde{n}|0\rangle$.[8 points]
(e) An harmonic oscillator is in the ground state of H. At a certain time, the Hamiltonian suddenly changes to $H + H'$. Plot the probability and change in energy for the final states $|\tilde{n}\rangle$ for $\tilde{n} = 0, \cdots, 5$ for $\Delta = 2$. [8 points]

Problem 3. We consider scattering off a spherical potential well given by

$$V(r) = \begin{cases} -V_0 & r \leq a \\ 0 & r > a \end{cases} \qquad V_0, a > 0$$

The particles' mass is m. We restrict ourselves to low energies, where it is sufficient to consider s wave scattering (angular momentum $l = 0$).
(a) Starting from the Schrödinger equation for this problem, derive the phase shift δ_0.[14 points]
(b) Calculate the total scattering cross section σ assuming a shallow potential well ($a\sqrt{2mV_0/\hbar^2} \ll 1$). [10 points]
(c)Show that the same total scattering cross section σ as in b) is also obtained when using the Born approximation. Note: part c) is really independent of parts a) and b). [16 points]

Problem 4. The normalized wavefunctions for the $2s$ and $2p$ states of the hydrogen atom are:

$$\begin{aligned} \psi_{2s} &= \frac{1}{\sqrt{32\pi a^3}} \left(N - r/a\right) e^{-r/2a} \\ \psi_{2p,0} &= \frac{1}{\sqrt{32\pi a^3}} \left(r/a\right) e^{-r/2a} \cos\theta \\ \psi_{2p,\pm 1} &= \frac{1}{\sqrt{64\pi a^3}} \left(r/a\right) e^{-r/2a} \sin\theta \, e^{\pm i\phi}. \end{aligned}$$

where a is the Bohr radius and N is a certain rational number.
(a) Calculate N. (Show your work; no credit for just writing down the answer.) [10 points]
(b) Find an expression for the probability of finding the electron at a distance greater than a from the nucleus, if the atom is in the $2p, +1$ state. (You may leave this answer in the form of a single integral over one variable.) [10 points]
(c) Now suppose the atom is perturbed by a constant uniform electric field $\vec{E} = E_0\hat{z}$. Find the energies of the $2s$ and $2p$ states to first order in E_0. [20 points]

NIU Physics PhD Candidacy Exam – Fall 2009 – Quantum Mechanics

DO ONLY THREE OUT OF FOUR QUESTIONS

Problem 1.

Consider the effects of the hyperfine splitting of the ground state of the Hydrogen atom in the presence of an external magnetic field $\vec{B} = B_0\hat{z}$. Let the electron spin operator be $\vec{S}$ and the proton spin operator be $\vec{I}$, and call the total angular momentum operator $\vec{J} = \vec{S} + \vec{I}$. Then the Hamiltonian for the system is:

$$H = \frac{E_\gamma}{\hbar^2}\vec{S}\cdot\vec{I} + 2\frac{\mu_B}{\hbar}\vec{B}\cdot\vec{S}, \tag{1}$$

where E_γ is the energy of the famous 21 cm line and μ_B is the Bohr magneton. The states of the system may be written in terms of angular momentum eigenstates of S_z, I_z or J^2, J_z, so clearly label which basis you are using in each of your answers.

(a) In the limit that B_0 is so large that E_γ can be neglected, find the energy eigenstates and eigenvalues. [12 points]

(b) In the limit that B_0 is so small that it can be neglected, find the energy eigenstates and eigenvalues. [15 points]

(c) Find the energy eigenvalues for general B_0, and show that the special limits obtained in parts (a) and (b) follow. [13 points]

Problem 2.

A quantum mechanical spinless particle of mass m is confined to move freely on the circumference of a circle of radius R in the x, y plane.

(a) Find the allowed energy levels of the particle, and the associated wavefunctions. [16 points]

(b) Now suppose the particle has a charge q and is placed in a constant electric field which is also in the x, y plane. Calculate the shifts in energy levels to second order in the electric field, treated as a perturbation. [16 points]

(c) Show that the degeneracies are not removed to *any* order in the electric field treated as a perturbation. [8 points]

Problem 3.

Given a 2D harmonic oscillator with Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) + kmxy \tag{2}$$

(a) How does $\langle x \rangle$ change with time, that is determine $d\langle x \rangle/dt$? [10 points]
(b) For $k = 0$, what are the energies of the ground state and first and second excited states? What are the degeneracies of each state? [15 points]
(c) For $k > 0$, using first order perturbation theory, what are the energy shifts of the ground state and the first excited states? [15 points]

Problem 4.

In many systems, the Hamiltonian is invariant under rotations. An example is the hydrogen atom where the potential $V(r)$ in the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(r), \tag{3}$$

depends only on the distance to the origin.

An infinitesimally small rotation along the z-axis of the wavefunction is given by

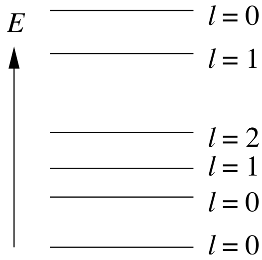
$$R_{z,d\varphi}\psi(x, y, z) = \psi(x - yd\varphi, y + xd\varphi, z), \tag{4}$$

(a) Show that this rotation can be expressed in terms of the angular momentum component L_z. [10 points]
(b) Starting from the expression of L_z in Cartesian coordinates, show that L_z can be related to the derivative with respect to φ in spherical coordinates. Derive the φ-dependent part of the wavefunction corresponding to an eigenstate of L_z. [10 points]
(c) Show that if $R_{z,d\varphi}$ commutes with the Hamiltonian, then there exist eigenfunctions of H that are also eigenfunctions of $R_{z,d\varphi}$. [10 points]
(d) Using the fact that L_i with $i = x, y, z$ commute with the Hamiltonian, show that $\mathbf{L}^2$ commutes with the Hamiltonian. [10 points]

2000 Spring Ph.D. Candidacy Exam, Quantum Mechanics
DO ONLY 3 OUT OF 4 OF THE QUESTIONS

Problem 1.

The diagram shows the six lowest energy levels and the associated angular monenta for a spinless particle moving in a certain three-dimensional central potential. There are no "accidental" degeneracies in this energy spectrum. Give the number of nodes (changes in sign) in the radial wave function associated with each level. Justify your answer.



Problem 2.

Assume that the mu-neutrino ν_μ and the tau-neutrino ν_τ are composed of a mixture of two mass eigenstates ν_1 and ν_2. The mixing ratio is given by

$$\begin{pmatrix} \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{1}$$

In free space, the states ν_1 and ν_2 evolve according to

$$\begin{pmatrix} |\nu_1(x,t)\rangle \\ |\nu_2(x,t)\rangle \end{pmatrix} = e^{ipx/\hbar} \begin{pmatrix} e^{-iE_1 t/\hbar}|\nu_1(0)\rangle \\ e^{-iE_2 t/\hbar}|\nu_2(0)\rangle \end{pmatrix} \tag{2}$$

Show that the transition probability for a mu-neutrino into a tau-neutrino is given by

$$P(\mu \to \tau) = \sin^2(2\theta)\sin^2\frac{(E_2 - E_1)t}{2\hbar}. \tag{3}$$

Problem 3.

Let the potential $V = 0$ for $r < a_0$ (the Bohr radius) and $V = \infty$ for $r > a_0$. V is a function of r only.
a) What is the energy of an electron in the lowest energy state of this potential?
b) How does that compare to the energy of the $1s$ state of Hydrogen?
c) What is the approximate energy of the lowest energy state with angular momentum greater than 0 (you can leave this result in integral form)?

Problem 4.
The Hamiltonian for a two-dimensional harmonic oscillator is given by

$$H_0 = (a^\dagger a + b^\dagger b + 1)\hbar\omega, \tag{4}$$

with the coordinates given by

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \quad \text{and} \quad y = \sqrt{\frac{\hbar}{2m\omega}}(b^\dagger + b). \tag{5}$$

a)Give the eigenvalues of this Hamiltonian.

The Hamiltonian is perturbed by

$$H' = \alpha xy, \tag{6}$$

where α is a small constant.
b) Express H' in terms of the operators a and b and their conjugates.
c) Using degenerate perturbation theory, show how the eigenvalues are changed by H' for the states with eigenenergy $2\hbar\omega$ for H_0.

2008 Fall Ph.D. Candidacy Exam, Quantum Mechanics
DO ONLY 3 OUT OF 4 OF THE QUESTIONS

Problem 1.

We consider a system made of the orthonormalized spin states $|+\rangle$ and $|-\rangle$, where $S_z|\pm\rangle = \pm(\hbar/2)|\pm\rangle$. Initially both of these states are energy eigenstates with the same energy ϵ.

a) An interaction V couples these spin states, giving rise to the matrix elements

$$\langle 1|V|1\rangle = \langle 2|V|2\rangle = 0 \qquad \text{and} \qquad \langle 1|V|2\rangle = \Delta$$

Give the Hamiltonian H of the interacting system.

b) Determine the energy eigenvalues of H.

c) Show that the states $|A\rangle$ and $|B\rangle$

$$|A\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + \frac{\Delta^*}{|\Delta|}|-\rangle\right) \qquad \text{and} \qquad |B\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - \frac{\Delta^*}{|\Delta|}|-\rangle\right)$$

are orthonormalized eigenstates of the interacting system.

d) Determine the time evolution of a state $|\psi(t)\rangle$ for which $|\psi(t=0)\rangle = |+\rangle$.

e) For the state $|\psi(t)\rangle$, calculate the probability that a measurement of S_z at time t yields $\pm\hbar/2$.

Problem 2.

A particle of mass m is constrained to move between two concentric impermeable spheres of radii $r = a$ and $r = b$. There is no other potential. Find the ground state energy and normalized wave function.

Problem 3.

Given a two-dimensional oscillator with Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) + kmxy, \tag{1}$$

a) What is the time dependence of $d\langle x\rangle/dt$?

b) What is the time dependence of $d\langle p_x\rangle/dt$?

c) For $k = 0$, what are the energies of the ground state and the first and second excited states? What are the degeneracies of each state?

d) For $k > 0$, using first-order perturbation theory, what are the energy shifts of the ground state and the first excited states?

Problem 4.

a) We want to study the spin-orbit coupling for a level with $l = 3$. How do you expect that this level will split under the interaction $\zeta\mathbf{L}\cdot\mathbf{S}$. Give also the degeneracies.

b) Show that for an arbitrary angular momentum operator (integer and half-integer), we can write

$$J_\pm|jm_j\rangle = \sqrt{(j \mp m_j)(j \pm m_j + 1)}|jm_j\rangle \tag{2}$$

(take $\hbar = 1$).

c) Since m_j is a good quantum number for the spin-orbit coupling, we can consider the different m_j values separately. Give the matrix for $\zeta\mathbf{L}\cdot\mathbf{S}$ in the $|lm, \frac{1}{2}\sigma\rangle$ basis with $\sigma = \pm\frac{1}{2}$ for $m_j = 5/2$. Find the eigenvalues and eigenstates of this matrix.

d) Write down the matrix for the spin-orbit coupling in the $|jm_j\rangle$ basis for $m_j = 5/2$.

e) Obtain the same eigenstates as in question c) by starting for the $m_j = 7/2$ state using the step operators.

2008 Spring Ph.D. Candidacy Exam, Quantum Mechanics
DO ONLY 3 OUT OF 4 OF THE QUESTIONS

Problem 1.

Consider a particle with mass m confined to a three-dimensional spherical potential well

$$V(r) = \begin{cases} 0, & r \leq a \\ V_0, & r > a \end{cases} \tag{1}$$

a) Give the Schrödinger equation for this problem.

b) Determine the explicit expressions for the ground state energy and the ground state wave function in the limit $V_0 \to \infty$.

c) For the more general case $0 < V_0 < \infty$, determine the transcendental equation from which we can obtain the eigenenergies of the particle for angular momentum $l = 0$.

d) Which condition must be fulfilled such that the transcendental equation derived in c) can be solved? (Hint: consider a graphical solution of the equation.) Compare this result with a particle in a one-dimensional rectangular well of depth V_0.

Problem 2.

The ground state energy and Bohr radius for the Hydrogen atom are

$$E_1 = -\frac{\hbar^2}{2ma_B^2}, \quad a_B = \frac{4\pi\varepsilon_0\hbar^2}{e^2 m}. \tag{2}$$

a) Calculate the ground state energy (in eV) and Bohr radius (in nm) of positronium (a hydrogen-like system consisting of an electron and a positron).

b) What is the degeneracy of the positronium ground state due to the spin? Write down the possible eigenvalues of the total spin together with the corresponding wavefunctions.

c) The ground state of positronium can decay by annihilation into photons. Calculate the energy and angular momentum released in the process and prove there must be at least two photons in the final state.

Problem 3.

A particle of mass m moves in one dimension inside a box of length L. Use first order perturbation theory to calculate the lowest order correction to the energy levels arising from the relativistic variation of the particle mass. You can assume that the effect of relativity is small. Note that the free particle relativistic Hamiltonian is $\hat{H}_{\rm rel} = \sqrt{m^2c^4 + p^2c^2} - mc^2$.

Problem 4.

a) Prove the variational theorem that states that for any arbitrary state

$$\langle\psi|\hat{H}|\psi\rangle \geq E_0. \tag{3}$$

b) Consider the Hamiltonian for a particle moving in one dimension

$$\hat{H} = \frac{\hat{p}^2}{2m} + V_0 \left(\frac{\hat{x}}{a}\right)^6, \tag{4}$$

where m is mass, a a length scale, and V_0 an energy scale. Is the wavefunction

$$\psi(x) = C(x^2 - a^2)e^{-(x/d)^4}, \tag{5}$$

where C is a normalization constant, d an adjustable parameter with dimensions of length, a good choice for the variational approximation to the *ground state*? Why or why not?

c) Depending on the answer of the previous part: If it is a good choice, make a rough order-of-magnitude estimate of the optimal choice for d. If it is not a good choice, propose a better variational wavefunction, including an estimate of the length scale.

2007 Fall Ph.D. Qualifier, Quantum Mechanics
DO ONLY 3 OUT OF 4 OF THE QUESTIONS

Problem 1.

The spin-orbit coupling of an electron of angular momentum l and spin $s = \frac{1}{2}$ is described by the Hamiltonian

$$H = \lambda \mathbf{l} \cdot \mathbf{s}, \tag{1}$$

where λ is the spin-orbit coupling parameter.

(a) Write down the matrix H and diagonalize it to show that the state is split into two states with total angular momentum $j = l \pm \frac{1}{2}$. Find the eigenenergies.

(b) Show that the eigenenergies can also be determined using the relation $\mathbf{j} = \mathbf{l} + \mathbf{s}$.

[The raising and lowering operators for l are $l_+|l, m_l\rangle = \sqrt{(l - m_l)(l + m_l + 1)}|l, m_l + 1\rangle$ and $l_-|l, m_l\rangle = \sqrt{(l + m_l)(l - m_l + 1)}|l, m_l - 1\rangle$ and similarly for s].

Problem 2.

(a) Write down the nonrelativistic Hamiltonian for a Helium atom with two electrons.

(b) Write down the ground state wavefunction (include the spin part) and give the ground state energy E_0 (in eV) in the absence of electron-electron interactions.

(c) Write down the matrix element for the lowest-order correction E_1 to the ground-state energy due to the electron-electron interaction. The $1s$ orbital is given by

$$\varphi_{100} = \frac{1}{\sqrt{8\pi}} \left(\frac{2Z}{a_0} \right)^{3/2} e^{-Zr/a_0} \tag{2}$$

The matrix element can be evaluated giving $\frac{5}{8} \frac{Ze^2}{4\pi\varepsilon_0 a_0}$.

(d) Find the ratio of the correction E_1 to E_0.

The Rydberg constant is $R = \frac{Z^2\hbar^2}{2ma_0^2}$ and the Bohr radius is $a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$.

Problem 3.

Two identical spin 1/2 fermions described by position coordinates $\vec{r}_i$ $(i = 1, 2)$ are bound in a three-dimensional isotropic harmonic oscillator potential

$$V(\vec{r}_i) = \frac{1}{2} m \omega^2 r_i^2. \tag{3}$$

(a) Write the wave functions of the system in terms of the single-particle spin eigenstates and the one-dimensional harmonic oscillator wave functions, for each of the energy eigenstates up to and including energy $4\hbar\omega$.

(b) Assume that in addition there is a weak spin-independent interaction V between the particles:

$$V(\vec{r}_1 - \vec{r}_2) = -\lambda \delta^{(3)}(\vec{r}_1 - \vec{r}_2) \tag{4}$$

Find the energies of the system correct to first order in λ for each of the unperturbed states found in part (a). You may leave your results in terms of definite integrals over known functions.

Problem 4.

Consider normal 1-dimensional particle in box potential ($V(x) = \infty$ for $|x| > L/2$ and $V(x) = 0$ inside box. Two identical particles are confined to the box (assume only orbital degrees of freedom, ignore spin).

(a) What is the normalized unperturbed ground state for

- two identical bosons of mass m confined in the box
- two identical fermions of mass m confined in the box
- And what are the unperturbed ground state energies of the two cases?

(b) Now a perturbation is applied. A small rectangular bump appears in the box between $-a/2$ and $+a/2$. This perturbation is $V_{pert} = +|V_0|$ for $|x| < a/2$ and is zero otherwise.
Use first-order perturbation theory to obtain the new ground state energies for the two cases.

2007 Spring Ph.D. Qualifier, Quantum Mechanics
DO ONLY 3 OUT OF 4 OF THE QUESTIONS

Problem 1.

A particle moves in a 1 dimensional potential described by an attractive delta function at the origin. The potential is;

$$V(x) = -W\delta(x)$$

(a) Discuss and determine the wavefunctions valid for bound state solutions of this system.

(b) Show that there is only one bound state and determine its energy.

Problem 2.

In this problem, $|0\rangle$, $|n\rangle$ are the shorthand for the eigenstates of the 1 dimensional simple harmonic oscillator (SHO) Hamiltonian, with $|0\rangle$, denoting the ground state. The $\hat{a}^\dagger$ and $\hat{a}$ are the SHO raising and lowering operators. (sometimes termed creation and destruction (annihilation) operators).

(a) Prove that the following state vector $|z\rangle$ is an eigenstate of the lowering operator $\hat{a}$ and that its eigenvalue is z.

$$|z\rangle = e^{z\hat{a}^\dagger}|0\rangle$$

The z is an arbitrary complex number. (Note, knowledge of the expansion of e^x will be useful.)

(b) Evaluate $\langle z_1|z_2\rangle$, where z_1 and z_2 are arbitrary complex numbers, and use this result to normalize state $|z\rangle$.

Problem 3.

Let the potential $V = 0$ for $r < a_0$ (the Bohr radius) and $V = \infty$ for $r > a_0$. V is a function of r only.

(a) What is the energy of an electron in the lowest energy state of this potential?

(b) How does this compare to the kinetic energy of the $1s$ state of Hydrogen?

(c) What is the approximate energy of the lowest energy state with angular momentum greater than 0 (you can leave this result in integral form)?

Problem 4.

In a magnetic resonance experiment a specimen containing nuclei of spin $I = \frac{1}{2}$ and magnetic moment $\mu = \hbar\gamma I$ is placed in a static magnetic field B_0 directed along the z-axis and a field B_1 which rotates in the xy-plane with angular frequency ω.

(a) Write down the Hamiltonian for the system.

(b) If the wave function is written

$$\psi(t) = c_+(t)\chi_{\frac{1}{2}} + c_-(t)\chi_{-\frac{1}{2}}$$

where $\chi_{\frac{1}{2}}$ and $\chi_{-\frac{1}{2}}$ are the spin eigenfunctions, show that

$$i\frac{dc_+}{dt} = \frac{1}{2}\omega_0 c_+ + \frac{1}{2}\omega_1 c_- e^{-i\omega t}$$

and

$$i\frac{dc_-}{dt} = -\frac{1}{2}\omega_0 c_- + \frac{1}{2}\omega_1 c_+ e^{i\omega t}$$

and where $\omega_0 = \gamma B_0$ and $\omega_1 = \gamma B_1$. Assuming that the system starts in the state $\chi_{-\frac{1}{2}}$, i.e. $c_+(0) = 0$ and $c_-(0) = 1$, solve these equations to show that subsequently the probability that the system is in the state $\chi_{\frac{1}{2}}$ is

$$|c_+|^2 = \omega_1^2 \frac{\sin^2 \frac{1}{2}[(\omega - \omega_0)^2 + \omega_1^2]^{\frac{1}{2}} t}{(\omega - \omega_0)^2 + \omega_1^2}$$